

Logistic Regression, cont.

Decision Trees

Machine Learning – 10701/15781

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Logistic Regression

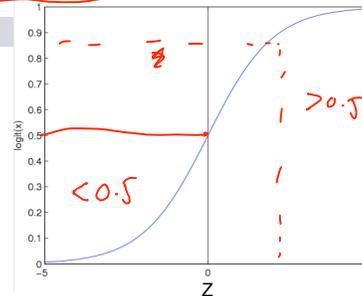
Logistic function (or Sigmoid):

$$\frac{1}{1 + \exp(-z)}$$

Learn $P(Y|\mathbf{X})$ directly!

- Assume a particular functional form
- Sigmoid applied to a linear function of the data:

$$P(Y = 1|X_i) = \frac{1}{1 + \exp(w_0 + \sum_{i=1}^n w_i X_i)}$$



Features can be discrete or continuous!

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Loss functions: Likelihood v. Conditional Likelihood

- Generative (Naïve Bayes) Loss function:

$$D = \langle x^j, y^j \rangle_{j=1 \dots N}$$

Data likelihood

$$\begin{aligned} \ln P(D | \mathbf{w}) &= \sum_{j=1}^N \ln P(x^j, y^j | \mathbf{w}) \\ &= \sum_{j=1}^N \ln P(y^j | x^j, \mathbf{w}) + \sum_{j=1}^N \ln P(x^j | \mathbf{w}) \end{aligned}$$

- Discriminative models cannot compute $P(\mathbf{x} | \mathbf{w})!$
- But, discriminative (logistic regression) loss function:

Conditional Data Likelihood

$$\ln P(D_Y | D_X, \mathbf{w}) = \sum_{j=1}^N \ln P(y^j | x^j, \mathbf{w})$$

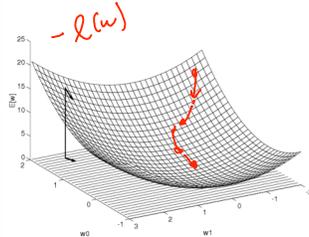
- Doesn't waste effort learning $P(X)$ – focuses on $P(Y|X)$ all that matters for classification

Handwritten notes:
 for generating data not important for classification
 discriminative likelihood
 i = training example
 $y^j = 1$ if spam
 $= 0$ if not spam
 x^j : list of words
 3 or 4-11

Optimizing concave function – Gradient ascent

(Conjugate G.D. better.)

- Conditional likelihood for Logistic Regression is concave → Find optimum with gradient ascent



Gradient: $\nabla_{\mathbf{w}} l(\mathbf{w}) = \left[\frac{\partial l(\mathbf{w})}{\partial w_0}, \dots, \frac{\partial l(\mathbf{w})}{\partial w_n} \right]^T$

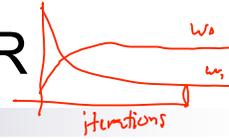
Update rule: $\Delta \mathbf{w} = \eta \nabla_{\mathbf{w}} l(\mathbf{w})$

Handwritten notes:
 step size
 Learning rate, $\eta > 0$

$$w_i^{(t+1)} \leftarrow w_i^{(t)} + \eta \frac{\partial l(\mathbf{w})}{\partial w_i}$$

- Gradient ascent is simplest of optimization approaches
 - e.g., Conjugate gradient ascent much better (see reading)

Gradient Descent for LR



Gradient ascent algorithm: iterate until change $< \epsilon$

$$w_0^{(t+1)} \leftarrow w_0^{(t)} + \eta \sum_j [y^j - \tilde{P}(Y^j = 1 | \mathbf{x}^j, \mathbf{w}^{(t)})]$$

Handwritten notes: $w^{(t)}$ is w at t^{th} iteration. A bracket under the sum is labeled "no x_0^j , $x_0^j = 1$ ".

For $i = 1 \dots n$,

$$w_i^{(t+1)} \leftarrow w_i^{(t)} + \eta \sum_j x_i^j [y^j - \tilde{P}(Y^j = 1 | \mathbf{x}^j, \mathbf{w}^{(t)})]$$

repeat

$$\left. \begin{aligned} &e^{w_0 + \sum_i w_i x_i^j} \\ \times &e^{-\sum_i w_i x_i^j} \end{aligned} \right\}$$

That's all M(C)LE. How about MAP?

$$p(\mathbf{w} | Y, \mathbf{X}) \propto P(Y | \mathbf{X}, \mathbf{w})p(\mathbf{w})$$

- One common approach is to define priors on \mathbf{w}
 - Normal distribution, zero mean, identity covariance
 - "Pushes" parameters towards zero
- Corresponds to **Regularization**
 - Helps avoid very large weights and overfitting
 - More on this later in the semester
- MAP estimate

$$\mathbf{w}^* = \arg \max_{\mathbf{w}} \ln \left[p(\mathbf{w}) \prod_{j=1}^N P(y^j | \mathbf{x}^j, \mathbf{w}) \right]$$

M(C)AP as Regularization

$$\ln \left[p(\mathbf{w}) \prod_{j=1}^N P(y^j | \mathbf{x}^j, \mathbf{w}) \right]$$

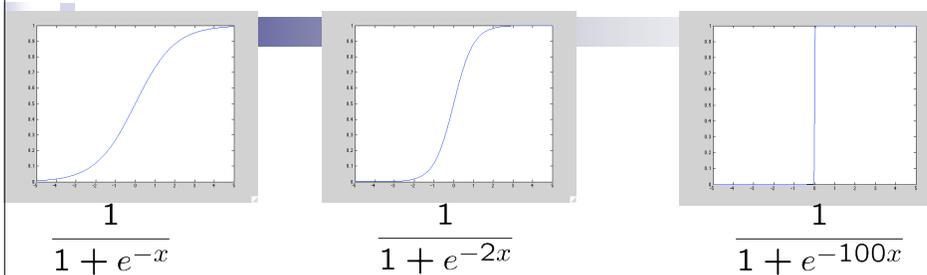
$$p(\mathbf{w}) = \prod_i \frac{1}{\kappa\sqrt{2\pi}} e^{-\frac{w_i^2}{2\kappa^2}}$$

Penalizes high weights, also applicable in linear regression

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Large parameters → Overfitting



- If data is linearly separable, weights go to infinity
- Leads to overfitting:

- Penalizing high weights can prevent overfitting...
 - again, more on this later in the semester

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Gradient of M(C)AP

$$\frac{\partial}{\partial w_i} \ln \left[p(\mathbf{w}) \prod_{j=1}^N P(y^j | \mathbf{x}^j, \mathbf{w}) \right]$$

$$p(\mathbf{w}) = \prod_i \frac{1}{\kappa\sqrt{2\pi}} e^{-\frac{w_i^2}{2\kappa^2}}$$

