New reading:
Chapter 7 of Koller&Friedman

Clique trees 2

Graphical Models – 10708

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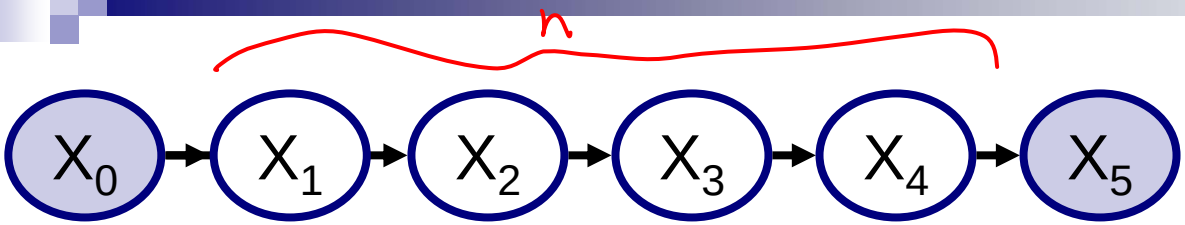
Announcements

- **Homework 2:**
  - **Out today/tomorrow**
  - **Programming part in groups of 2-3**
- **Class project**
  - **More details on Wednesday**

What if I want to compute $P(X_i|x_0,x_{n+1})$ for each i?



Compute:

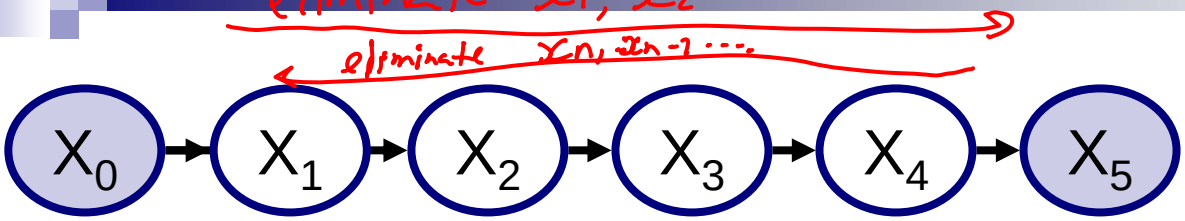
$$P(X_i \mid x_0, x_{n+1})$$

Variable elimination for each i? $O(n)$

Variable elimination for ~~each~~ every i, what's the complexity?

naive : $O(n^2)$

Reusing computation



Compute:

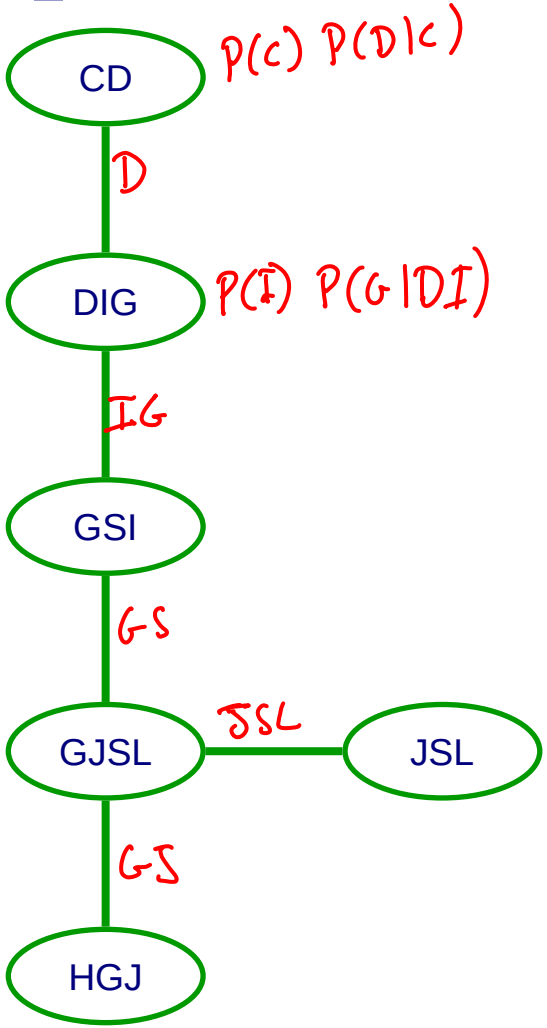
$$P(X_i \mid x_0, x_{n+1})$$

$$P(X_4, x_0, x_5) = \sum_{x_1,x_2,x_3} P(x_0)\, P(x_1 \mid x_0)\, P(x_2 \mid x_1) \ldots$$

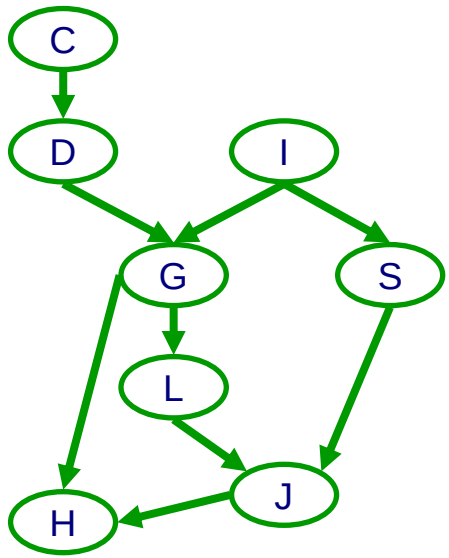
$$= P(x_0) \sum_{x_2,x_3} P(x_3 \mid x_2) \ldots \underbrace{\sum_{x_1} P(x_1 \mid x_0)\, P(x_2 \mid x_1)}_{g_1(x_2)}$$

$$P(X_3, x_0, x_5) = \sum_{x_1,x_2,x_4} P(x_0)\, P(x_1 \mid x_0) \ldots = \sum_{x_2,x_4} P(x_3 \mid x_2) \ldots \underbrace{\sum_{x_1} P(x_1 \mid x_0) P(x_2 \mid x_1)}_{g_1(x_2)}$$

