Dirichlet Process Mixtures

A gentle tutorial

Graphical Models – 10708
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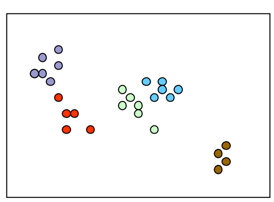
Motivation

- We are given a data set, and are told that it was generated from a mixture of Gaussians.



- Unfortunately, no one has any idea *how many* Gaussians produced the data.

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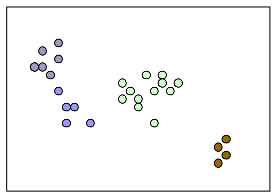


- Unfortunately, no one has any idea *how many* Gaussians produced the data.

What to do?

- We can guess the number of clusters, do EM for Gaussian Mixture Models, look at the results, and then try again…
- We can do hierarchical agglomerative clustering, and cut the tree at a visually appealing level…
- We want to cluster the data in a statistically principled manner, without resorting to hacks.

Review: Dirichlet Distribution

- Let $\Theta = \{\theta_1, \theta_2, \ldots, \theta_m\}$
- We write:

$$\Theta \sim \text{Dirichlet}(\alpha_1, \alpha_2, \ldots, \alpha_m)$$

- Distribution over possible parameter vectors for a multinomial distribution, and is in fact the conjugate prior for the multinomial.
- Beta distribution is the special case of a Dirichlet for 2 dimensions.
- Samples from the distribution lie in the *m-1* dimensional simplex
- Thus, it is in fact a "distribution over distributions."

Dirichlet Process

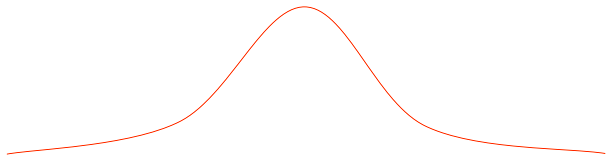
- A *Dirichlet Process* is also a distribution over distributions.
- We write:

$$G \sim DP(\alpha, G_0)$$

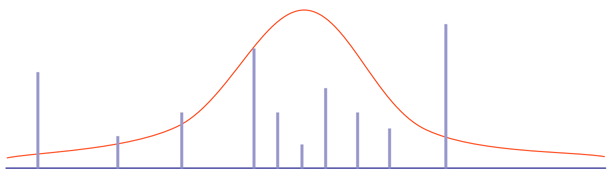
  - G_0 is a base distribution
  - α is a positive scaling parameter
- G has the same support as G_0

Dirichlet Process

- Consider Gaussian G_0

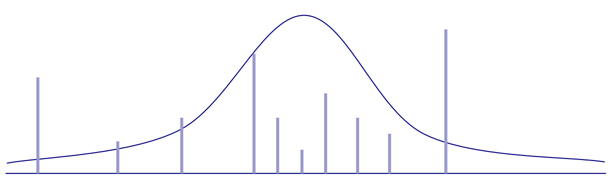


- $G \sim DP(\alpha, G_0)$



Dirichlet Process

- $G \sim DP(\alpha, G_0)$

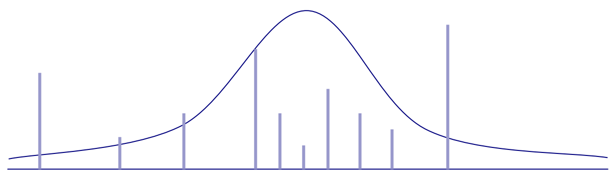


- G_0 is continuous, so the probability that any two samples are equal is precisely zero.
- However, G is a discrete distribution, made up of a countably infinite number of point masses [Blackwell]
  - Therefore, there is always a non-zero probability of two samples colliding