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MLE vs MAP

- Maximum conditional likelihood estimate

$$\mathbf{w}^* = \arg \max_{\mathbf{w}} \ln \left[\prod_{j=1}^N P(y^j | \mathbf{x}^j, \mathbf{w}) \right]$$

$$w_i^{(t+1)} \leftarrow w_i^{(t)} + \eta \sum_j x_i^j [y^j - \hat{P}(Y^j = 1 | \mathbf{x}^j, \mathbf{w})]$$

- Maximum conditional a posteriori estimate

$$\mathbf{w}^* = \arg \max_{\mathbf{w}} \ln \left[p(\mathbf{w}) \prod_{j=1}^N P(y^j | \mathbf{x}^j, \mathbf{w}) \right]$$

$$w_i^{(t+1)} \leftarrow w_i^{(t)} + \eta \left\{ -\lambda w_i^{(t)} + \sum_j x_i^j [y^j - \hat{P}(Y^j = 1 | \mathbf{x}^j, \mathbf{w})] \right\}$$

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G. Naïve Bayes vs. Logistic Regression 1

[Ng & Jordan, 2002]

- Generative and Discriminative classifiers
 - focuses on setting when GNB leads to linear classifier
 - variance σ_i (depends on feature i , not on class k)
- Asymptotic comparison (# training examples \rightarrow infinity)
 - when GNB model correct
 - GNB, LR produce identical classifiers
 - when model incorrect
 - LR is less biased – does not assume conditional independence
 - **therefore LR expected to outperform GNB**

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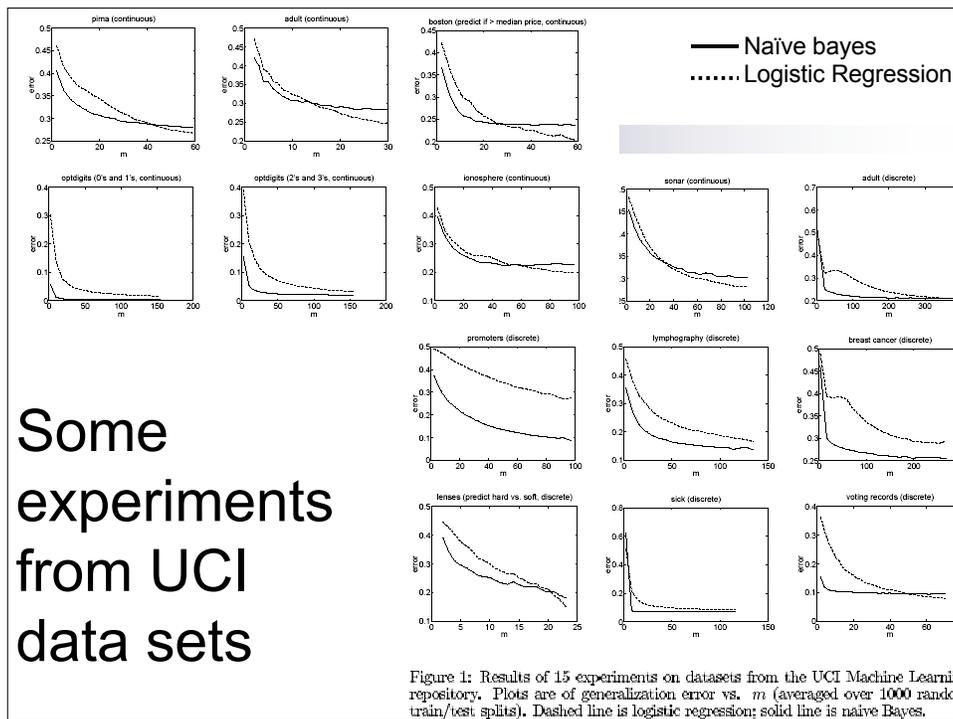
G. Naïve Bayes vs. Logistic Regression 2

[Ng & Jordan, 2002]

- Generative and Discriminative classifiers
 - focuses on setting when GNB leads to linear classifier
- Non-asymptotic analysis
 - convergence rate of parameter estimates, $n = \#$ of attributes in X
 - Size of training data to get close to infinite data solution
 - GNB needs $O(\log n)$ samples
 - LR needs $O(n)$ samples
 - **GNB converges more quickly to its (perhaps less helpful) asymptotic estimates**

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What you should know about Logistic Regression (LR)

- Gaussian Naïve Bayes with class-independent variances representationally equivalent to LR
 - Solution differs because of objective (loss) function
- In general, NB and LR make different assumptions
 - NB: Features independent given class \rightarrow assumption on $P(\mathbf{X}|Y)$
 - LR: Functional form of $P(Y|\mathbf{X})$, no assumption on $P(\mathbf{X}|Y)$
- LR is a linear classifier
 - decision rule is a hyperplane
- LR optimized by conditional likelihood
 - no closed-form solution
 - concave \rightarrow global optimum with gradient ascent
 - Maximum conditional a posteriori corresponds to regularization
- Convergence rates
 - GNB (usually) needs less data
 - LR (usually) gets to better solutions in the limit

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Linear separability

- A dataset is **linearly separable** iff \exists a **separating hyperplane**:
 - $\exists \mathbf{w}$, such that:
 - $w_0 + \sum_i w_i x_i > 0$; if $\mathbf{x}=\{x_1, \dots, x_n\}$ is a positive example
 - $w_0 + \sum_i w_i x_i < 0$; if $\mathbf{x}=\{x_1, \dots, x_n\}$ is a negative example

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Not linearly separable data

- Some datasets are **not linearly separable!**

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Addressing non-linearly separable data – Option 1, non-linear features

- Choose non-linear features, e.g.,
 - Typical linear features: $w_0 + \sum_i w_i x_i$
 - Example of non-linear features:
 - Degree 2 polynomials, $w_0 + \sum_i w_i x_i + \sum_{ij} w_{ij} x_i x_j$
- Classifier $h_{\mathbf{w}}(\mathbf{x})$ still linear in parameters \mathbf{w}
 - As easy to learn
 - Data is linearly separable in higher dimensional spaces
 - More discussion later this semester

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Addressing non-linearly separable data – Option 2, non-linear classifier

- Choose a classifier $h_{\mathbf{w}}(\mathbf{x})$ that is non-linear in parameters \mathbf{w} , e.g.,
 - Decision trees, neural networks, nearest neighbor,...
- More general than linear classifiers
- But, can often be harder to learn (non-convex/concave optimization required)
- But, but, often very useful
- (BTW. Later this semester, we'll see that these options are not that different)

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A small dataset: Miles Per Gallon

