Reading:
Chapter 2 of Koller&Friedman

BN Semantics

Graphical Models – 10708
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September 14^{th}, 2005

Announcements

- Homework 1:
  - Out later today
  - Due October 3rd – **beginning of class!**
  - It's hard – start early, ask questions
- Collaboration policy
  - OK to discuss in groups
  - Tell us on your paper who you talked with
  - Each person must write their own unique paper
  - No searching the web, papers, etc. for answers, we trust you want to learn
- We are looking into room changes
- We cannot take official auditors for this class ☹
  - Too many people already

Basic concepts for random variables

- Atomic outcome: assignment $x_1, \ldots, x_n$ to $X_1, \ldots, X_n$
- Conditional probability: $P(X,Y)=P(X)P(Y|X)$
- Bayes rule: $P(X|Y)= \frac{P(Y|X)\,P(X)}{P(Y)}$
- Chain rule:
  - $P(X_1, \ldots, X_n) = P(X_1)P(X_2|X_1)\cdots P(X_k|X_1, \ldots, X_{k-1})$

Conditionally independent random variables

- **Sets** of variables **X**, **Y**, **Z**
- X is independent of Y given Z if
  - $P \models (\mathbf{X=x} \perp \mathbf{Y=y} \mid \mathbf{Z=z})$, $\forall$ $\mathbf{x} \in Val(\mathbf{X})$, $\mathbf{y} \in Val(\mathbf{Y})$, $\mathbf{z} \in Val(\mathbf{Z})$

- Shorthand:
  - **Conditional independence:** $P \models (\mathbf{X} \perp \mathbf{Y} \mid \mathbf{Z})$
  - For $P \models (\mathbf{X} \perp \mathbf{Y} \mid \emptyset)$, write $P \models (\mathbf{X} \perp \mathbf{Y})$
- **Notation:** $I(P)$ – independence properties entailed by P

- **Proposition:** P statisfies $(\mathbf{X} \perp \mathbf{Y} \mid \mathbf{Z})$ if and only if
  - $P(\mathbf{X},\mathbf{Y}|\mathbf{Z}) = P(\mathbf{X}|\mathbf{Z})\, P(\mathbf{Y}|\mathbf{Z})$

Properties of independence

- **Symmetry:**
  - $(\mathbf{X} \perp \mathbf{Y} \mid \mathbf{Z}) \Leftrightarrow (\mathbf{Y} \perp \mathbf{X} \mid \mathbf{Z})$
- **Decomposition:**
  - $(\mathbf{X} \perp \mathbf{Y},\mathbf{W} \mid \mathbf{Z}) \Rightarrow (\mathbf{X} \perp \mathbf{Y} \mid \mathbf{Z})$
- **Weak union:**
  - $(\mathbf{X} \perp \mathbf{Y},\mathbf{W} \mid \mathbf{Z}) \Rightarrow (\mathbf{X} \perp \mathbf{Y} \mid \mathbf{Z},\mathbf{W})$
- **Contraction:**
  - $(\mathbf{X} \perp \mathbf{W} \mid \mathbf{Y},\mathbf{Z}) \; \& \; (\mathbf{X} \perp \mathbf{Y} \mid \mathbf{Z}) \Rightarrow (\mathbf{X} \perp \mathbf{Y},\mathbf{W} \mid \mathbf{Z})$
- **Intersection:**
  - $(\mathbf{X} \perp \mathbf{Y} \mid \mathbf{W},\mathbf{Z}) \; \& \; (\mathbf{X} \perp \mathbf{W} \mid \mathbf{Y},\mathbf{Z}) \Rightarrow (\mathbf{X} \perp \mathbf{Y},\mathbf{W} \mid \mathbf{Z})$
  - Only for positive distributions! ($P(\alpha)>0$, $\forall\alpha$, $\alpha\neq\emptyset$)

Bayesian networks

- One of the most exciting advancements in statistical AI in the last 10-15 years
- Compact representation for exponentially-large probability distributions
- Fast marginalization too
- Exploit conditional independencies

Let's start on BNs…

- Consider $P(X_i)$
  - Assign probability to each $x_i \in Val(X_i)$
  - Independent parameters $|Val(X_i)|-1$

- Consider $P(X_1,\ldots,X_n)$
  - How many independent parameters if $|Val(X_i)|=k$?

k^n-1

What if variables are independent?

- What if variables are independent? $X_1, \ldots, X_n$
  - $(X_i \perp X_j), \forall\ i,j$
  - Not enough!!! (See homework 1 ☺)
  - Must assume that $(\mathbf{X} \perp \mathbf{Y}), \forall\ \mathbf{X},\mathbf{Y}$ subsets of $\{X_1,\ldots,X_n\}$

- Can write
  - $P(X_1,\ldots,X_n) = \prod_{i=1\ldots n} P(X_i)$

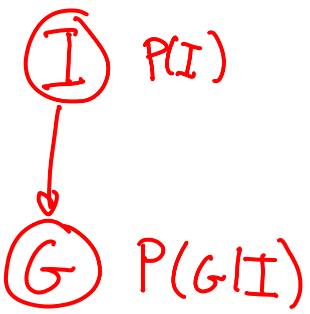
- How many independent parameters now?

$n \cdot (K-1)$

Conditional parameterization – two nodes

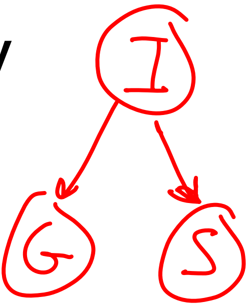
- Grade is determined by Intelligence

$$P(I, G) = P(I) \cdot P(G \mid I)$$



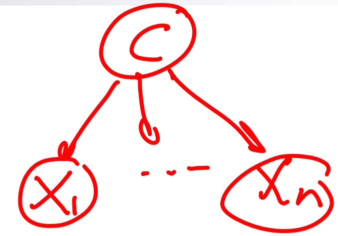
Conditional parameterization – three nodes

- Grade and SAT score are determined by Intelligence
- $(G \perp S \mid I)$



$$P(I,G,S) = P(I)\,P(G \mid I)\,\underbrace{P(S \mid I,G)}_{P(S \mid I)}$$

The naïve Bayes model – Your first real Bayes Net



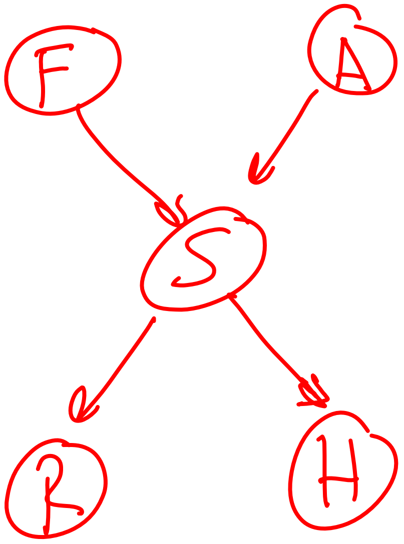
- Class variable: C
- Evidence variables: $X_1,\ldots,X_n$
- assume that $(\mathbf{X} \perp \mathbf{Y} \mid C)$, $\forall$ $\mathbf{X},\mathbf{Y}$ subsets of $\{X_1,\ldots,X_n\}$

$$P(C, X_1, \ldots, X_n) = P(C)\, P(X_1 \mid C) \cdots \underbrace{P(X_n \mid C, X_1, \ldots, X_{n-1})}_{P(X_n \mid C)}$$