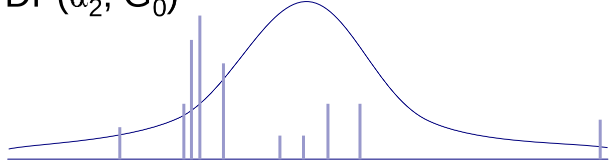
Dirichlet Process

- $G \sim DP(\alpha_1, G_0)$

α values determine how close G is to G_0



- $G \sim DP(\alpha_2, G_0)$

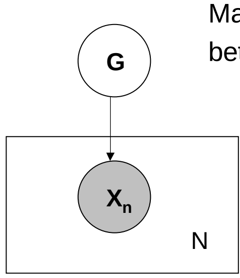


Sampling from a DP

$G \sim DP(\alpha, G_0)$

$X_n \mid G \sim G$ for n = {1, …, N} (iid)

Marginalizing out G introduces dependencies between the X_n variables



$$P(X_1, \ldots, X_N) = \int P(G) \prod_{n=1}^{N} P(X_n|G) dG$$

Sampling from a DP

$$P(X_1, \dots, X_N) = \int P(G) \prod_{n=1}^{N} P(X_n|G)dG$$

Assume we view these variables in a specific order, and are interested in the behavior of X_n given the previous n - 1 observations.

$$X_n|X_1, \dots, X_{n-1} = \begin{cases} X_i & \text{with probability } \frac{1}{n-1+\alpha} \\ \text{new draw from } G_0 & \text{with probability } \frac{\alpha}{n-1+\alpha} \end{cases}$$

Let there be K unique values for the variables:

$$X_k^* \text{ for } k \in \{1, \dots, K\}$$

Sampling from a DP

$$X_n|X_1, \dots, X_{n-1} = \begin{cases} X_i & \text{with probability } \frac{1}{n-1+\alpha} \\ \text{new draw from } G_0 & \text{with probability } \frac{\alpha}{n-1+\alpha} \end{cases}$$

$$P(X_1, \dots, X_N) = P(X_1)P(X_2|X_1)\dots P(X_N|X_1, \dots, X_{N-1})$$

Chain rule

$$= \frac{\alpha^K}{\alpha(1+\alpha)\dots(N-1+\alpha)} \prod_{k=1}^{K} G_0(X_k^*)$$

P(partition) **P(draws)**

Notice that the above formulation of the joint does not depend on the order we consider the variables. We can arrive at a mixture model by assuming exchangeability and applying DeFinetti's Theorem (1935).

Chinese Restaurant Process

$$X_n | X_1, \ldots, X_{n-1} = \begin{cases} X_i & \text{with probability } \frac{1}{n-1+\alpha} \\ \text{new draw from } G_0 & \text{with probability } \frac{\alpha}{n-1+\alpha} \end{cases}$$

Let there be *K* unique values for the variables:

$$X_k^* \text{ for } k \in \{1, \ldots, K\}$$

Can rewrite as:

$$X_n | X_1, \ldots, X_{n-1} = \begin{cases} X_k^* & \text{with probability } \frac{\text{num}_{n-1}(X_k^*)}{n-1+\alpha} \\ \text{new draw from } G_0 & \text{with probability } \frac{\alpha}{n-1+\alpha} \end{cases}$$

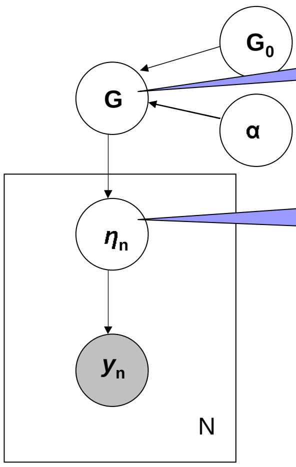
Chinese Restaurant Process

$$X_n | X_1, \ldots, X_{n-1} = \begin{cases} X_k^* & \text{with probability } \frac{\text{num}_{n-1}(X_k^*)}{n-1+\alpha} \\ \text{new draw from } G_0 & \text{with probability } \frac{\alpha}{n-1+\alpha} \end{cases}$$

Consider a restaurant with infinitely many tables, where the X_n's represent the patrons of the restaurant. From the above conditional probability distribution, we can see that a customer is more likely to sit at a table if there are already many people sitting there. However, with probability proportional to α, the customer will sit at a new table.