Suppose we want to predict MPG

mpg	cylinders	displacement	horsepower	weight	acceleration	modelyear	maker
good	4	low	low	low	high	75to78	asia
bad	6	medium	medium	medium	medium	70to74	america
bad	4	medium	medium	medium	low	75to78	europa
bad	8	high	high	high	low	70to74	america
bad	6	medium	medium	medium	medium	70to74	america
bad	4	low	medium	low	medium	70to74	asia
bad	4	low	medium	low	low	70to74	asia
bad	8	high	high	high	low	75to78	america
:	:	:	:	:	:	:	:
:	:	:	:	:	:	:	:
:	:	:	:	:	:	:	:
:	:	:	:	:	:	:	:
bad	8	high	high	high	low	70to74	america
good	8	high	medium	high	high	79to83	america
bad	8	high	high	high	low	75to78	america
good	4	low	low	low	low	79to83	america
bad	6	medium	medium	medium	high	75to78	america
good	4	medium	low	low	low	79to83	america
good	4	low	low	medium	high	79to83	america
bad	8	high	high	high	low	70to74	america
good	4	low	medium	low	medium	75to78	europa
bad	5	medium	medium	medium	medium	75to78	europa

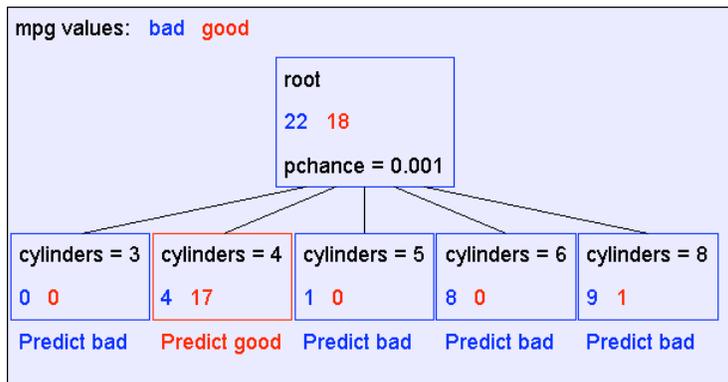
40 Records

From the UCI repository (thanks to Ross Quinlan)

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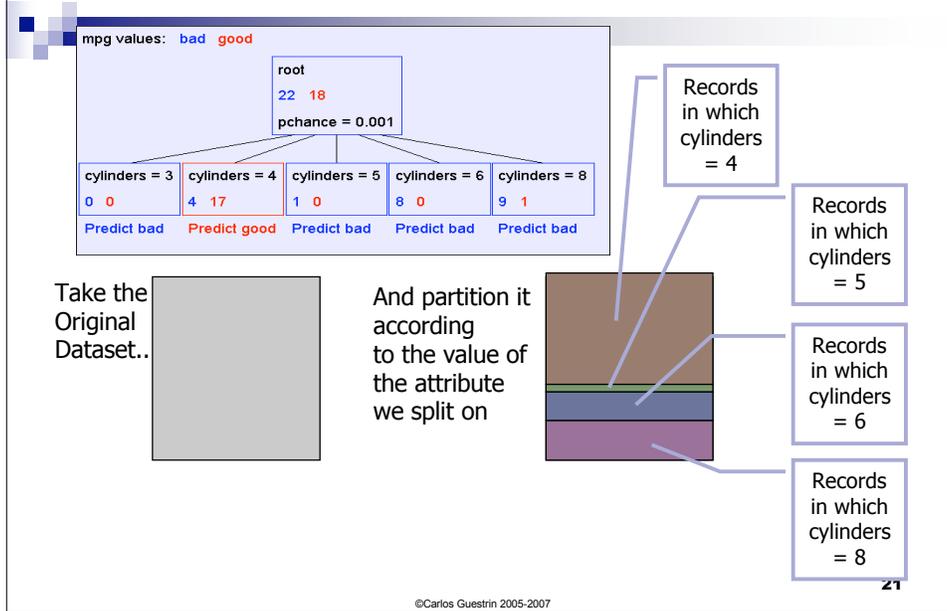
A Decision Stump



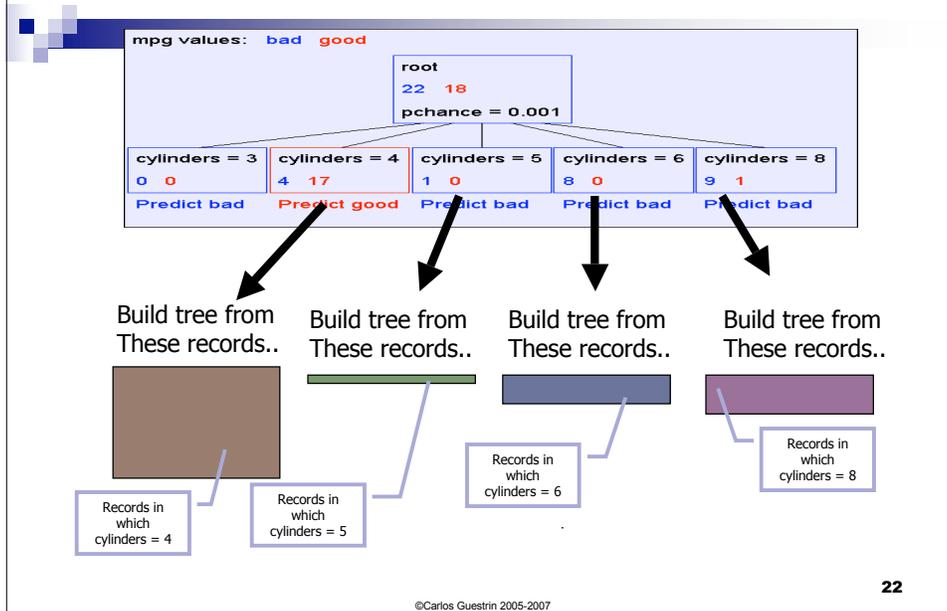
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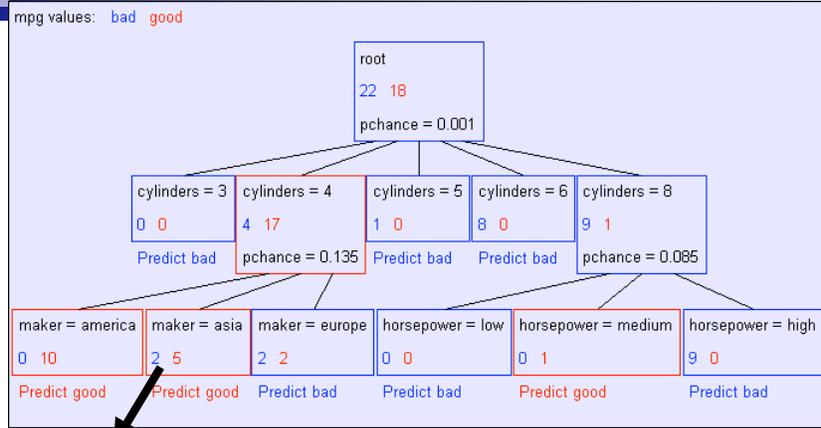
Recursion Step