Cluster graph

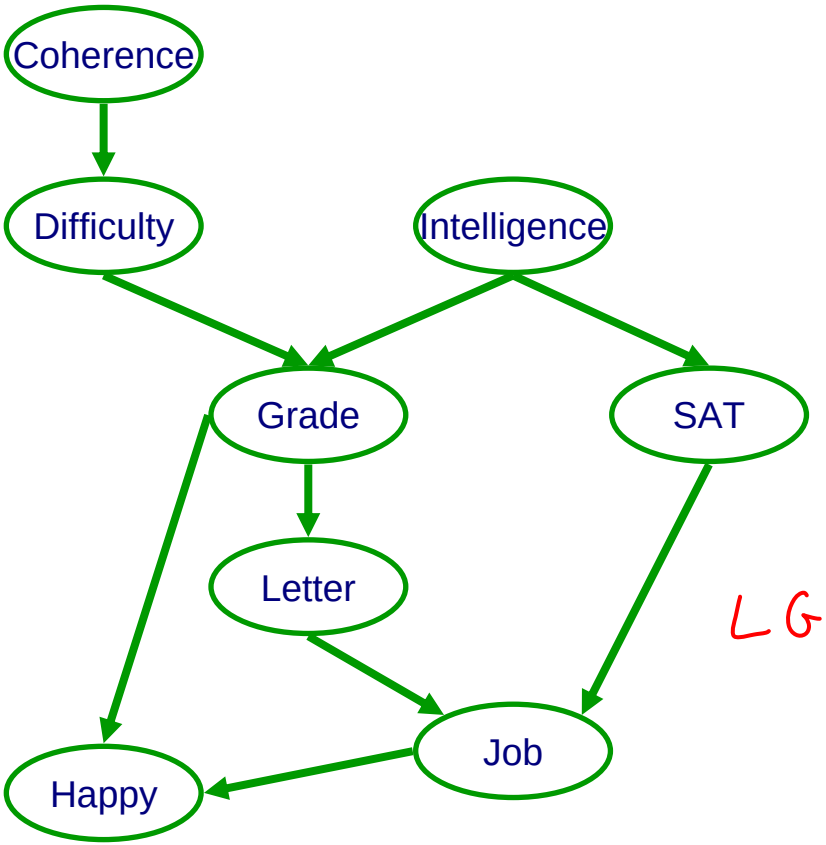


- **Cluster graph**: For set of factors *F*
  - Undirected graph
  - Each node i associated with a cluster $\mathbf{C}_i$
  - *Family preserving*: for each factor $f_j \in F$, $\exists$ node i such that scope[f_j] $\subseteq \mathbf{C}_i$
  - Each edge i – j is associated with a separator $\mathbf{S}_{ij} = \mathbf{C}_i \cap \mathbf{C}_j$



Factors generated by VE

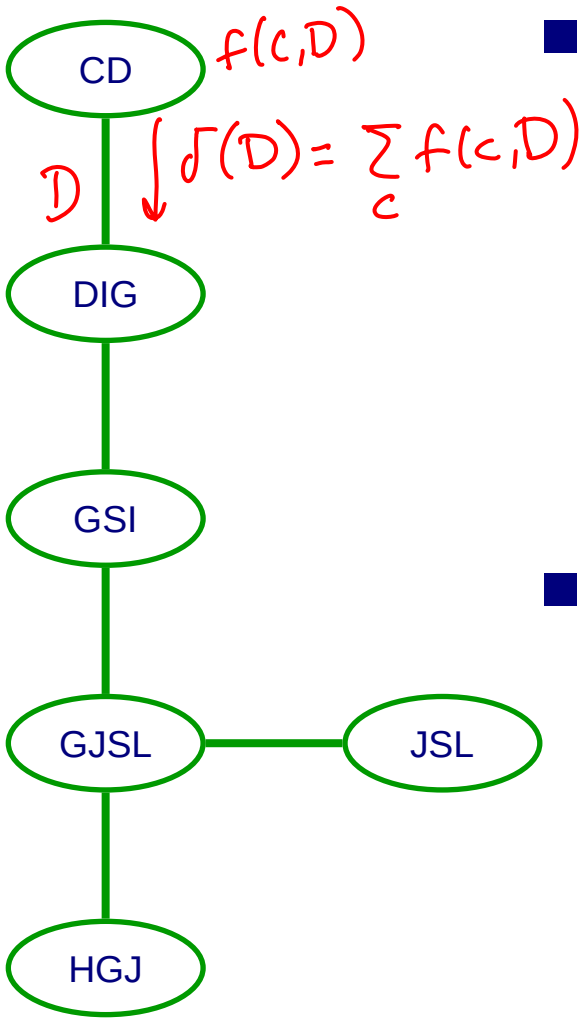
P(C) P(D|C) P(I)
P(G|D,I), P(S|I),
P(L|G) P(J|L,S)
P(H|G,J)