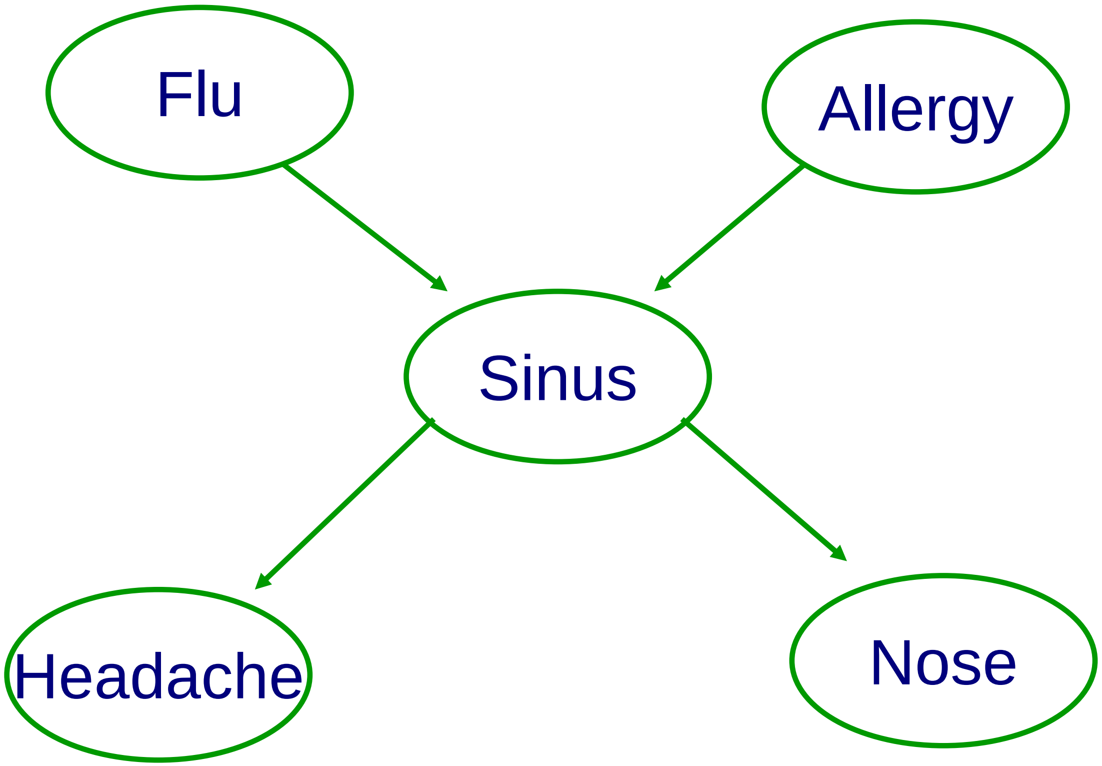
$$= P(C) \prod_i P(X_i \mid C)$$

Causal structure

- Suppose we know the following:
  - The flu causes sinus inflammation
  - Allergies cause sinus inflammation
  - Sinus inflammation causes a runny nose
  - Sinus inflammation causes headaches
- How are these connected?



Possible queries



- Inference

  $P(F \mid H=t, N=t)$

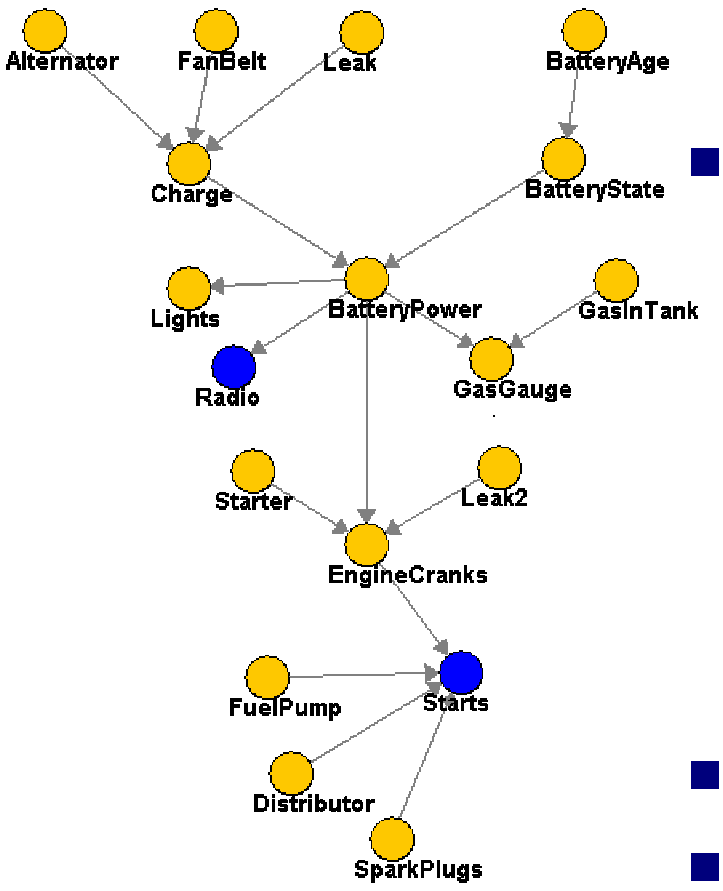
- Most probable explanation

  obs: $H=t$ & $N=t$

  $$\max_{f,a,s} P(F=f, A=a, S=s \mid H=t \;\&\; N=t)$$

- Active data collection

Car starts BN

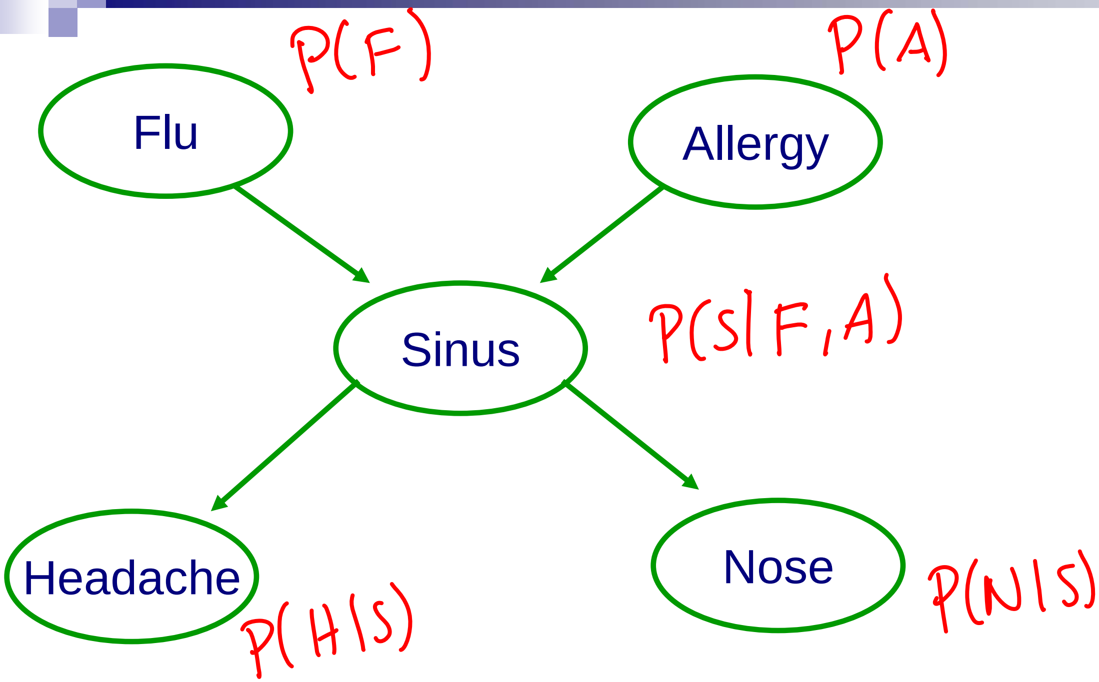


- 18 binary attributes
- Inference
  - P(BatteryAge|Starts=f)