Recursion Step



Second level of tree



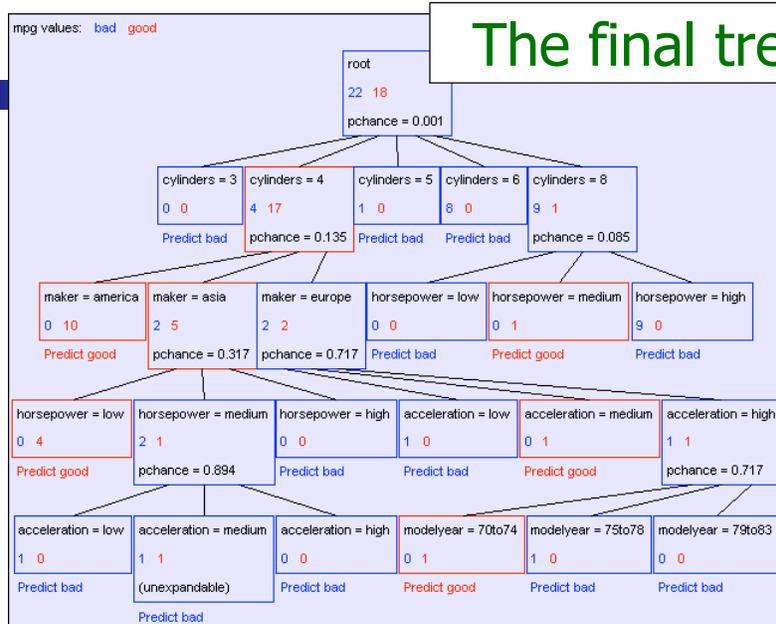
Recursively build a tree from the seven records in which there are four cylinders and the maker was based in Asia

(Similar recursion in the other cases)

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The final tree

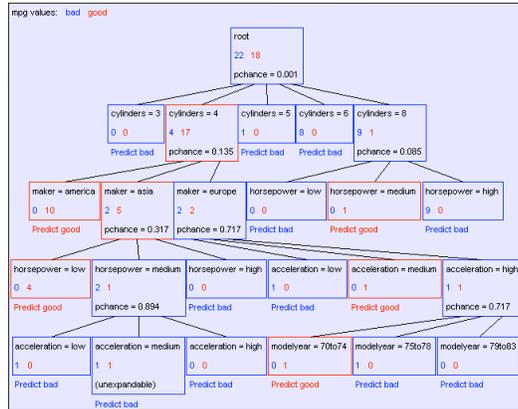


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Classification of a new example

- Classifying a test example – traverse tree and report leaf label



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Announcements

- **Pittsburgh won the Super Bowl !!**
 - Two years ago...
- Recitation this Thursday
 - Logistic regression, discriminative v. generative

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Are all decision trees equal?

- Many trees can represent the same concept
- But, not all trees will have the same size!
 - e.g., $\phi = A \wedge B \vee \neg A \wedge C$ ((A and B) or (not A and C))

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Learning decision trees is hard!!!

- Learning the simplest (smallest) decision tree is an NP-complete problem [Hyafil & Rivest '76]
- Resort to a greedy heuristic:
 - Start from empty decision tree
 - Split on **next best attribute (feature)**
 - Recurse

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Choosing a good attribute

X_1	X_2	Y
T	T	T
T	F	T
T	T	T
T	F	T
F	T	T
F	F	F
F	T	F
F	F	F

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Measuring uncertainty

- Good split if we are more certain about classification after split
 - Deterministic good (all true or all false)
 - Uniform distribution bad

$P(Y=A) = 1/2$	$P(Y=B) = 1/4$	$P(Y=C) = 1/8$	$P(Y=D) = 1/8$
----------------	----------------	----------------	----------------

$P(Y=A) = 1/4$	$P(Y=B) = 1/4$	$P(Y=C) = 1/4$	$P(Y=D) = 1/4$
----------------	----------------	----------------	----------------

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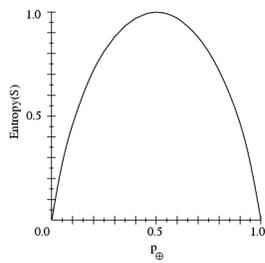
Entropy

Entropy $H(X)$ of a random variable Y

$$H(Y) = - \sum_{i=1}^k P(Y = y_i) \log_2 P(Y = y_i)$$

More uncertainty, more entropy!

Information Theory interpretation: $H(Y)$ is the expected number of bits needed to encode a randomly drawn value of Y (under most efficient code)



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Andrew Moore's Entropy in a nutshell



Low Entropy



High Entropy

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Andrew Moore's Entropy in a nutshell



Low Entropy

..the values (locations of soup) sampled entirely from within the soup bowl



High Entropy

..the values (locations of soup) unpredictable... almost uniformly sampled throughout our dining room

Information gain

- Advantage of attribute – decrease in uncertainty

- Entropy of Y before you split

- Entropy after split

- Weight by probability of following each branch, i.e., normalized number of records

$$H(Y | X) = - \sum_{j=1}^v P(X = x_j) \sum_{i=1}^k P(Y = y_i | X = x_j) \log_2 P(Y = y_i | X = x_j)$$

X ₁	X ₂	Y
T	T	T
T	F	T
T	T	T
T	F	T
F	T	T
F	F	F

- Information gain is difference $IG(X) = H(Y) - H(Y | X)$

Learning decision trees

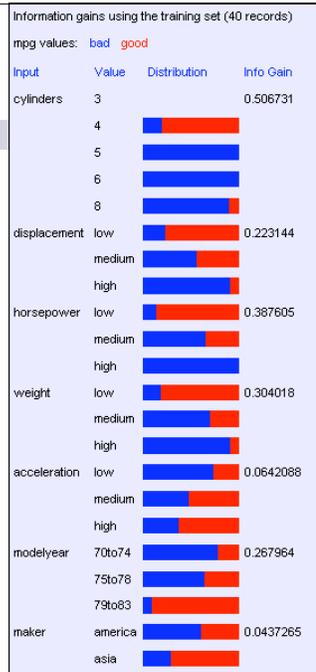
- Start from empty decision tree
- Split on **next best attribute (feature)**
 - Use, for example, information gain to select attribute
 - Split on $\arg \max_i IG(X_i) = \arg \max_i H(Y) - H(Y | X_i)$
- Recurse

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Suppose we want to predict MPG

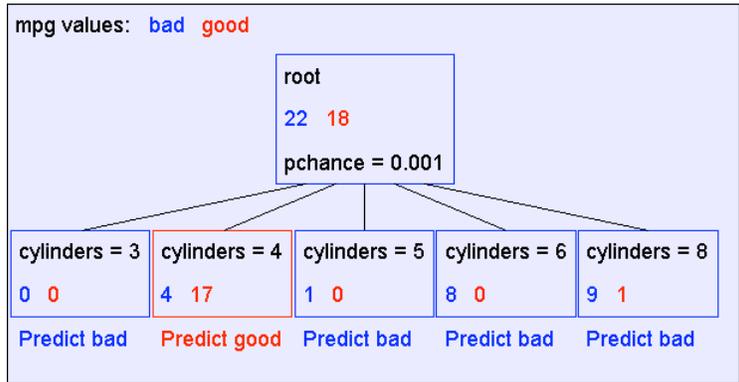
Look at all the information gains...