C
DC — D — DGI
GI — I
GSI — SI
GS
LG — GJL — GJLS — JLS
GJL — GJ
HGJ — GJ

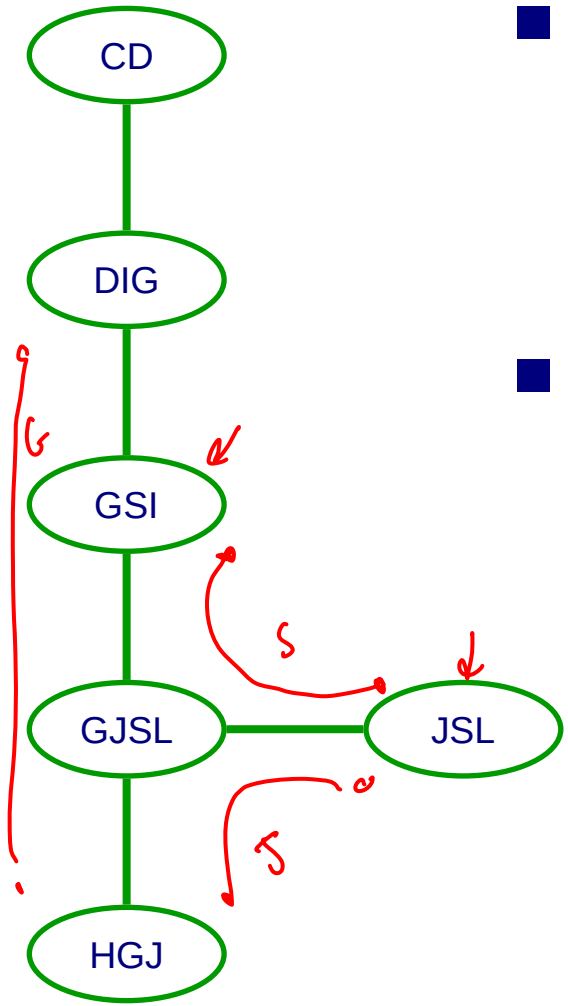
Elimination order:
{C,D,I,S,L,H,J,G}

Cluster graph for VE



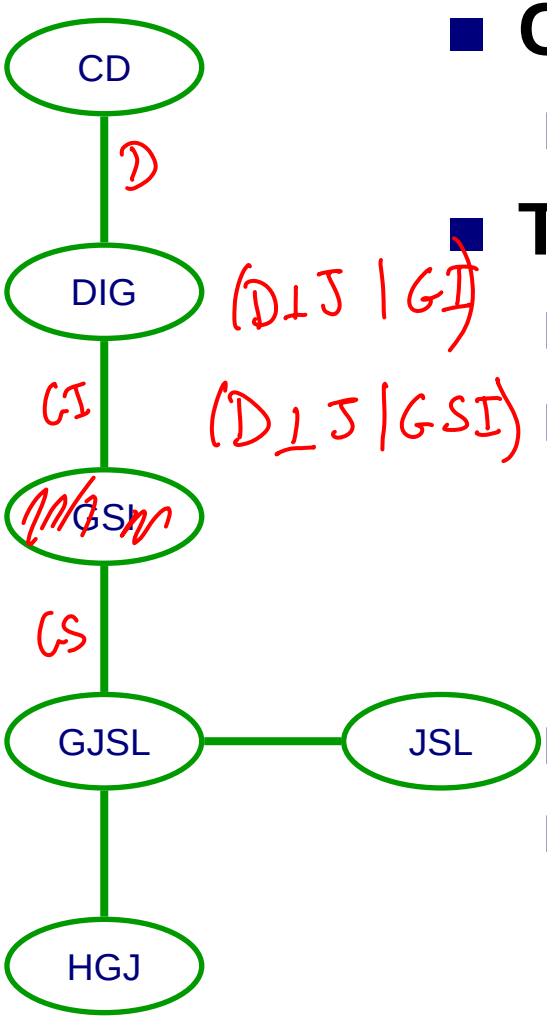
- **VE generates cluster tree!**
  - One clique for each factor used/generated
  - Edge i – j, if f_i used to generate f_j
  - "Message" from i to j generated when marginalizing a variable from f_i
  - Tree because factors only used once
- **Proposition**:
  - "Message" δ_{ij} from i to j
  - Scope[δ_{ij}] $\subseteq$ $\mathbf{S}_{ij}$

Running intersection property



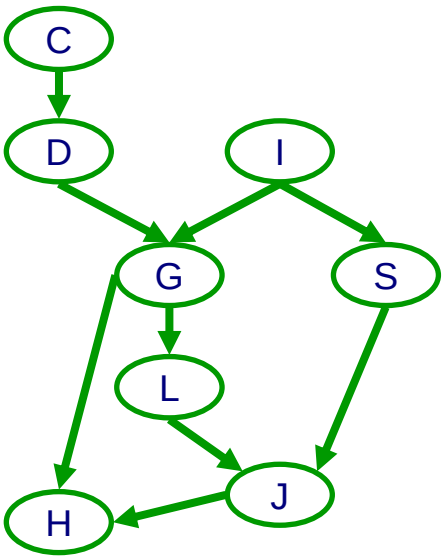
- **Running intersection property (RIP)**
  - Cluster tree satisfies RIP if whenever $X \in \mathbf{C}_i$ and $X \in \mathbf{C}_j$ then X is in every cluster in the (unique) path from $\mathbf{C}_i$ to $\mathbf{C}_j$
- **Theorem**:
  - Cluster tree generated by VE satisfies RIP

Clique tree & Independencies



- **Clique tree (or Junction tree)**
  - A cluster tree that satisfies the RIP
- **Theorem**:
  - Given some BN with structure *G* and factors *F*
  - For a clique tree *T* for *F* consider $\mathbf{C}_i$ – $\mathbf{C}_j$ with separator $\mathbf{S}_{ij}$:
    - **X** – any set of vars in $\mathbf{C}_i$ side of the tree
    - **Y** – any set of vars in $\mathbf{C}_i$ side of the tree
  - Then, $(\mathbf{X} \perp \mathbf{Y} \mid \mathbf{S}_{ij})$ in BN
  - Furthermore, $I(T) \subseteq I(G)$

Variable elimination in a clique tree 1

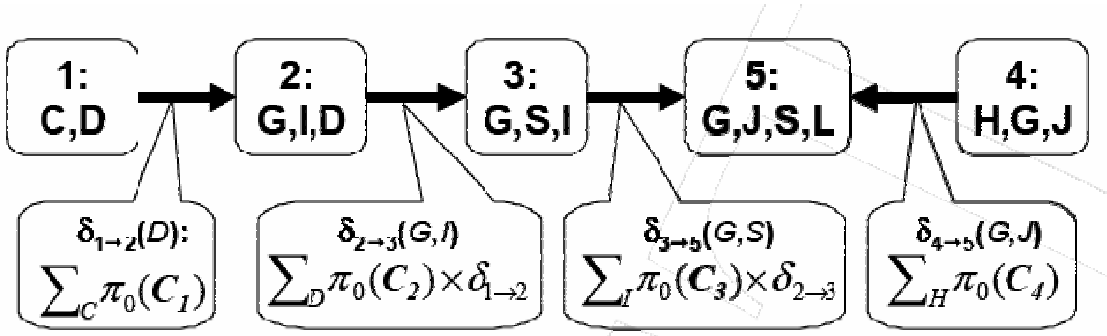


- **Clique tree for a BN**
  - Each CPT assigned to a clique
  - Initial potential $\pi_0(\mathbf{C}_i)$ is product of CPTs

Variable elimination in a clique tree 2

- **VE in clique tree to compute $P(X_i)$**
  - Pick a root (any node containing X_i)
  - Send messages recursively from leaves to root
    - Multiply incoming messages with initial potential
    - Marginalize vars that are not in separator
  - Clique *ready* if received messages from all neighbors

Beliefs from messages

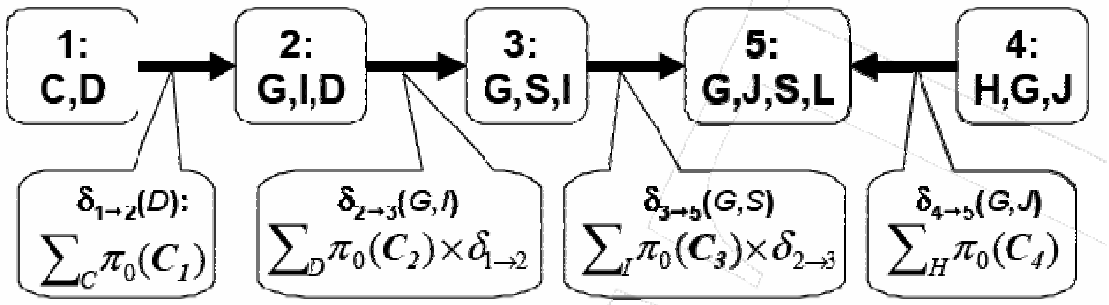


- **Theorem**: When **clique C_i is ready**
  - Receive messages from all neighbors
  - Belief $\pi_i(\mathbf{C}_i)$ is product of initial factor with messages:

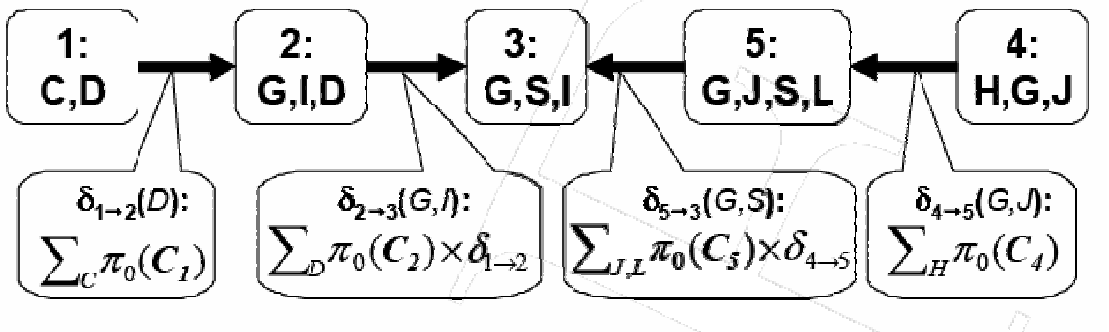
Choice of root

- **Message does not depend on root!!!**

Root: node 5

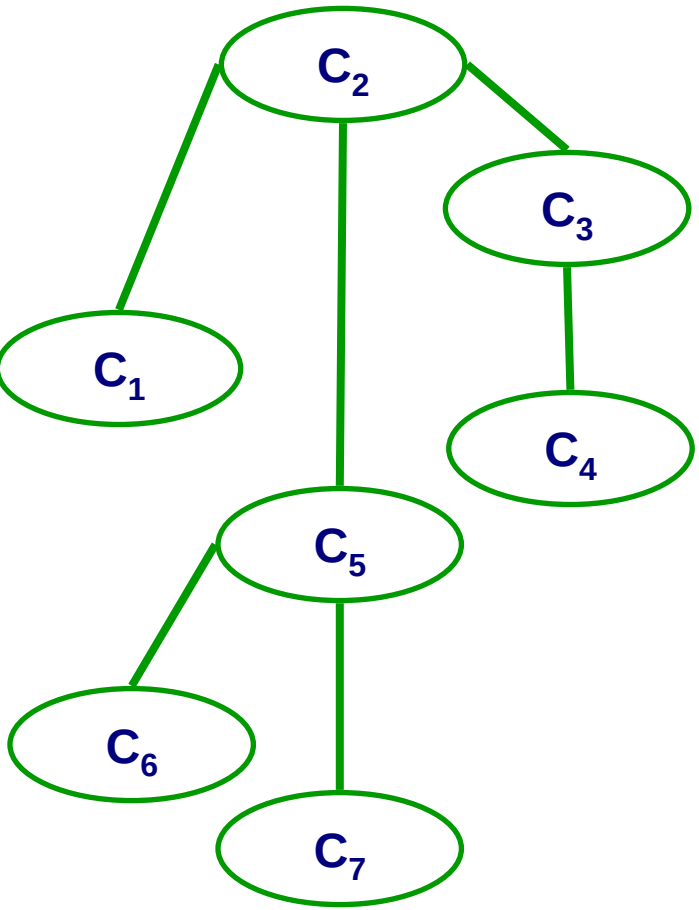


Root: node 3



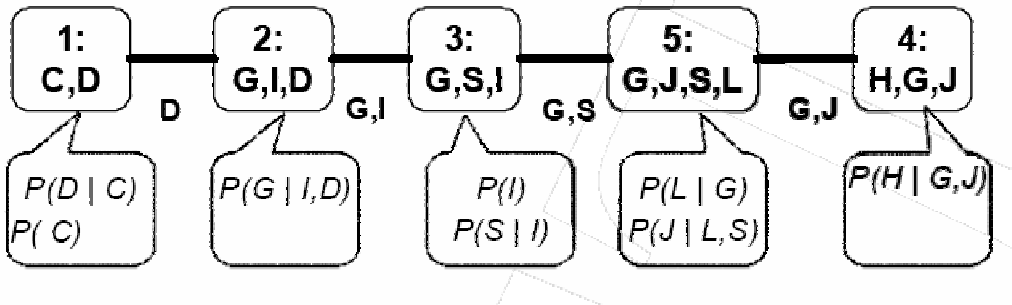
"Cache" computation: Obtain belief for all roots in linear time!!

Shafer-Shenoy Algorithm (a.k.a. VE in clique tree for all roots)



- Clique $\mathbf{C}_i$ *ready to transmit* to neighbor $\mathbf{C}_j$ if received messages from all neighbors but j
  - Leaves are always ready to transmit
- While $\exists$ $\mathbf{C}_i$ ready to transmit to $\mathbf{C}_j$
  - Send message $\delta_{i \to j}$
- Complexity: Linear in # cliques
  - One message sent each direction in each edge
- **Corollary**: At convergence
  - Every clique has correct belief

Calibrated Clique tree



- Initially, neighboring nodes don't agree on "distribution" over separators
- **Calibrated clique tree**:
  - At convergence, tree is *calibrated*
  - Neighboring nodes agree on distribution over separator

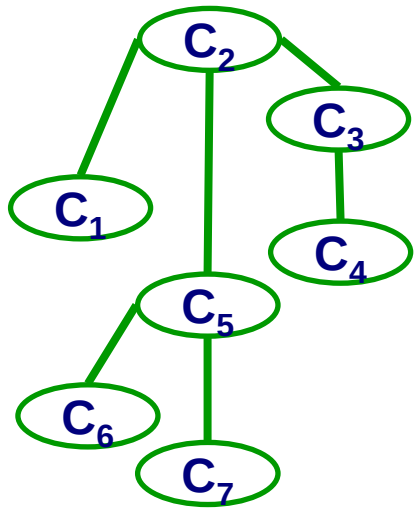
Message passing with division

- Computing messages by multiplication:

- Computing messages by division:

Lauritzen-Spiegelhalter Algorithm (a.k.a. belief propagation)

Simplified description
see reading for details



- Initialize all separator potentials to 1
  - $\mu_{ij} \leftarrow 1$
- All messages ready to transmit
- While $\exists\ \delta_{i\rightarrow j}$ ready to transmit
  - $\mu_{ij}' \leftarrow$
  - If $\mu_{ij}' \neq \mu_{ij}$
    - $\delta_{i\rightarrow j} \leftarrow$
    - $\pi_j \leftarrow \pi_j \times \delta_{i\rightarrow j}$
    - $\mu_{ij} \leftarrow \mu_{ij}'$
    - $\forall$ neighbors k of j, k$\neq$ i, $\delta_{j\rightarrow k}$ ready to transmit
- Complexity: Linear in # cliques
  - for the "right" schedule over edges (leaves to root, the root to leaves)
- **Corollary**: At convergence, every clique has correct belief

VE versus BP in clique trees

- VE messages (the one that multiplies)
- BP messages (the one that divides)

Clique tree invariant

- **Clique tree potential**:
  - Product of clique potentials divided by separators potentials

- Clique tree **invariant**:
  - P(**X**) = π_T

Belief propagation and clique tree invariant

- **Theorem**: Invariant is maintained by BP algorithm!