Also known as the "clustering effect," and can be seen in the setting of social clubs. [Aldous]

Dirichlet Process Mixture



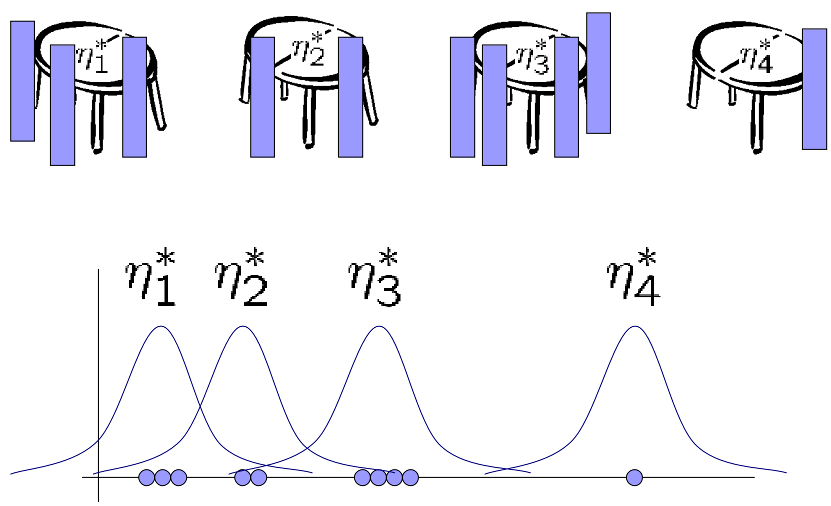
countably infinite number of point masses

draw N times from G to get parameters for different mixture components

If η_n were drawn from e.g. a Gaussian, no two values would be the same, but since they are drawn from a distribution drawn from a Dirichlet Process, we expect a clustering of the η_n

unique values for η_n = # mixture components

CRP Mixture



Stick Breaking

- So far, we've just mentioned properties of a distribution G drawn from a Dirichlet Process
- In 1994, Sethuraman developed a constructive way of forming G, known as "stick breaking"

Stick Breaking

$$V_1, V_2, \ldots, V_i, \ldots \sim \text{Beta}(1, \alpha)$$

$$f(V_i = v_i | \alpha) = \alpha(1 - v_i)^{\alpha - 1}$$

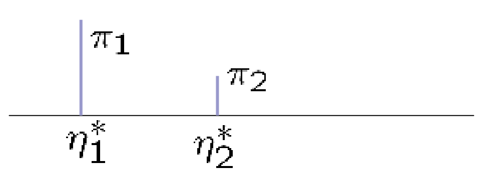
$$\eta_1^*, \eta_2^*, \ldots, \eta_i^*, \ldots \sim G_0$$

$$\pi_i(\mathbf{v}) = v_i \prod_{j=1}^{i-1} (1 - v_j)$$

$$G = \sum_{i=1}^{\infty} \pi_i(\mathbf{v}) \delta_{\eta_i^*}$$

1. Draw η_1* from G_0
2. Draw v_1 from Beta(1, α)
3. $\pi_1 = v_1$
4. Draw η_2* from G_0
5. Draw v_2 from Beta(1, α)
6. $\pi_2 = v_2(1 - v_1)$

…



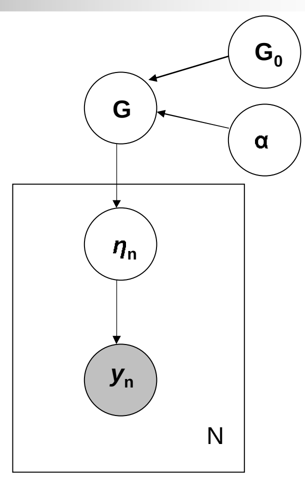
Formal Definition

- Let α be a positive, real-valued scalar
- Let G_0 be a non-atomic probability distribution over support set A
- We say $G \sim DP(\alpha, G_0)$, if for all natural numbers k and k-partitions $\{A_1, \ldots, A_k\}$,

$$(G(A_1), \ldots, G(A_k)) \sim \text{Dirichlet}(\alpha G_0(A_1), \ldots, \alpha G_0(A_k))$$