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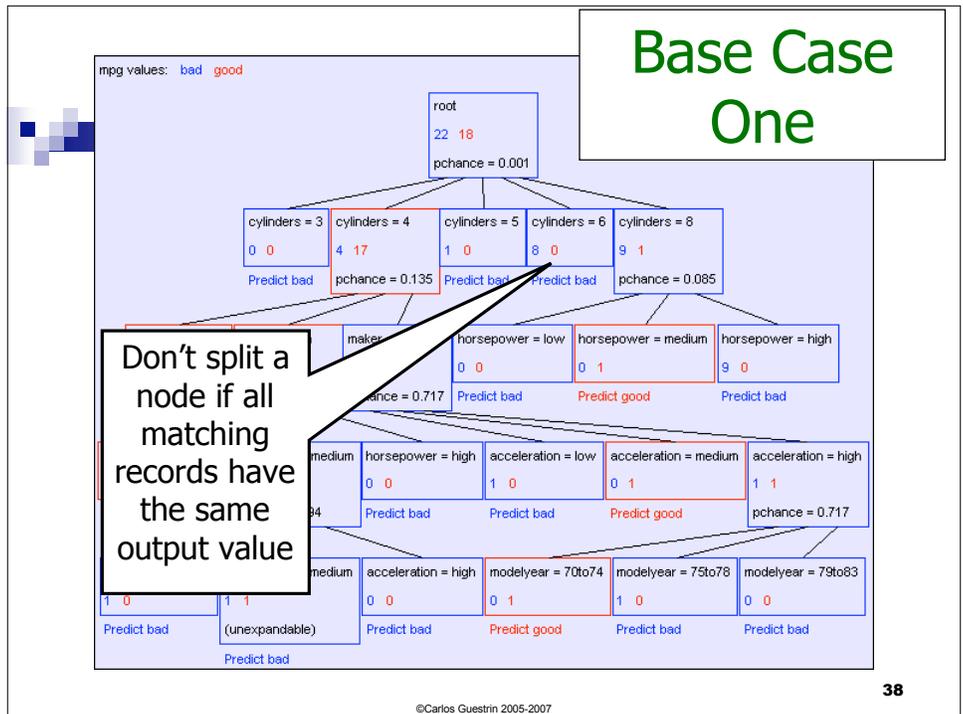
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A Decision Stump



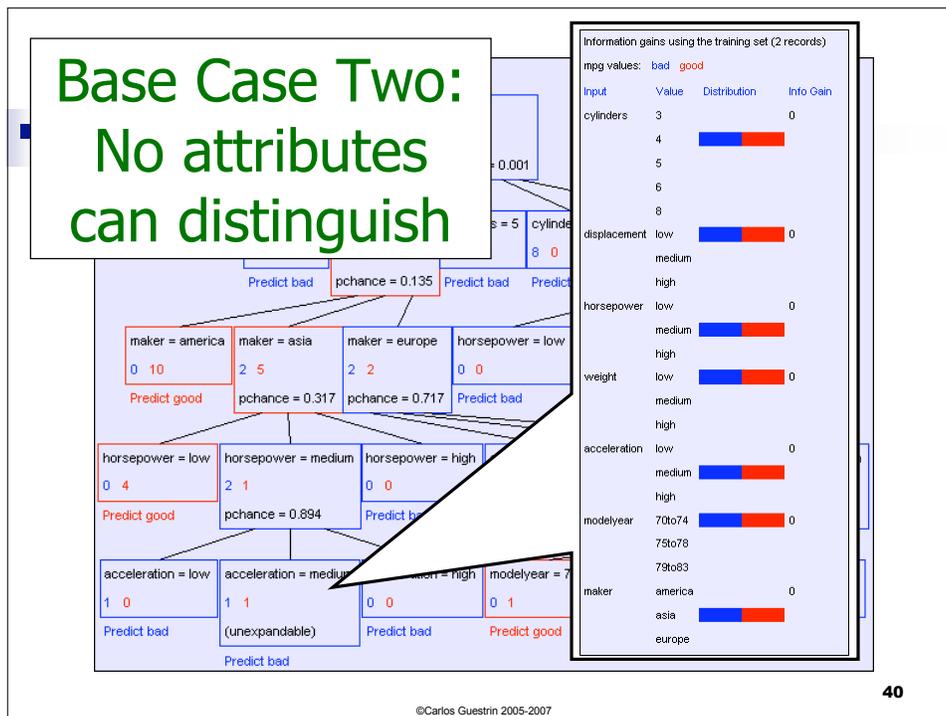
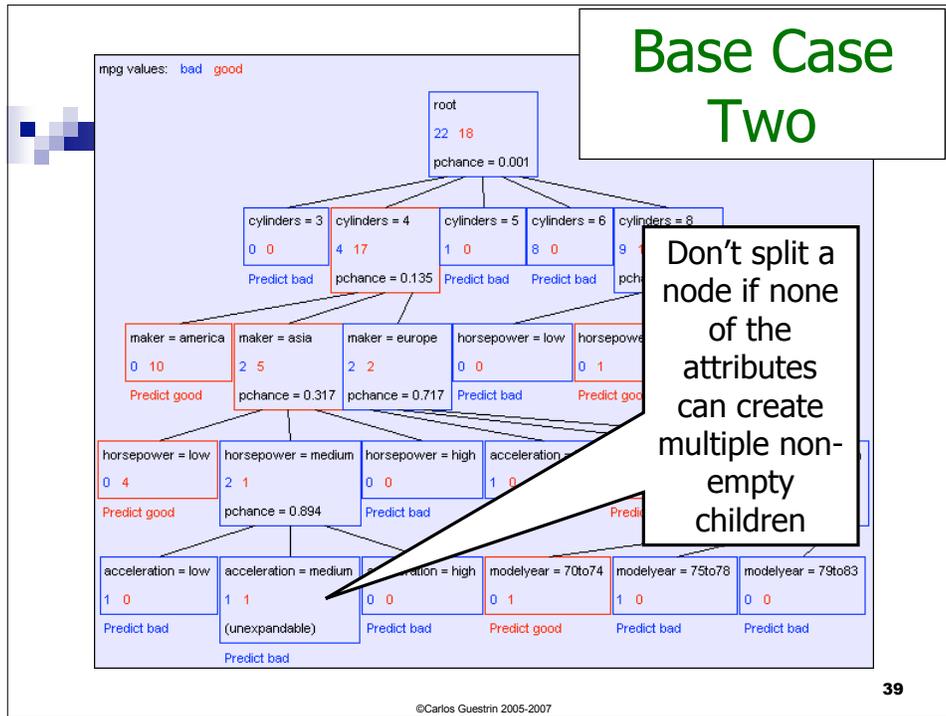
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Base Cases