- BP reparameterizes potentials and messages
  - At convergence, potentials and messages are marginal distributions

Subtree correctness

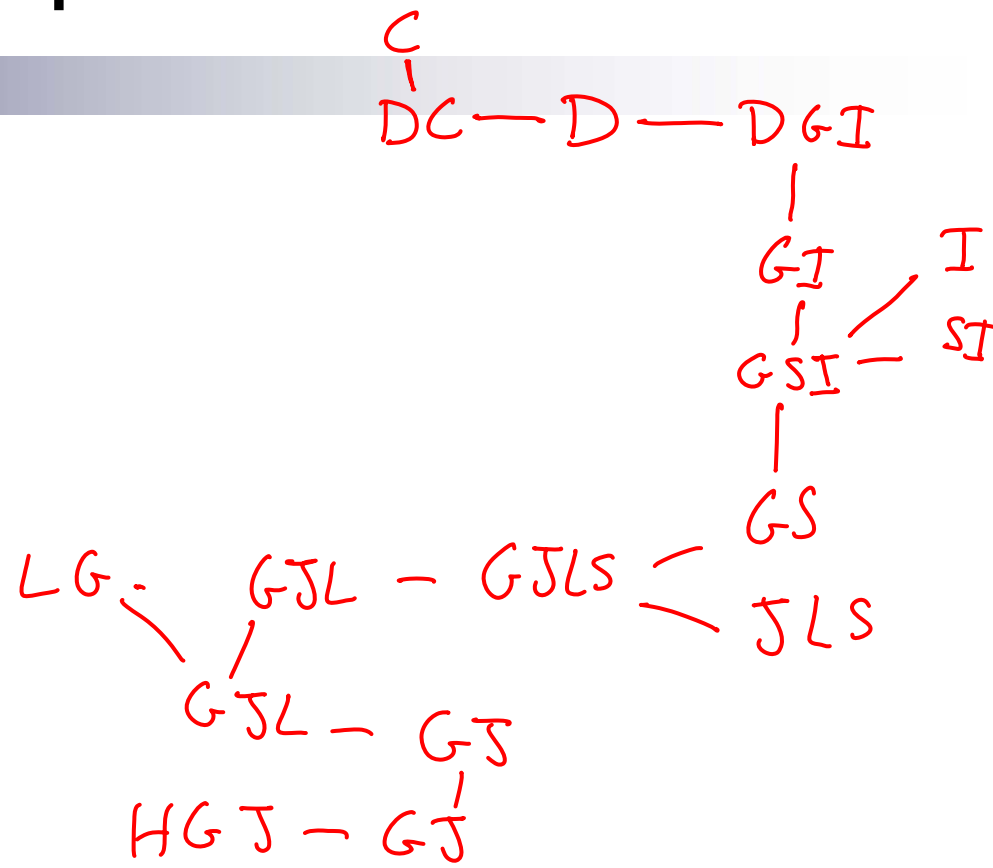
- **Informed message** from i to j, if all messages into i (other than from j) are informed
  - Recursive definition (leaves always send informed messages)
- **Informed subtree**:
  - All incoming messages informed
- **Theorem**:
  - Potential of connected informed subtree *T'* is marginal over scope[*T'*]
- **Corollary**:
  - At convergence, clique tree is *calibrated*
    - π_i = P(scope[π_i])
    - μ_{ij} = P(scope[μ_{ij}])

Answering queries with clique trees

- Query within clique
- Incremental updates – Observing evidence Z=z
  - Multiply some clique by indicator **1**(Z=z)
- Query outside clique
  - Use variable elimination!

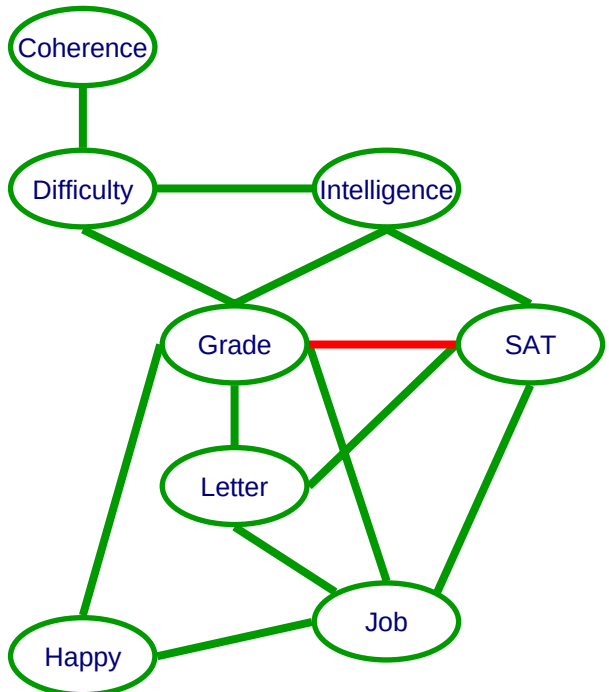
Constructing a clique tree from VE

- Select elimination order $\prec$
- Connect factors that would be generated if you run VE with order $\prec$
- Simplify!
  - Eliminate factor that is subset of neighbor



Find clique tree from chordal graph

- Triangulate moralized graph to obtain chordal graph
- Find maximal cliques
  - NP-complete in general
  - Easy for chordal graphs
  - Max-cardinality search from last lecture
- Generate weighted graph over cliques
  - Edge weights (i,j) is separator size – $|\mathbf{C}_i \cap \mathbf{C}_j|$
- Maximum spanning tree finds clique tree satisfying RIP!!!