$$P(BA \mid Starts=f) = \sum_{A,F,\ldots,S} P(A,F,S\ldots,BA \mid S=f)$$

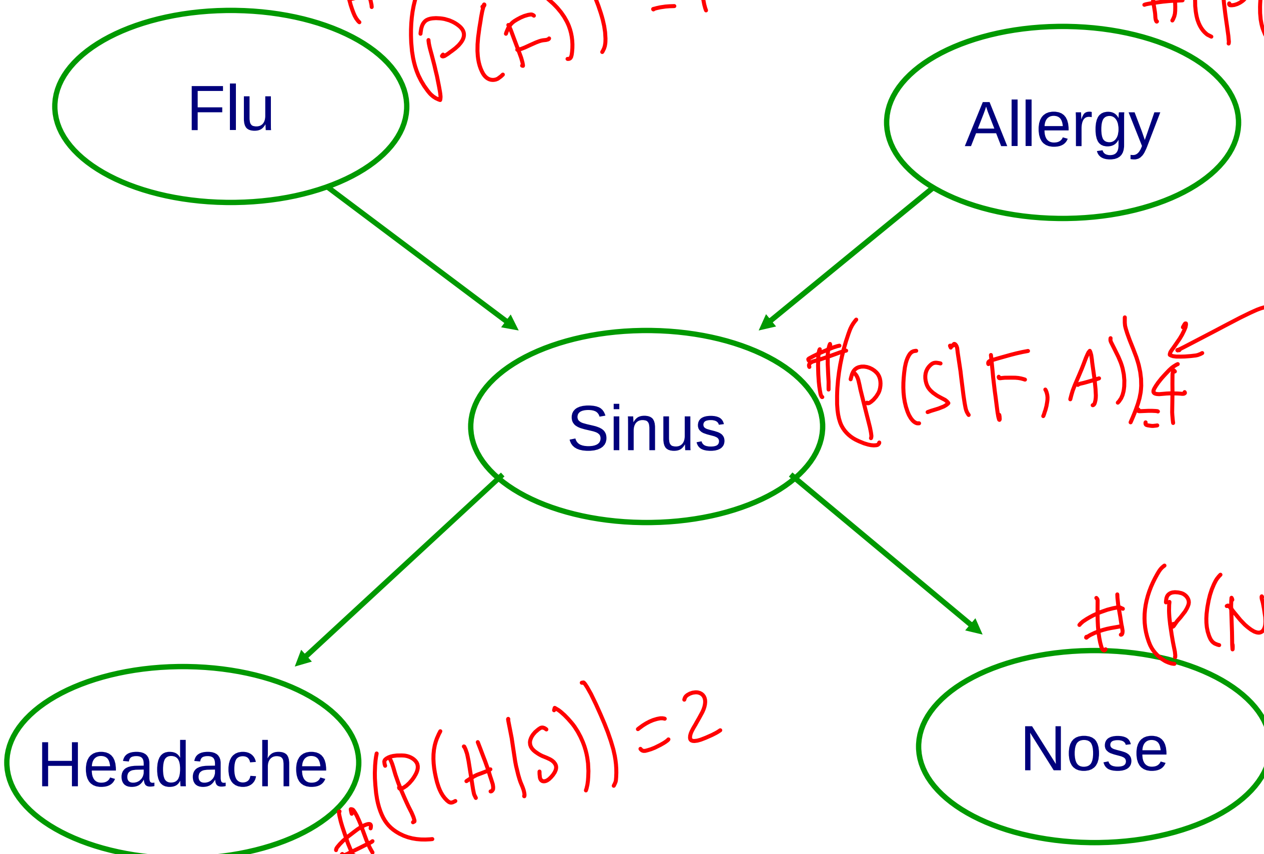
- 2^{18} terms, why so fast?
- Not impressed?
  - HailFinder BN – more than 3^{54} = 58149737003040059690390169 terms

Factored joint distribution - Preview



$P(F,A,S,H,N) =$ prod. factors

Number of parameters



par $(P(F)) = 1$

$\#(P(A)) = 1$

$\#(P(S|F,A)) = 4$

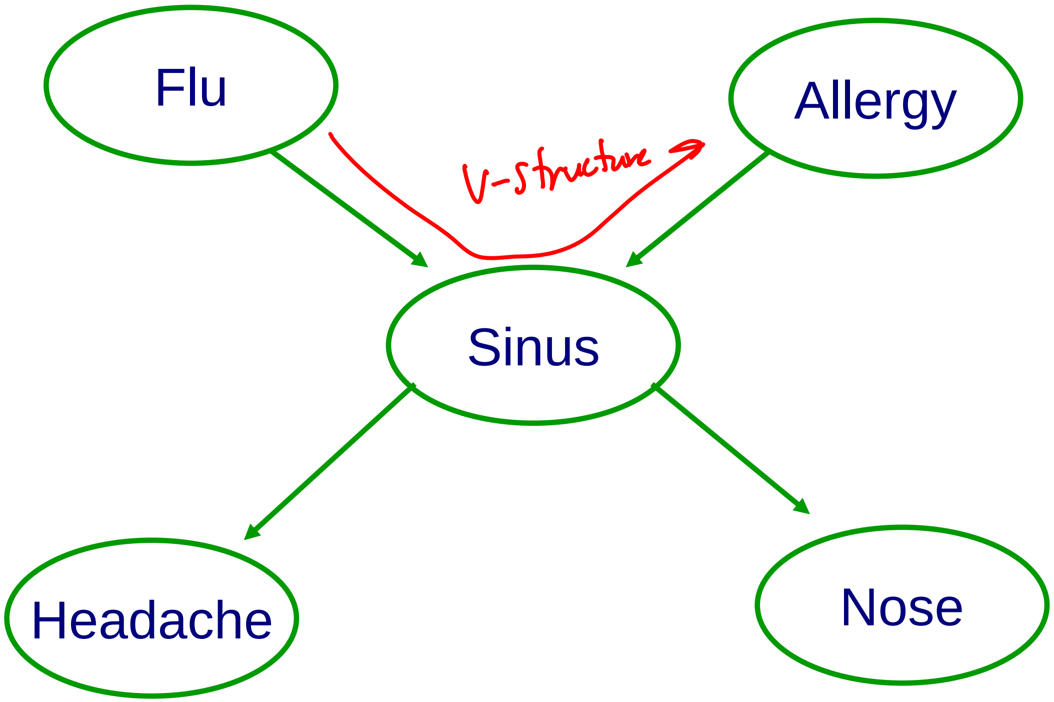
	$F \wedge A$	$F \wedge \neg A$	$\neg F \wedge A$	$\neg F \wedge \neg A$
S	0.8	0.5	0.6	0.1
$\neg S$	0.2	0.5	0.4	0.9

$\#(P(N|S)) = 2$

$\#(P(H|S)) = 2$

total = 10 , full join # pars $= 2^5 - 1$
$= 31$

Key: Independence assumptions



$(F \perp A)$

not $(F \perp N)$

$(F \perp N \mid S)$

not $(H \perp N)$

$(H \perp N \mid S)$

not $(F \perp A \mid H)$

Knowing sinus separates the variables from each other

(Marginal) Independence

- Flu and Allergy are (marginally) independent

Flu = t	
Flu = f	

- More Generally:

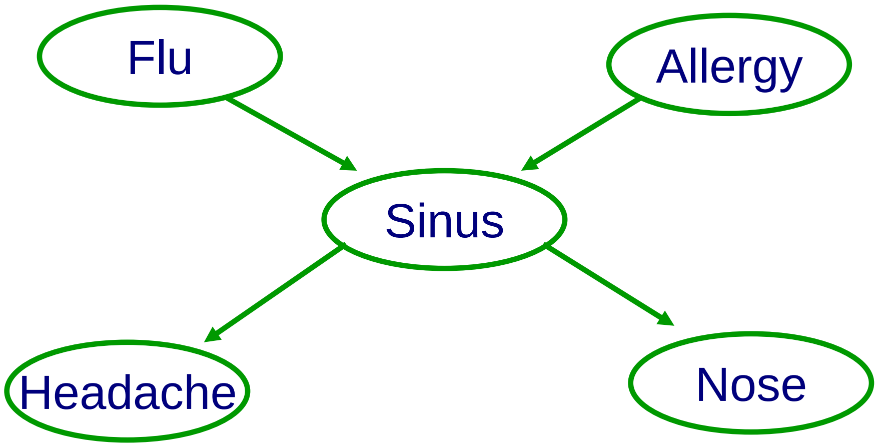
Allergy = t	
Allergy = f	

	Flu = t	Flu = f
Allergy = t		
Allergy = f		

Conditional independence

- Flu and Headache are not (marginally) independent
- Flu and Headache are independent given Sinus infection
- More Generally:

The independence assumption



Local Markov Assumption:
A variable X is independent of its non-descendants given its parents

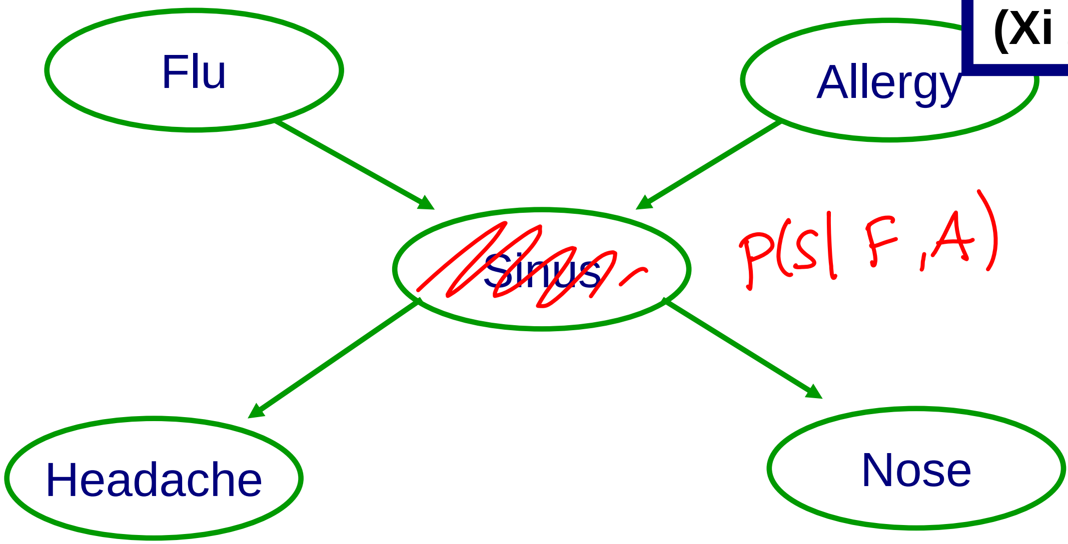
($X_i \perp \text{NonDescendants}_{X_i} \mid \text{Pa}_{X_i}$)

$(F \perp A)$

$(H \perp \{F, A, N\} \mid S)$

Explaining away

Local Markov Assumption:
A variable X is independent of its non-descendants given its parents
($X_i \perp NonDescendants_{X_i} \mid Pa_{X_i}$)



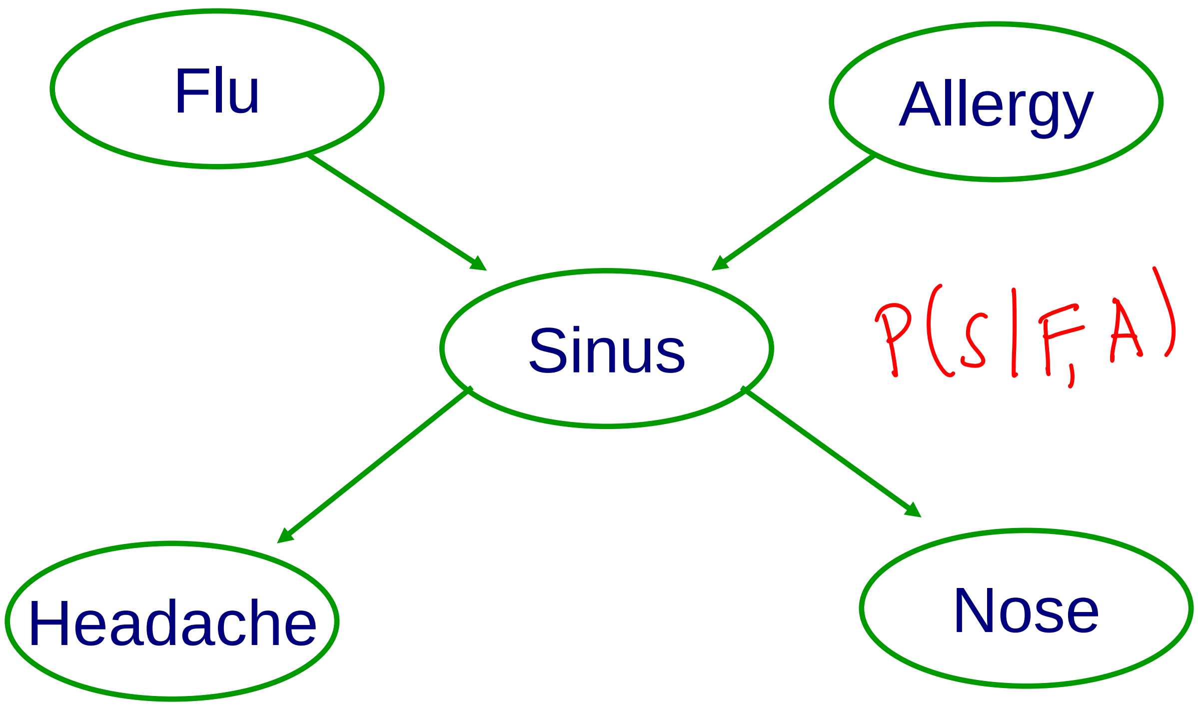
$P(S \mid F, A)$

$P(A \mid S=t) \geq P(A)$

$P(A=t \mid S=t, F=t)$

$\leq P(A \mid S=t)$

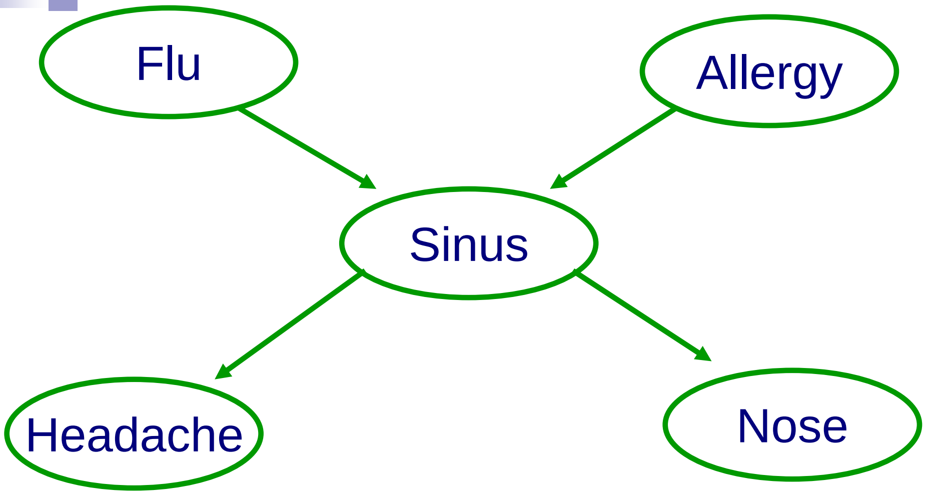
What about probabilities?
Conditional probability tables (CPTs)



$P(S \mid F, A)$

	$F \wedge A$	$\neg F \wedge A$		
S				
$\neg S$				

Joint distribution

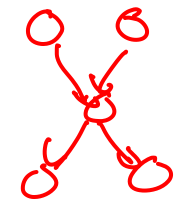


$$P(A,F,S,H,N) = P(F)\,P(A)\,P(S|F,A)\,P(H|S)\,P(N|S)$$

Why can we decompose? Markov Assumption!