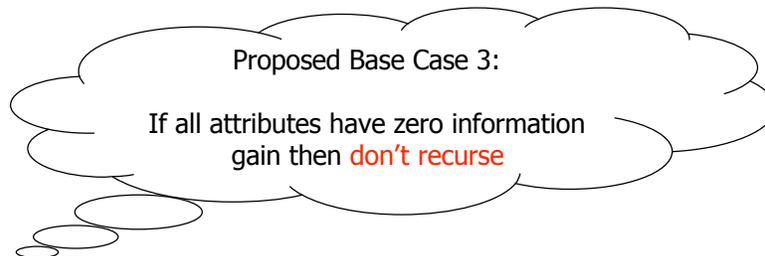
- Base Case One: If all records in current data subset have the same output then **don't recurse**
- Base Case Two: If all records have exactly the same set of input attributes then **don't recurse**

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Base Cases: An idea

- Base Case One: If all records in current data subset have the same output then **don't recurse**
- Base Case Two: If all records have exactly the same set of input attributes then **don't recurse**



• *Is this a good idea?*

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The problem with Base Case 3

a	b	y
0	0	0
0	1	1
1	0	1
1	1	0

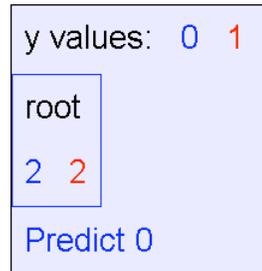
$y = a \text{ XOR } b$

The information gains:

Information gains using the training set (4 records)
y values: 0 1

Input	Value	Distribution	Info Gain
a	0		0
	1		0
b	0		0
	1		0

The resulting decision tree:



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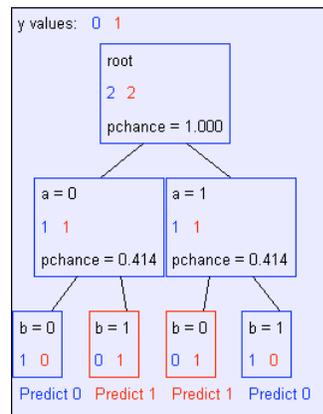
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If we omit Base Case 3:

a	b	y
0	0	0
0	1	1
1	0	1
1	1	0

$y = a \text{ XOR } b$

The resulting decision tree:



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Basic Decision Tree Building Summarized

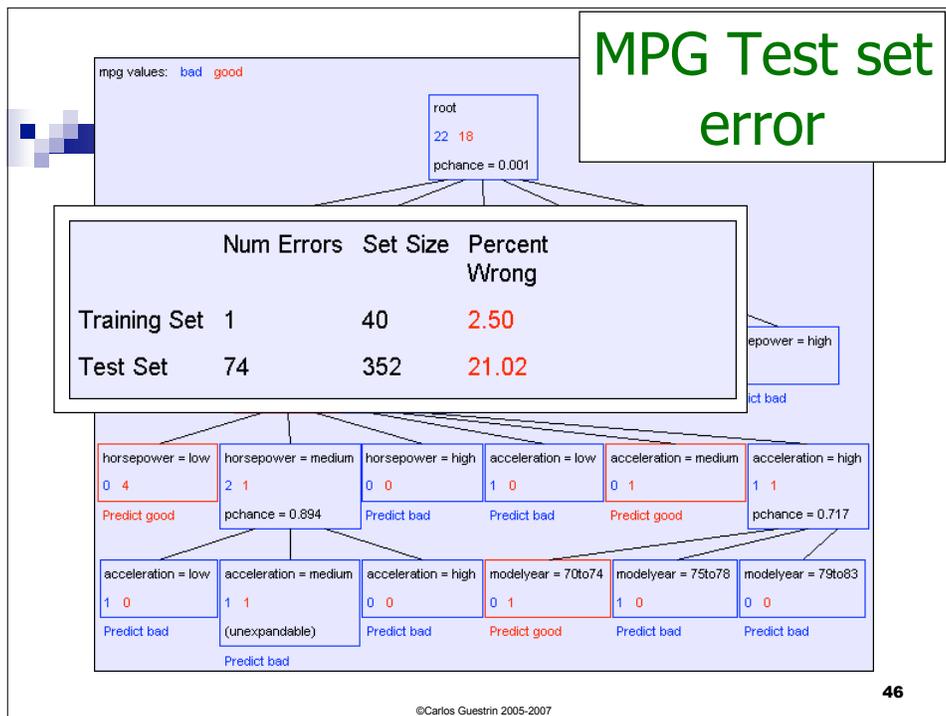
BuildTree(DataSet, Output)

- If all output values are the same in DataSet, return a leaf node that says “predict this unique output”
- If all input values are the same, return a leaf node that says “predict the majority output”
- Else find attribute X with highest Info Gain
- Suppose X has n_X distinct values (i.e. X has arity n_X).
 - Create and return a non-leaf node with n_X children.
 - The i 'th child should be built by calling
BuildTree(DS_i , Output)

Where DS_i built consists of all those records in DataSet for which X = i th distinct value of X.