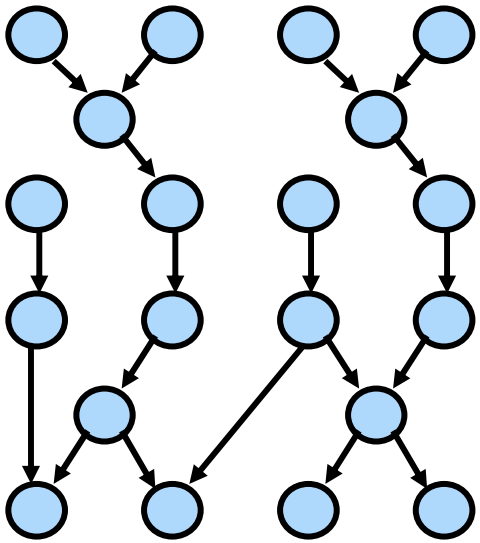
Clique trees versus VE

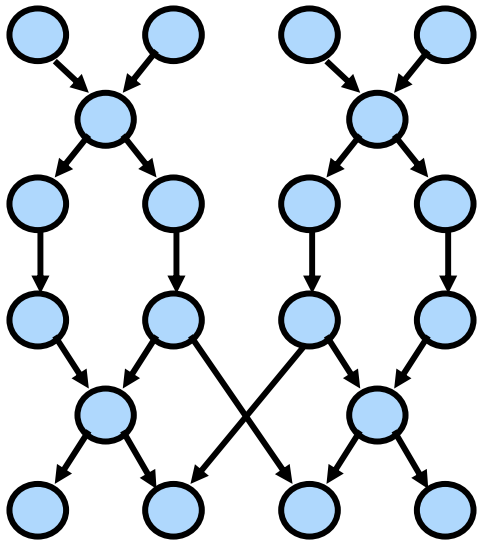
- Clique tree advantages
  - Multi-query settings
  - Incremental updates
  - Pre-computation makes complexity explicit
- Clique tree disadvantages
  - Space requirements – no factors are “deleted”
  - Slower for single query
  - Local structure in factors may be lost when they are multiplied together into initial clique potential

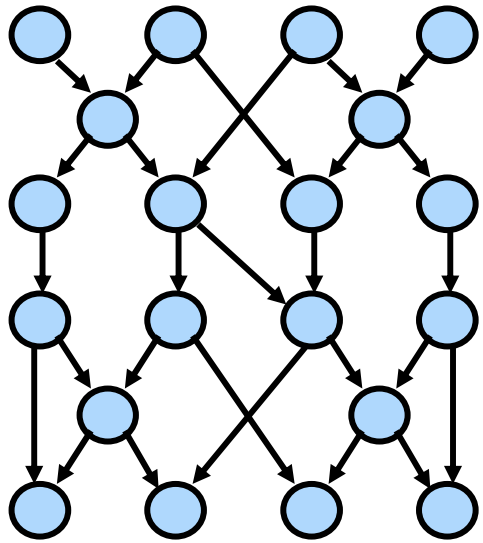
Clique tree summary

- Solve marginal queries for all variables in only twice the cost of query for one variable
- Cliques correspond to maximal cliques in induced graph
- Two message passing approaches
  - VE (the one that multiplies messages)
  - BP (the one that divides by old message)
- Clique tree invariant
  - Clique tree potential is always the same
  - We are only reparameterizing clique potentials
- Constructing clique tree for a BN
  - from elimination order
  - from triangulated (chordal) graph
- Running time (only) exponential in size of largest clique
  - Solve **exactly** problems with thousands (or millions, or more) of variables, and cliques with tens of nodes (or less)