A general Bayes net

- Set of random variables $X_1, \ldots, X_n$
- Directed acyclic graph DAG



- CPTs $P(X_i \mid Pa_{X_i})$
- Joint distribution:

$$P(X_1, \ldots, X_n) = \prod_{i=1}^{n} P\left(X_i \mid \mathbf{Pa}_{X_i}\right)$$

- **Local Markov Assumption:**
  - A variable X is independent of its non-descendants given its parents – **(Xi ⊥ NonDescendantsXi | PaXi)**

Questions????

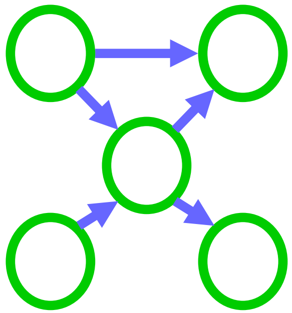
- What distributions can be represented by a BN?

- What BNs can represent a distribution?

- What are the independence assumptions encoded in a BN?
  - in addition to the local Markov assumption

Today: The Representation Theorem – Joint Distribution to BN

BN:



Encodes independence assumptions

If conditional independencies in BN are subset of conditional independencies in *P*

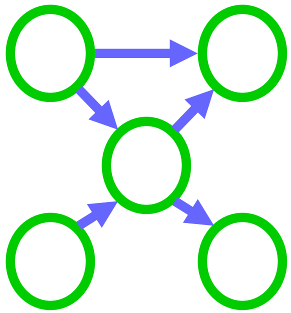
Obtain

Joint probability distribution:

$$P(X_1, \ldots, X_n) = \prod_{i=1}^{n} P\left(X_i \mid \mathbf{Pa}_{X_i}\right)$$

Today: The Representation Theorem – BN to Joint Distribution

BN:



Encodes independence assumptions

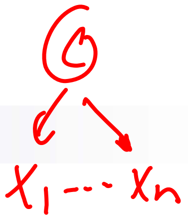
If joint probability distribution:

$$P(X_1, \ldots, X_n) = \prod_{i=1}^{n} P\left(X_i \mid \mathbf{Pa}_{X_i}\right)$$

Obtain

Then conditional independencies in BN are subset of conditional independencies in *P*

Let's start proving it for naïve Bayes – From joint distribution to BN



- Independence assumptions:
  - X_i independent given C

start here

- Let's assume that *P* satisfies independencies must prove that *P* factorizes according to BN:
  - $P(C,X_1,\ldots,X_n) = P(C) \prod_i P(X_i|C)$

prove

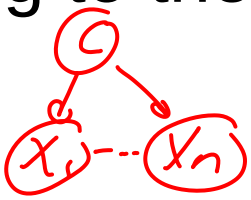
- Use chain rule!

$$P(C, X_1, \ldots, X_n) = P(C)\, P(X_1|C) \cdots \underbrace{P(X_n|C, X_1, \ldots, X_{n-1})}_{P(X_n|C)}$$

$$= P(C) \prod_i P(X_i|C)$$

Let's start proving it for naïve Bayes – From BN to joint distribution 1

- Let's assume that *P* factorizes according to the BN:
  - $P(C,X_1,\ldots,X_n) = P(C) \prod_i P(X_i|C)$

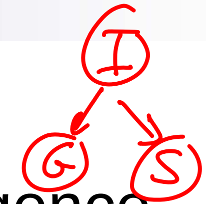


- Prove the independence assumptions:
  - X_i independent given C
  - Actually, $(\mathbf{X} \perp \mathbf{Y} \mid C)$, $\forall$ **X**,**Y** subsets of $\{X_1,\ldots,X_n\}$

$$P(X_1 | C, X_2) = P(X_1 | C)$$

$$P(X_1, X_2 | C, X_3, X_4, X_5) = P(X_1, X_2 | C)$$

Let's start proving it for naïve Bayes – From BN to joint distribution 2



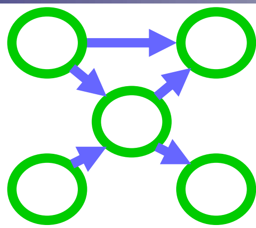
- Let's consider a simpler case
  - Grade and SAT score are determined by Intelligence
  - P(I,G,S) = P(I)P(G|I)P(S|I)
  - Prove that P(G,S|I) = P(G|I) P(S|I)

$I(G,S \mid I)$

$$P(G,S \mid I) = \frac{P(G,S,I)}{P(I)} = \frac{\cancel{P(I)} \cdot P(G \mid I) \cdot P(S \mid I)}{\cancel{P(I)}}$$

q.e.d.

Today: The Representation Theorem

BN:  **Encodes independence assumptions**

If conditional independencies in BN are subset of conditional independencies in *P*

Obtain

Joint probability distribution:

$$P(X_1, \ldots, X_n) = \prod_{i=1}^{n} P\left(X_i \mid \mathbf{Pa}_{X_i}\right)$$

If joint probability distribution:

$$P(X_1, \ldots, X_n) = \prod_{i=1}^{n} P\left(X_i \mid \mathbf{Pa}_{X_i}\right)$$

Obtain

Then conditional independencies in BN are subset of conditional independencies in *P*

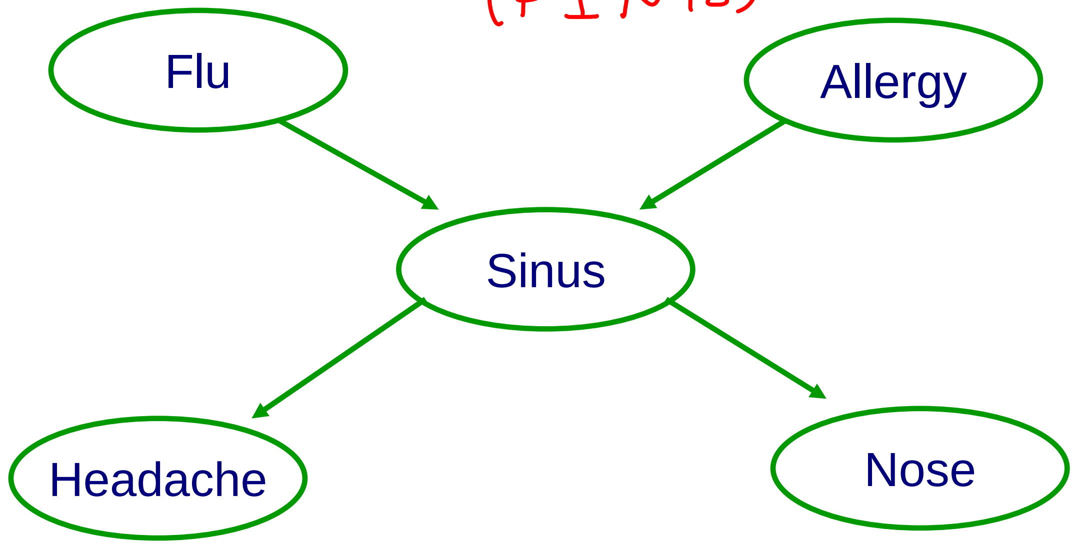
Local Markov assumption & I-maps

- Local independence assumptions in BN structure G: $I_\ell(G)$

- Independence assertions of *P*: $I(P)$

- BN structure G is an ***I-map*** (independence map) if: $I_\ell(G) \subseteq I(P)$