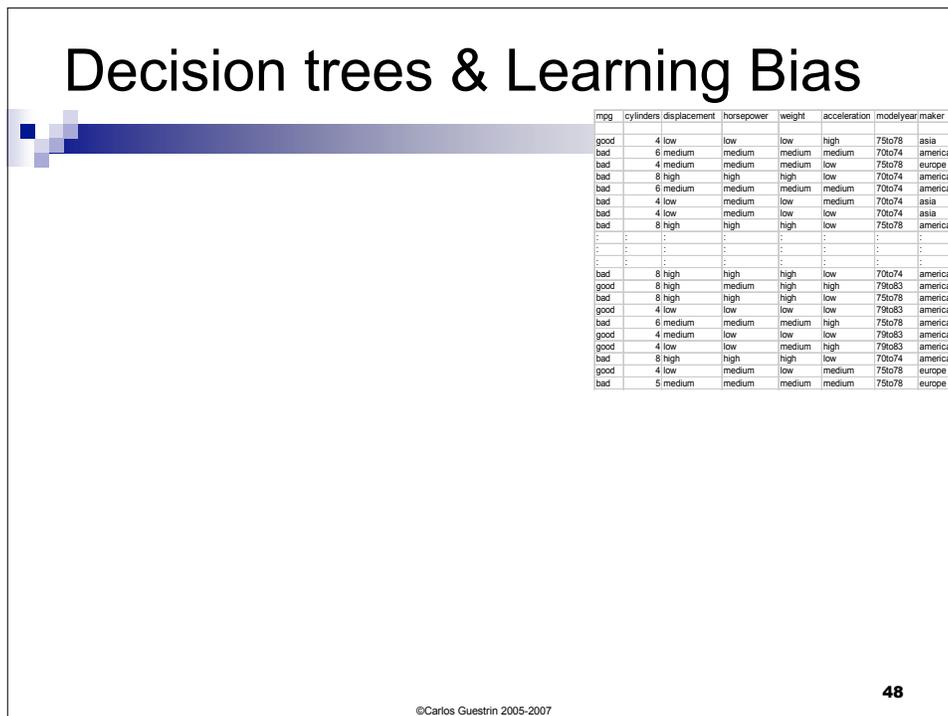
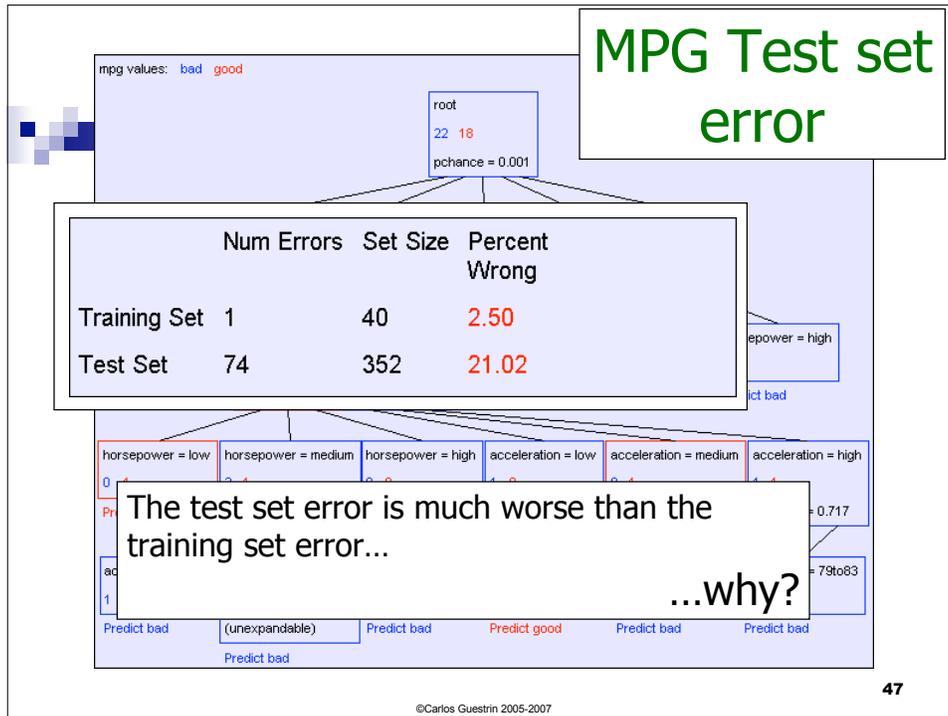
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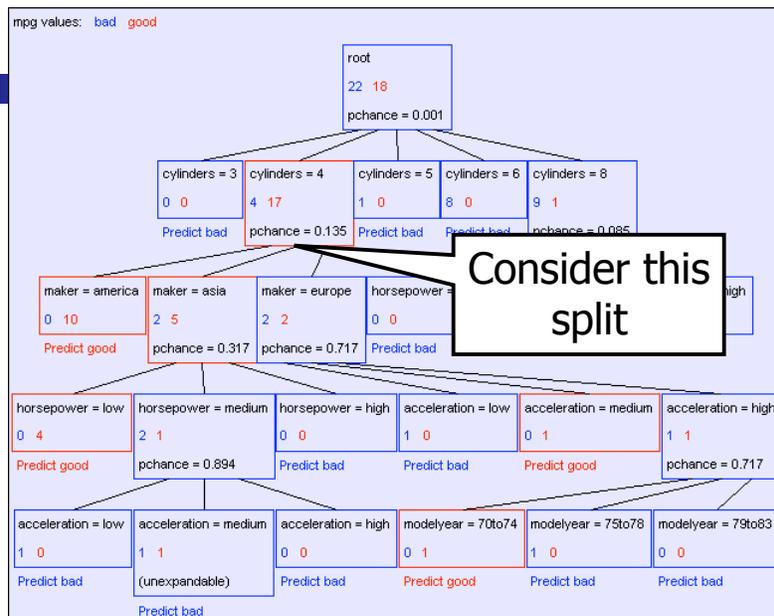


Decision trees will overfit

- Standard decision trees are have no learning biased
 - Training set error is always zero!
 - (If there is no label noise)
 - Lots of variance
 - Will definitely overfit!!!
 - Must bias towards simpler trees
- Many strategies for picking simpler trees:
 - Fixed depth
 - Fixed number of leaves
 - Or something smarter...

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A chi-square test

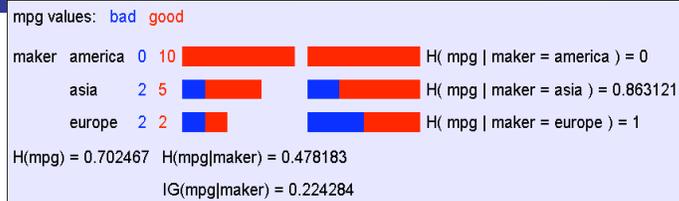


- Suppose that mpg was completely uncorrelated with maker.
- What is the chance we'd have seen data of at least this apparent level of association anyway?

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A chi-square test



- Suppose that mpg was completely uncorrelated with maker.
- What is the chance we'd have seen data of at least this apparent level of association anyway?