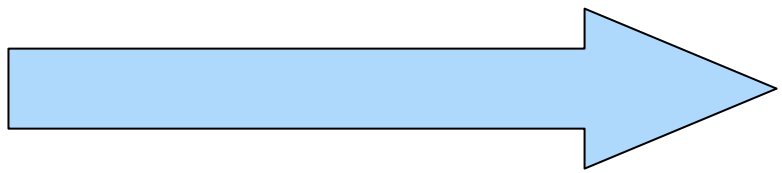
Global Structure: Treewidth w



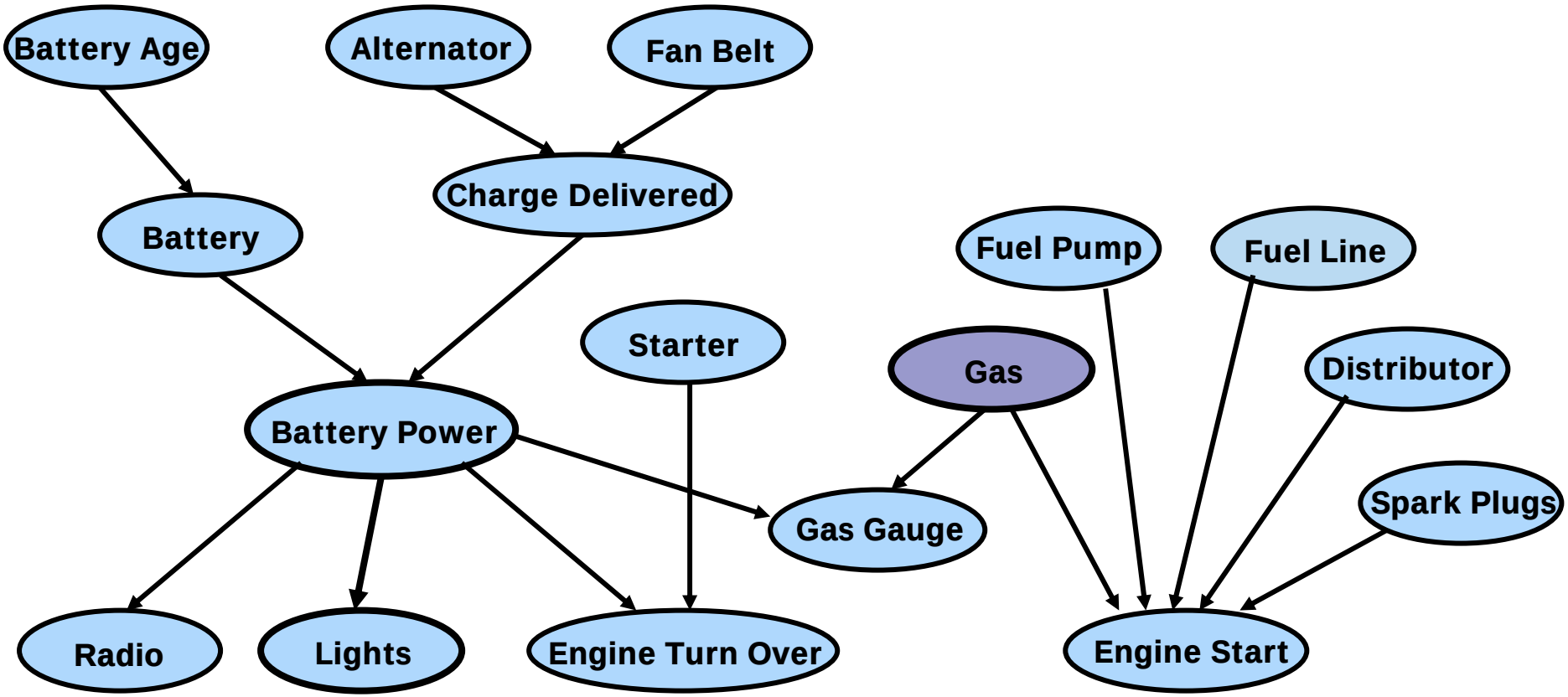




$O(n \exp(w))$

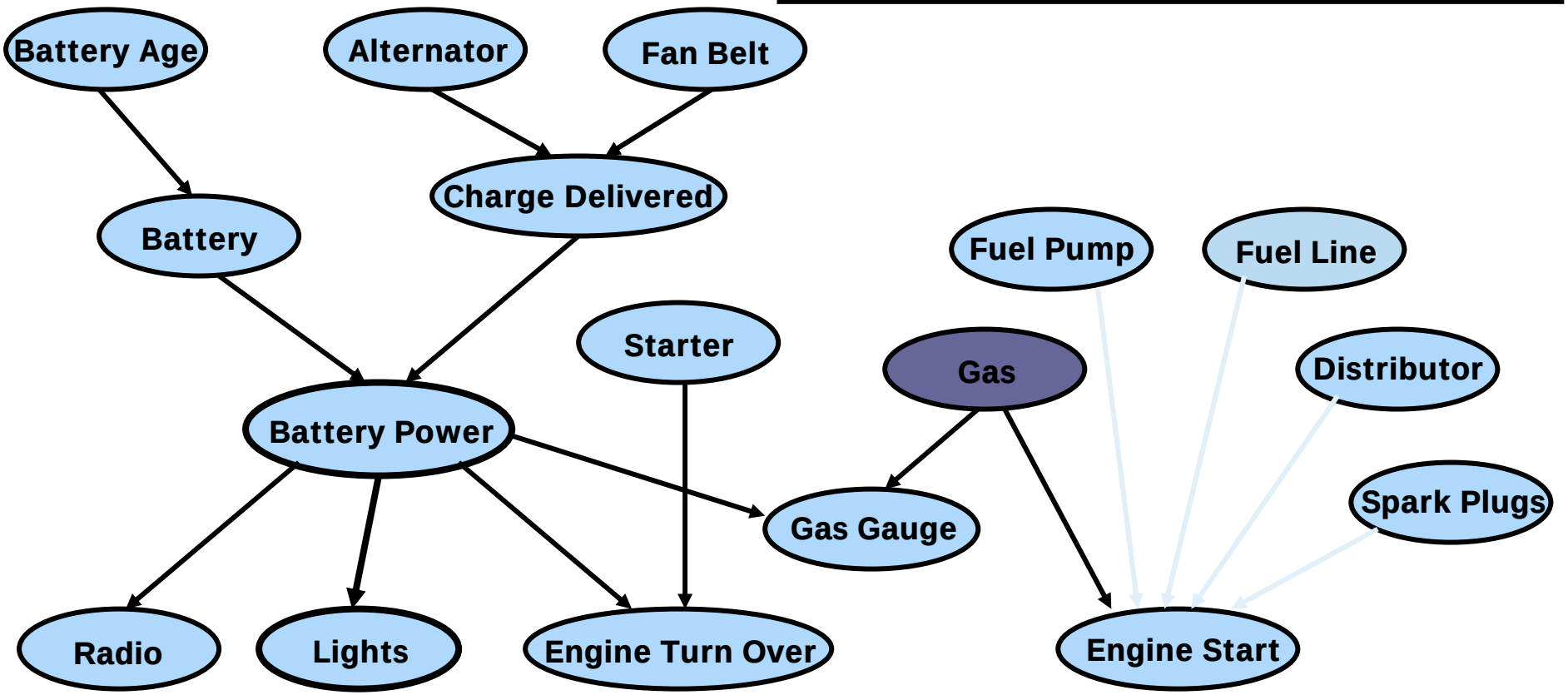


Local Structure 1: Context specific indepencence



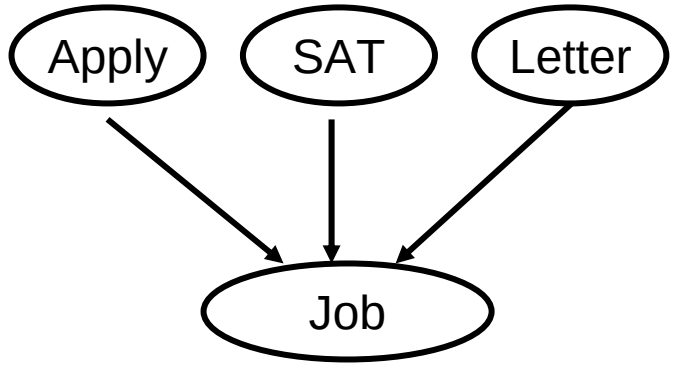
Local Structure 1: Context specific indepencence

Context Specific Independence (CSI)
After observing a variable, some vars become independent