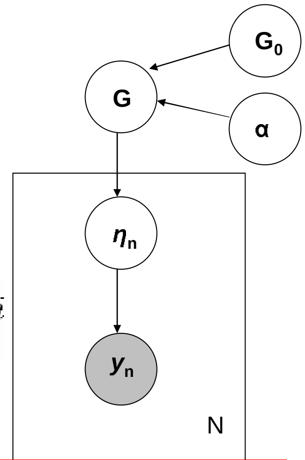
Inference in a DPM

- EM is generally used for inference in a mixture model, but G is nonparametric, making EM difficult
- Markov Chain Monte Carlo techniques [Neal 2000]
- Variational Inference [Blei and Jordan 2006]



Gibbs Sampling [Neal 2000]

- Algorithm 1:
  - Define H_i to be the *single observation posterior*
  - We marginalize out *G* from our model, and sample each η_n given everything else



$$\eta_i|\eta_{-i}, y_i \sim \sum_{j \neq i} q_{i,j}\delta(\eta_j) + r_i H_i$$

$$q_{i,j} = bF(y_i, \eta_j)$$

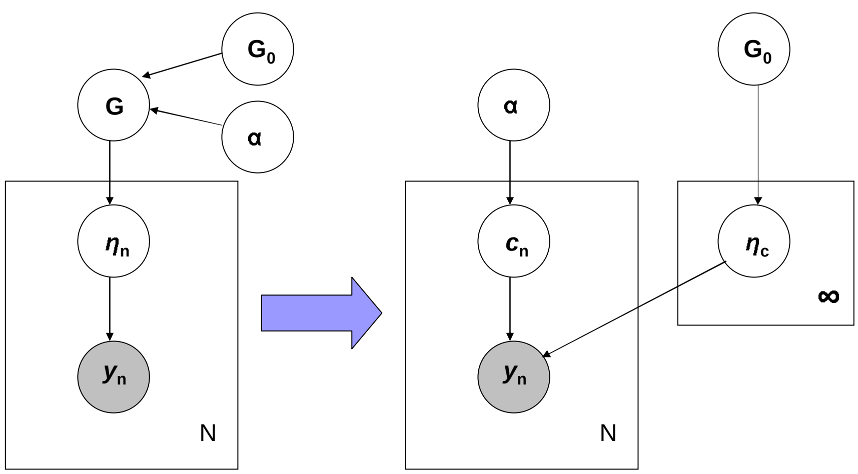
$$r_i = b\alpha \int F(y_i, \eta) dG_0(\eta)$$

SLOW TO CONVERGE!

Gibbs Sampling [Neal 2000]

- Algorithm 2:

[Grenager 2005]



Gibbs Sampling [Neal 2000]

- Algorithm 2 (cont.):
  - We sample from the distribution over an individual cluster assignment c_n given y_n, and all the other cluster assignments

1. Initialize cluster assignments $c_1, \ldots, c_N$
2. For i=1,…,N, draw c_i from:

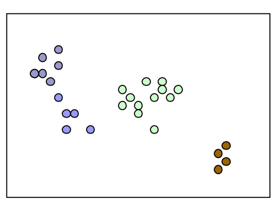
$$P(c_i = c | c_{-i}, y_i, \eta) \propto \frac{n_{-i,c}}{n-1+\alpha} F(y_i, \eta_c) \quad \text{if } c = c_j \text{ for some } j \neq i$$

$$P(c_i \neq c_j \forall j \neq i | c_{-i}, y_i) \propto \frac{\alpha}{n-1+\alpha} \int F(y_i, \eta) dG_0(\eta) \quad \text{otherwise}$$

3. For all c, draw $\eta_c \mid y_i$ (for all i such that $c_i = c$)

Conclusion

- We now have a statistically principled mechanism for solving our original problem.



- This was intended as a general and fairly shallow overview of Dirichlet Processes.

Acknowledgments

- Much thanks goes to David Blei.
- Some material for this presentation was inspired by slides from Teg Grenager and Zoubin Ghahramani.

References

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