By using a particular kind of chi-square test, the answer is 7.2%

(Such simple hypothesis tests are very easy to compute, unfortunately, not enough time to cover in the lecture, but in your homework, you'll have fun! :))

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Using Chi-squared to avoid overfitting

- Build the full decision tree as before
- But when you can grow it no more, start to prune:
 - Beginning at the bottom of the tree, delete splits in which $p_{chance} > MaxPchance$
 - Continue working your way up until there are no more prunable nodes

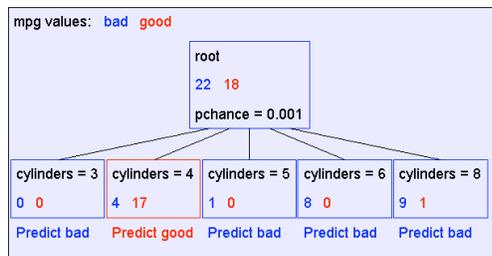
MaxPchance is a magic parameter you must specify to the decision tree, indicating your willingness to risk fitting noise

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Pruning example

- With *MaxPchance* = 0.1, you will see the following MPG decision tree:



Note the improved test set accuracy compared with the unpruned tree

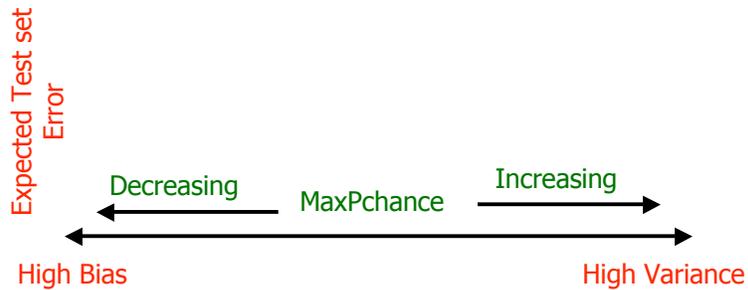
	Num Errors	Set Size	Percent Wrong
Training Set	5	40	12.50
Test Set	56	352	15.91

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MaxPchance

- Technical note MaxPchance is a regularization parameter that helps us bias towards simpler models



- We'll learn to choose the value of these magic parameters soon!

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Real-Valued inputs

- What should we do if some of the inputs are real-valued?

mpg	cylinders	displacemen	horsepower	weight	acceleration	modelyear	maker
good	4	97	75	2265	18.2	77	asia
bad	6	199	90	2648	15	70	america
bad	4	121	110	2600	12.8	77	europa
bad	8	350	175	4100	13	73	america
bad	6	198	95	3102	16.5	74	america
bad	4	108	94	2379	16.5	73	asia
bad	4	113	95	2228	14	71	asia
bad	8	302	139	3570	12.8	78	america
:	:	:	:	:	:	:	:
:	:	:	:	:	:	:	:
:	:	:	:	:	:	:	:
:	:	:	:	:	:	:	:
good	4	120	79	2625	18.6	82	america
bad	8	455	225	4425	10	70	america
good	4	107	86	2464	15.5	76	europa
bad	5	131	103	2830	15.9	78	europa

Infinite number of possible split values!!!

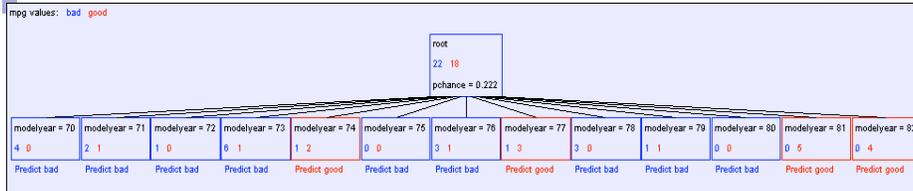
Finite dataset, only finite number of relevant splits!

Idea One: Branch on each possible real value

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“One branch for each numeric value” idea:



Hopeless: with such high branching factor will shatter the dataset and overfit

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Threshold splits

- Binary tree, split on attribute X
 - One branch: $X < t$
 - Other branch: $X \geq t$

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Choosing threshold split

- Binary tree, split on attribute X
 - One branch: $X < t$
 - Other branch: $X \geq t$
- Search through possible values of t
 - Seems hard!!!
- But only finite number of t 's are important
 - Sort data according to X into $\{x_1, \dots, x_m\}$
 - Consider split points of the form $x_i + (x_{i+1} - x_i)/2$

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A better idea: thresholded splits

- Suppose X is real valued
- Define $IG(Y|X:t)$ as $H(Y) - H(Y|X:t)$
- Define $H(Y|X:t) =$
$$H(Y|X < t) P(X < t) + H(Y|X \geq t) P(X \geq t)$$
 - $IG(Y|X:t)$ is the information gain for predicting Y if all you know is whether X is greater than or less than t
- Then define $IG^*(Y|X) = \max_t IG(Y|X:t)$
- For each real-valued attribute, use $IG^*(Y|X)$ for assessing its suitability as a split
- Note, may split on an attribute multiple times, with different thresholds

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Example with MPG

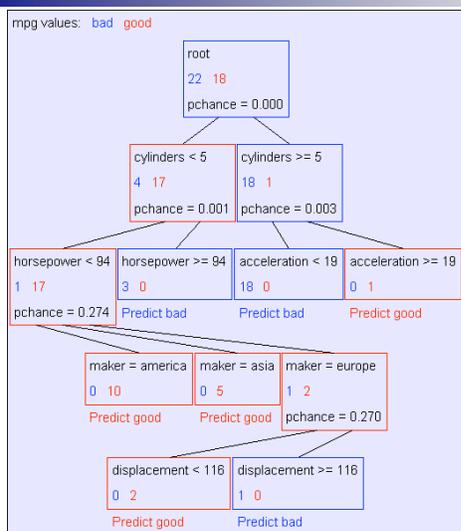
Information gains using the training set (40 records)
mpg values: bad good

Input	Value	Distribution	Info Gain
cylinders	< 5		0.48268
	>= 5		
displacement	< 198		0.428205
	>= 198		
horsepower	< 94		0.48268
	>= 94		
weight	< 2789		0.379471
	>= 2789		
acceleration	< 18.2		0.159982
	>= 18.2		
modelyear	< 81		0.319193
	>= 81		
maker	america		0.0437265
	asia		
	europa		

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Example tree using reals



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What you need to know about decision trees

- Decision trees are one of the most popular data mining tools
 - Easy to understand
 - Easy to implement
 - Easy to use
 - Computationally cheap (to solve heuristically)
- Information gain to select attributes (ID3, C4.5,...)
- Presented for classification, can be used for regression and density estimation too
- Decision trees will overfit!!!
 - Zero bias classifier → Lots of variance
 - Must use tricks to find “simple trees”, e.g.,
 - Fixed depth/Early stopping
 - Pruning
 - Hypothesis testing

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- Some of the material in the decision trees presentation is courtesy of Andrew Moore, from his excellent collection of ML tutorials:
 - <http://www.cs.cmu.edu/~awm/tutorials>

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