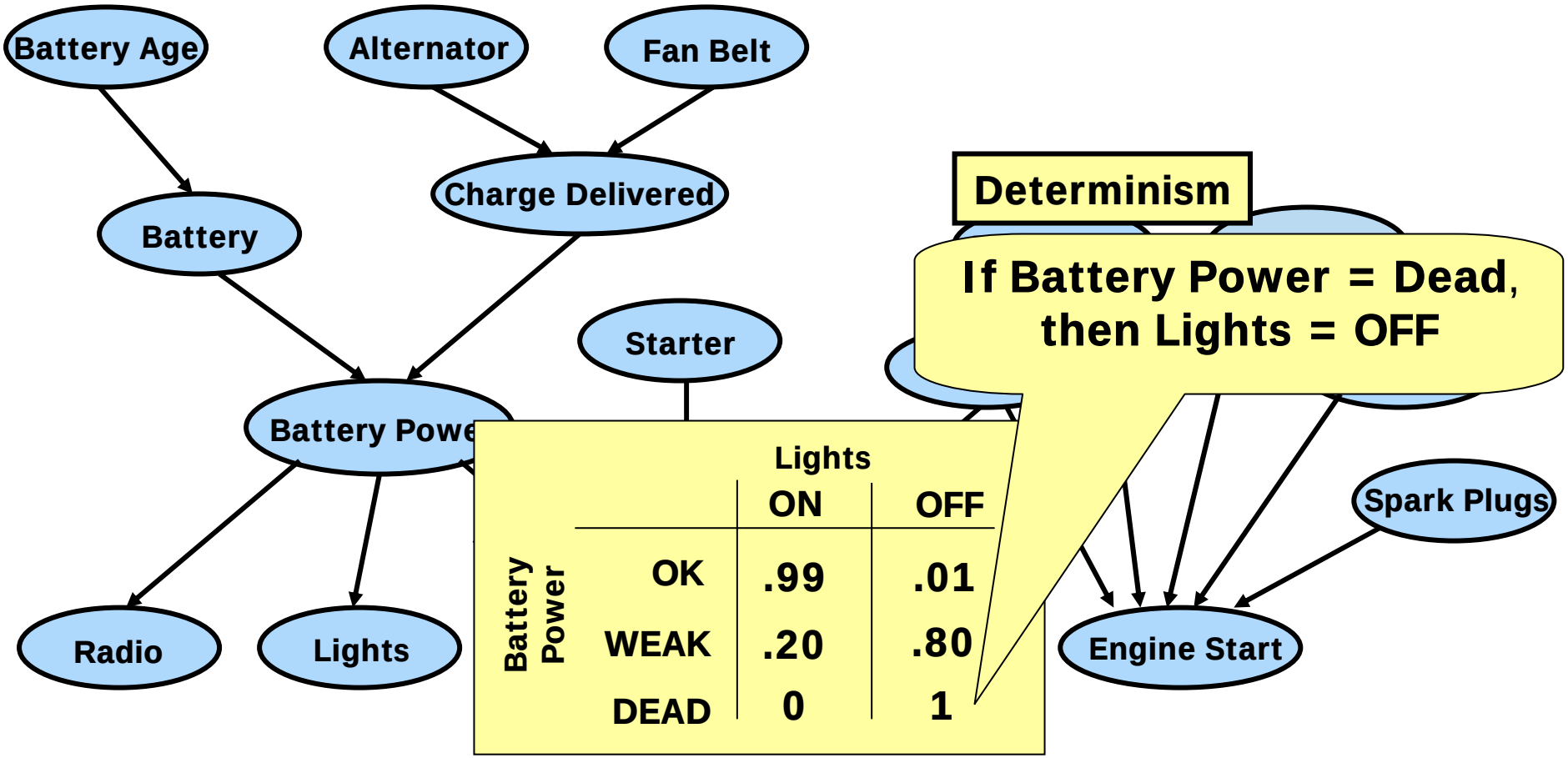


CSI example: Tree CPD

- Represent $P(X_i|\mathbf{Pa}_{Xi})$ using a decision tree
  - Path to leaf is an assignment to (a subset of) $\mathbf{Pa}_{Xi}$
  - Leaves are distributions over X_i given assignment of $\mathbf{Pa}_{Xi}$ on path to leaf
- **Interpretation of leaf**:
  - For specific assignment of $\mathbf{Pa}_{Xi}$ on path to this leaf – X_i is independent of other parents
- Representation can be exponentially smaller than equivalent table



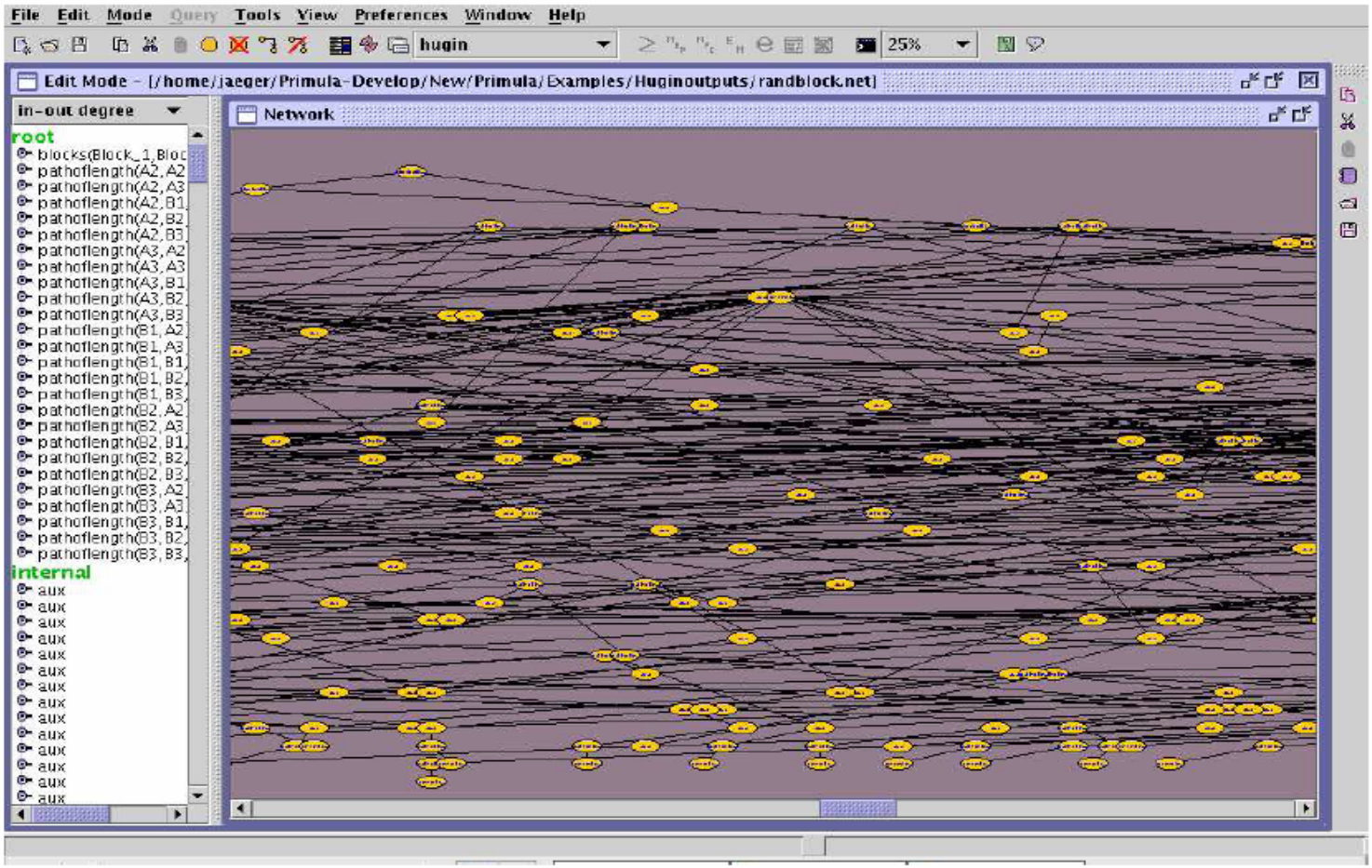
Local Structure 2: Determinism



Today's Models …

- **Often characterized by:**
  - Richness in local structure (determinism, CSI)
  - Massiveness in size (10,000's variables)
  - High connectivity (treewidth)
- **Enabled by:**
  - High level modeling tools: relational, first order
  - Advances in machine learning
  - New application areas (synthesis):
    - Bioinformatics (e.g. linkage analysis)
    - Sensor networks
- **Exploiting local structure a must!**

Exact inference in large models is possible…



- BN from a relational model