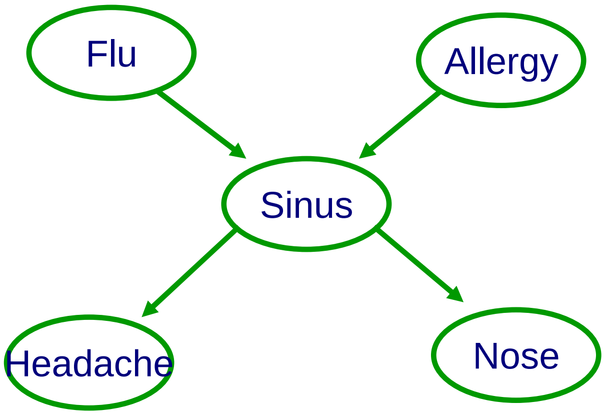
$(F \perp N \mid S)$



Local Markov Assumption:
A variable X is independent of its non-descendants given its parents
($X_i \perp \mathrm{NonDescendants}_{X_i} \mid \mathrm{Pa}_{X_i}$)

Factorized distributions

- Given
  - Random vars $X_1,\ldots,X_n$
  - *P* distribution over vars
  - BN structure *G* over same vars
- *P* factorizes according to *G* if



$$P(X_1,\ldots,X_n)=\prod_{i=1}^{n} P\left(X_i \mid \mathbf{Pa}_{X_i}\right)$$

BN Representation Theorem – I-map to factorization

If conditional independencies in BN are subset of conditional independencies in *P*

Obtain

Joint probability distribution:

$$P(X_1, \ldots, X_n) = \prod_{i=1}^{n} P\left(X_i \mid \mathbf{Pa}_{X_i}\right)$$

$\boxed{\textbf{\textit{G} is an I-map of \textit{P}}}$

$I_\ell(G) \subseteq I(P)$

$\boxed{\textbf{\textit{P} factorizes according to \textit{G}}}$

BN Representation Theorem – I-map to factorization: **Proof**

G* is an I-map of *P

Obtain

P* factorizes according to *G

$$P(X_1,\ldots,X_n) = \prod_{i=1}^{n} P\left(X_i \mid \mathbf{Pa}_{X_i}\right)$$

$$P(F,A,S,H,N) \overset{\text{topologically aligned chain rule}}{=}$$

$$P(F)\,P(A|F)\,P(S|FA)\,P(H|FAS)\,P(N|FASH)$$

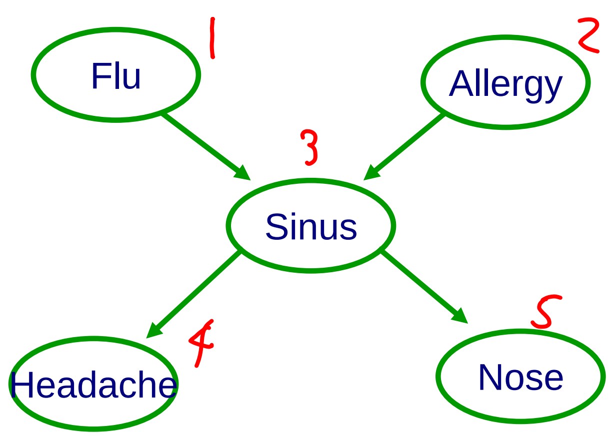
I-map I-map

$$P(F)\;P(A)\;P(S|FA)\;P(H|S)\;P(N|S)$$

q.e.d.

ALL YOU NEED:

Local Markov Assumption:
A variable X is independent of its non-descendants given its parents
(Xi $\perp$ NonDescendants$_{Xi}$ | Pa$_{Xi}$)



Defining a BN

- Given a set of variables and conditional independence ~~assumptions~~ assertions of P
- Choose an ordering on variables, e.g., $X_1, \ldots, X_n$
- For i = 1 to n
  - Add X_i to the network
  - Define parents of X_i, $\mathbf{Pa}_{X_i}$, in graph as the minimal subset of $\{X_1,\ldots,X_{i-1}\}$ such that local Markov assumption holds – X_i independent of rest of $\{X_1,\ldots,X_{i-1}\}$, given parents $\mathbf{Pa}_{X_i}$
  - Define/learn CPT – $P(X_i \mid \mathbf{Pa}_{X_i})$

BN Representation Theorem – Factorization to I-map

If joint probability distribution:

$$P(X_1, \ldots, X_n) = \prod_{i=1}^{n} P\left(X_i \mid \mathbf{Pa}_{X_i}\right)$$

Obtain

Then conditional independencies in BN are subset of conditional independencies in *P*

P* factorizes according to *G

G* is an I-map of *P

$I_\ell(G) \subseteq I(P)$

BN Representation Theorem – Factorization to I-map: **Proof**

If joint probability distribution:

$$P(X_1, \ldots, X_n) = \prod_{i=1}^{n} P\left(X_i \mid \mathbf{Pa}_{X_i}\right)$$

Obtain

Then conditional independencies in BN are subset of conditional independencies in *P*

P* factorizes according to *G

G* is an I-map of *P

Homework 1!!!! ☺

The BN Representation Theorem

G is an I map of P

If conditional independencies in BN are subset of conditional independencies in *P*

Obtain

P factorizes accord. to G

Joint probability distribution:

$$P(X_1, \ldots, X_n) = \prod_{i=1}^{n} P\left(X_i \mid \mathbf{Pa}_{X_i}\right)$$

Important because:
Every *P* has at least one BN structure *G*

G is I-map P

P fact. ac. G

If joint probability distribution:

$$P(X_1, \ldots, X_n) = \prod_{i=1}^{n} P\left(X_i \mid \mathbf{Pa}_{X_i}\right)$$

Obtain

Then conditional independencies in BN are subset of conditional independencies in *P*

Important because:
Read independencies of *P* from BN structure *G*

Independencies encoded in BN

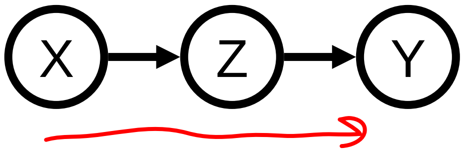
- We said: All you need is the local Markov assumption
  - $(X_i \perp NonDescendants_{Xi} \mid \mathbf{Pa}_{Xi})$
- But then we talked about other (in)dependencies
  - e.g., explaining away
- What are the independencies encoded by a BN?
  - Only assumption is local Markov
  - But many others can be derived using the algebra of conditional independencies!!!

Understanding independencies in BNs – BNs with 3 nodes

Local Markov Assumption:
A variable X is independent of its non-descendants given its parents

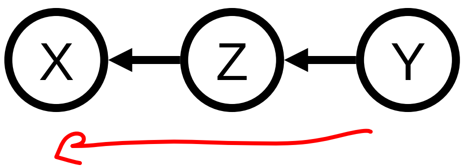
obs X

Indirect causal effect:



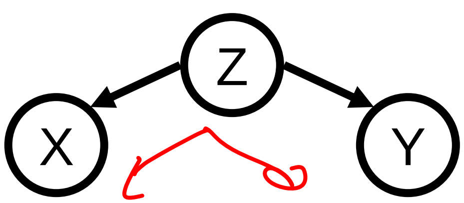
$(X \perp Y \mid Z)$
not $(X \perp Y)$

Indirect evidential effect:



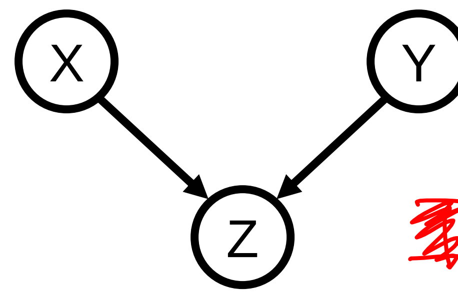
$(X \perp Y \mid Z)$
not $(X \perp Y)$

Common cause:



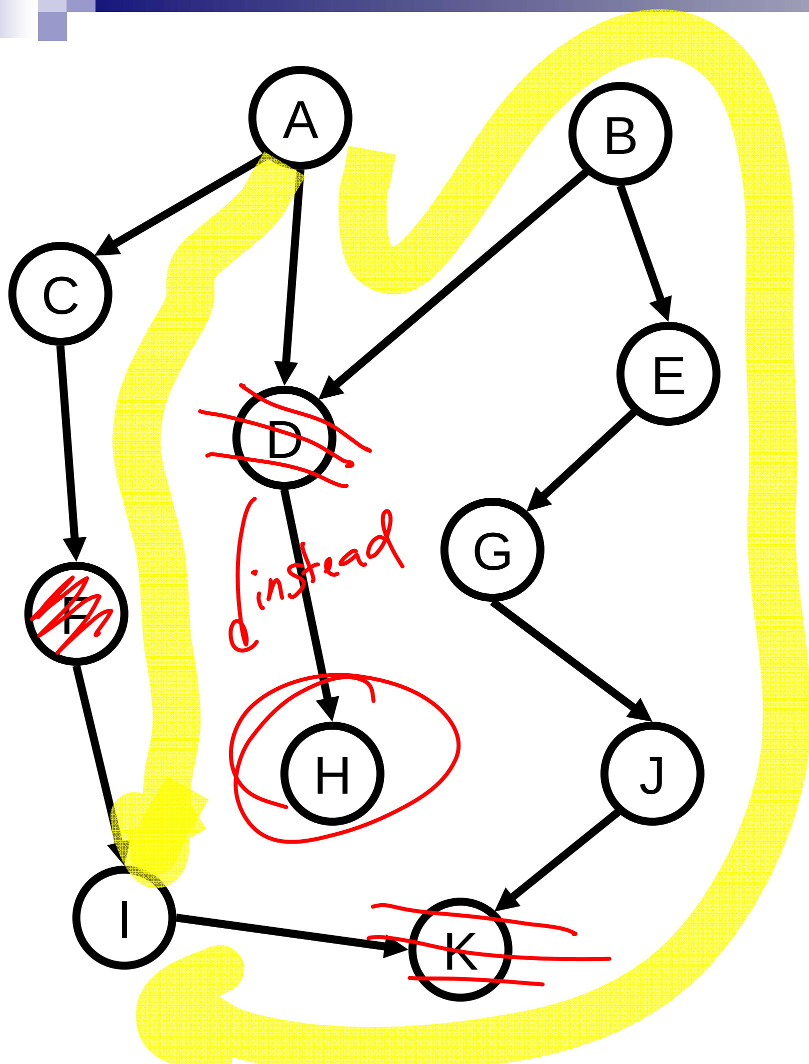
$(X \perp Y \mid Z)$
not $(X \perp Y)$

Common effect:



$(X \perp Y)$
not $(X \perp Y \mid Z)$

Understanding independencies in BNs – Some examples



$(H \perp A \mid D)$

$(A \perp B)$

not $(A \perp B \mid D)$

not $(A \perp B \mid H)$

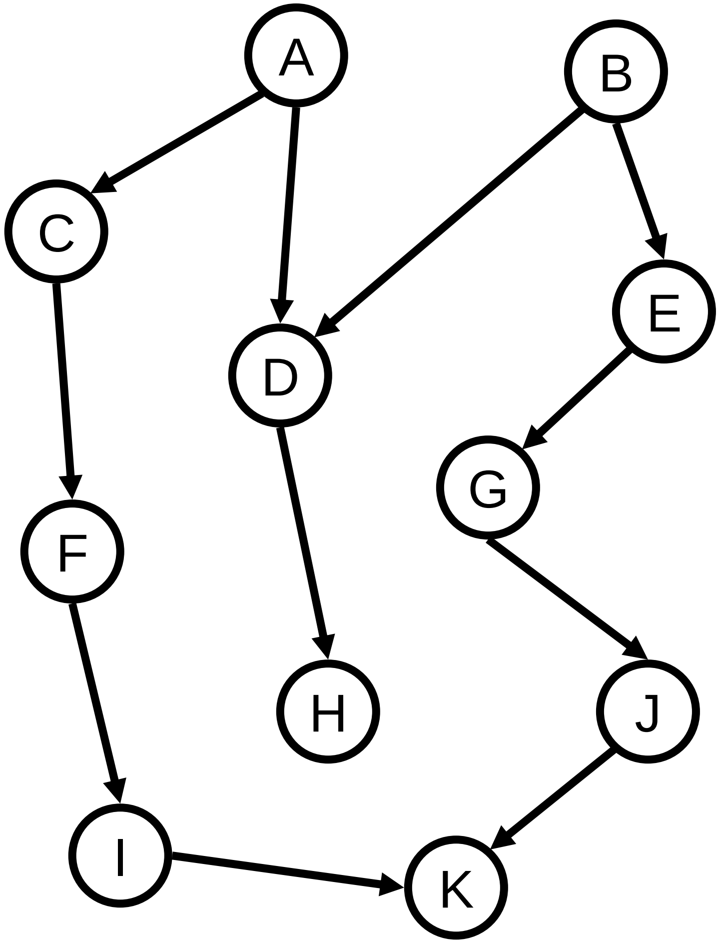
not $(A \perp B \mid K)$

①: $(A \perp I \mid F)$

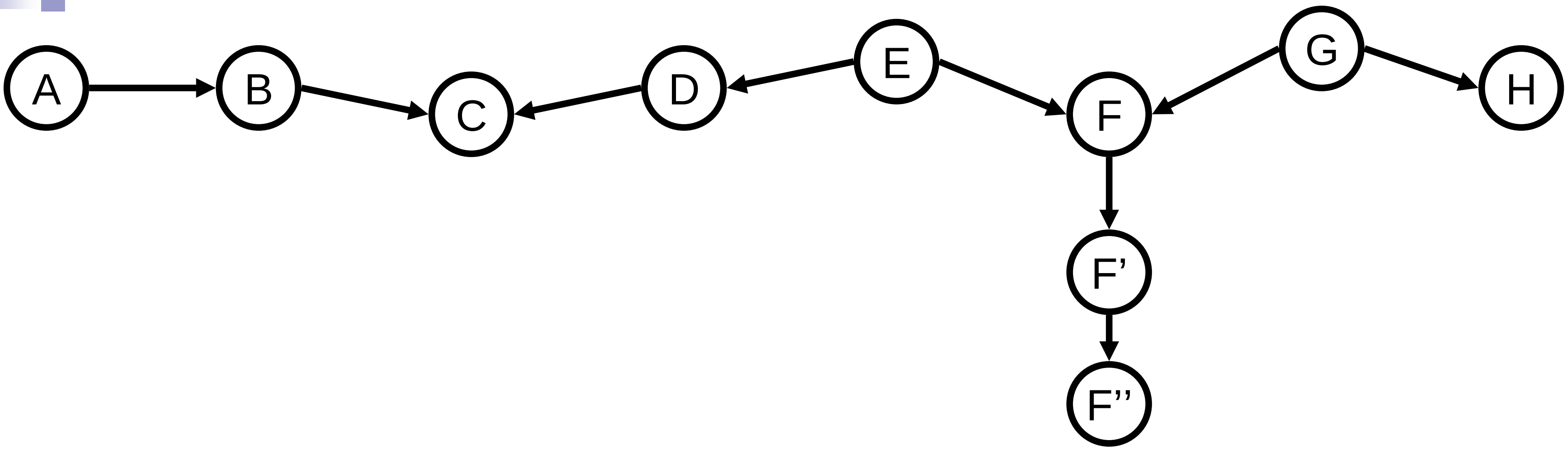
② not $(A \perp I \mid F, D, K)$

③ not $(A \perp I \mid F, H, K)$

Understanding independencies in BNs – Some more examples



An active trail – Example



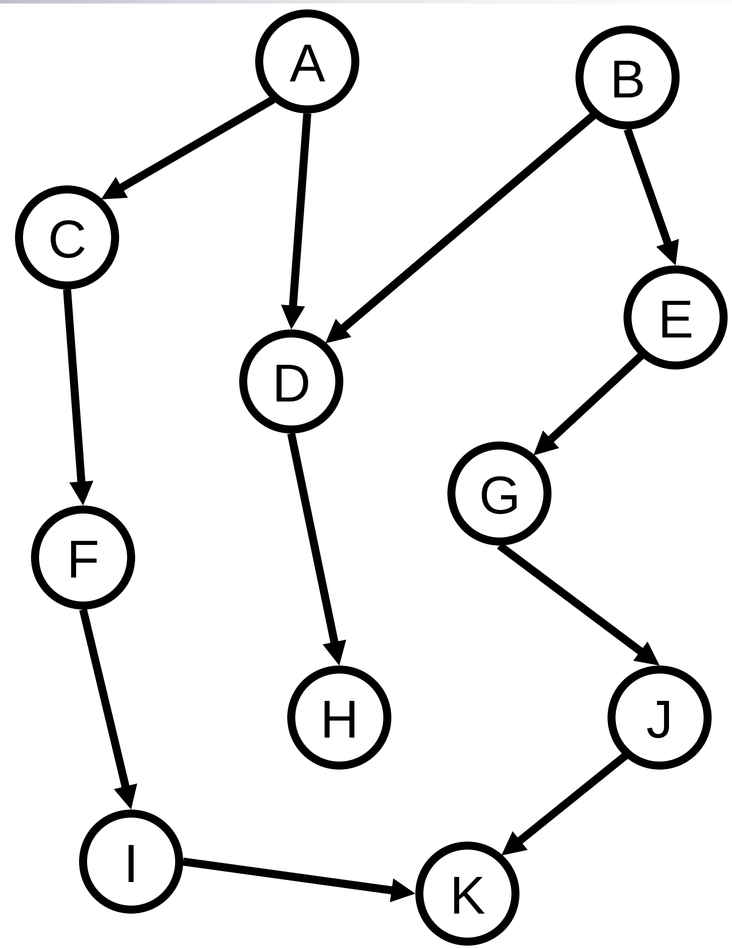
When are A and H independent?

Active trails formalized

- A path $X_1 - X_2 - \cdots - X_k$ is an **active trail** when variables $\boldsymbol{O} \subseteq \{X_1,\ldots,X_n\}$ are observed if for each consecutive triplet in the trail:
  - $X_{i-1} \rightarrow X_i \rightarrow X_{i+1}$, and X_i is **not observed** ($X_i \notin \boldsymbol{O}$)
  - $X_{i-1} \leftarrow X_i \leftarrow X_{i+1}$, and X_i is **not observed** ($X_i \notin \boldsymbol{O}$)
  - $X_{i-1} \leftarrow X_i \rightarrow X_{i+1}$, and X_i is **not observed** ($X_i \notin \boldsymbol{O}$)
  - $X_{i-1} \rightarrow X_i \leftarrow X_{i+1}$, and X_i **is observed** ($X_i \in \boldsymbol{O}$), or **one of its descendents**

Active trails and independence?

- **Theorem**: Variables $\mathbf{X_i}$ **and** $\mathbf{X_j}$ **are independent given** $\boldsymbol{Z}\subseteq\{X_1,\ldots,X_n\}$ if the is **no active trail** between X_i and X_j when variables $\boldsymbol{Z}\subseteq\{X_1,\ldots,X_n\}$ are observed

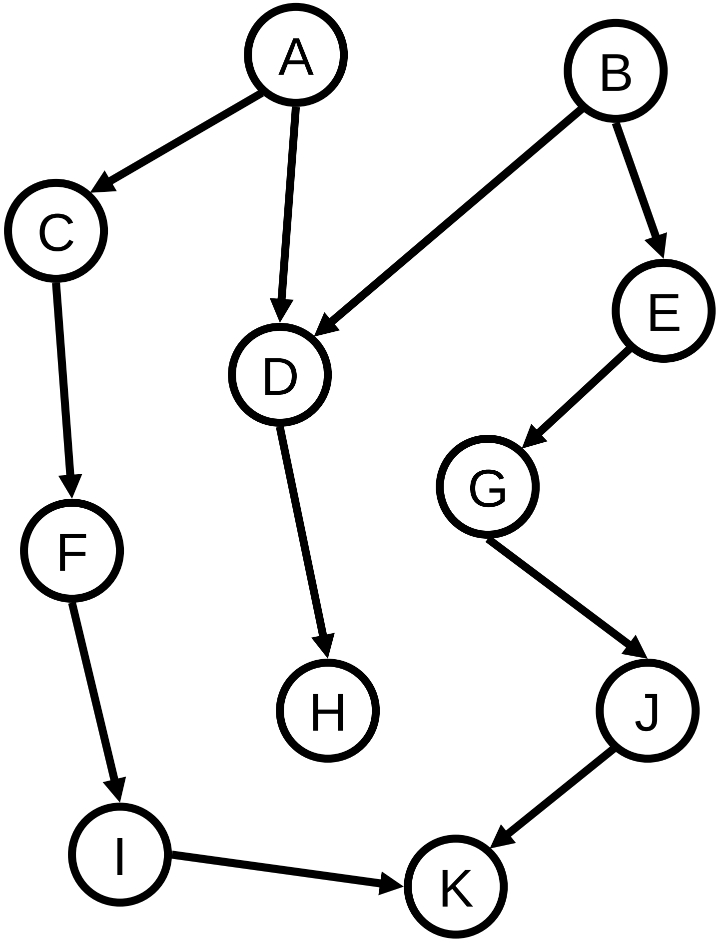


More generally: Soundness of d-separation

- Given BN structure G
- Set of independence assertions obtained by d-separation:
  - $I(G) = \{(\mathbf{X} \perp \mathbf{Y} | \mathbf{Z}) : \text{d-sep}_G(\mathbf{X};\mathbf{Y}|\mathbf{Z})\}$
- **Theorem: Soundness of d-separation**
  - If P factorizes over G then $I(G) \subseteq I(P)$
- **Interpretation:** d-separation only captures true independencies
- Proof discussed when we talk about undirected models

Existence of dependency when not d-separated

- **Theorem:** If X and Y are not d-separated given **Z**, then X and Y are dependent given **Z** under some *P* that factorizes over *G*
- **Proof sketch**:
  - Choose an active trail between X and Y given **Z**
  - Make this trail dependent
  - Make all else uniform (independent) to avoid “canceling” out influence



More generally: Completeness of d-separation

- **Theorem: Completeness of d-separation**
  - For "almost all" distributions that P factorize over to G, we have that $I(G) = I(P)$
  - *"almost all" distributions*: except for a set of measure zero of parameterizations of the CPTs (assuming no finite set of parameterizations has positive measure)
- **Proof sketch:**

Interpretation of completeness

- **Theorem: Completeness of d-separation**
  - For "almost all" distributions that *P* factorize over to *G*, we have that I(*G*) = I(*P*)
- BN graph is usually sufficient to capture all independence properties of the distribution!!!!
- But only for complete independence:
  - $P \models (\mathbf{X}=\mathbf{x} \perp \mathbf{Y}=\mathbf{y} \mid \mathbf{Z}=\mathbf{z}), \forall\ \mathbf{x} \in Val(\mathbf{X}), \mathbf{y} \in Val(\mathbf{Y}), \mathbf{z} \in Val(\mathbf{Z})$
- Often we have context-specific independence (CSI)
  - $\exists\ \mathbf{x} \in Val(\mathbf{X}), \mathbf{y} \in Val(\mathbf{Y}), \mathbf{z} \in Val(\mathbf{Z})$: $P \models (\mathbf{X}=\mathbf{x} \perp \mathbf{Y}=\mathbf{y} \mid \mathbf{Z}=\mathbf{z})$
  - Many factors may affect your grade
  - But if you are a frequentist, all other factors are irrelevant ☺

What you need to know

- Independence & conditional independence
- Definition of a BN
- The representation theorems
  - Statement
  - Interpretation
- d-separation and independence
  - soundness
  - existence
  - completeness

Acknowledgements

- JavaBayes applet
  - http://www.pmr.poli.usp.br/ltd/Software/javabayes/Home/index.html