# Markov Decision Processes (MDPs)

Machine Learning – 10701/15781

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# Thus far this semester

- Regression: $f: X \mapsto \mathbb{R}$
- Classification: $f: X \mapsto Y \in \{hot, very\ hot, boiling\ hot\}$
- Density estimation:

$$f: X \mapsto [0,1]$$

$$\int_X f(x)\, dx = 1$$

# Learning to act

- Reinforcement learning
- An agent
  - Makes sensor observations X
  - Must select action A ( Y )
  - Receives rewards
    - positive for “good” states
    - negative for “bad” states

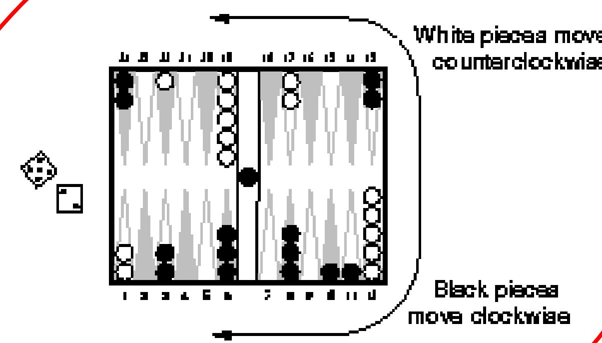
[Ng et al. ’05]

fully observable

X → true state of worlds

# Learning to play backgammon [Tesauro '95]

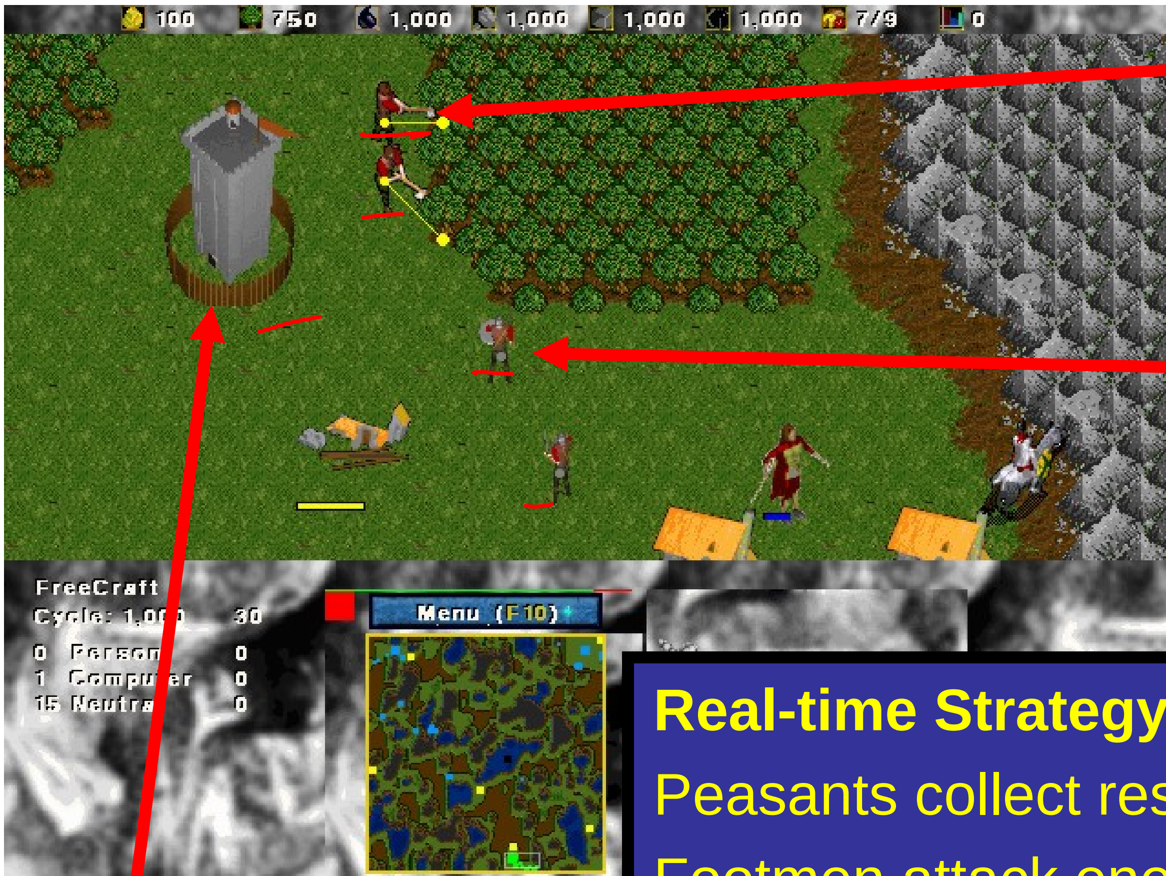
- Combines reinforcement learning with neural networks
- Played 300,000 games against itself
- Achieved grandmaster level!



# Roadmap to learning about reinforcement learning

- When we learned about Bayes nets:
  - First talked about formal framework:
    - representation
    - inference
  - Then learning for BNs
- For reinforcement learning:
  - Formal framework
    - Markov decision processes
  - Then learning

# FreeCraft




peasant

footman

building

**Real-time Strategy Game**

Peasants collect resources and build

Footmen attack enemies

Buildings train peasants and footmen

# States and actions

- State space:
  - Joint state **x** of entire system

$X = (P_1 = 0,\ P_2 = 7,\ E_1 = 17 \ldots)$

- Action space:
  - Joint action **a**= {$a_1$,…, $a_n$} for all agents

enemy is "nature", or has a predefined strategy that doesn't on how you act. explicitly

# States change over time

- Like an HMM, state changes over time

$$\begin{array}{ccccccc} & A_1 & & A_2 & & A_{T-1} & \\ X_1 & \longrightarrow & X_2 & \longrightarrow & \cdots & \longrightarrow & X_T \end{array}$$

- Next state depends on current state and action selected
  - e.g., action="build castle" likely to lead to a state where you have a castle
- Transition model:
  - Dynamics of the entire system P(**x**'|**x**,**a**)


$$P(x^{(t)} \mid x^{(t-1)}, a^{(t-1)})$$

dynamics

you control

# Some states and actions are better than others

- Each state **x** is associated with a reward
  - positive reward for successful attack
  - negative for loss


- Reward function:
  - Total reward R(**x**)

$R(x,a) \leftarrow$ depend on state & action

e.g., $R(x) = \begin{cases} 0 & \text{if playing} \\ 1 & \text{if won} \\ -1 & \text{if lost} \end{cases}$ } at end

# Discounted Rewards

An assistant professor gets paid, say, 20K per year.

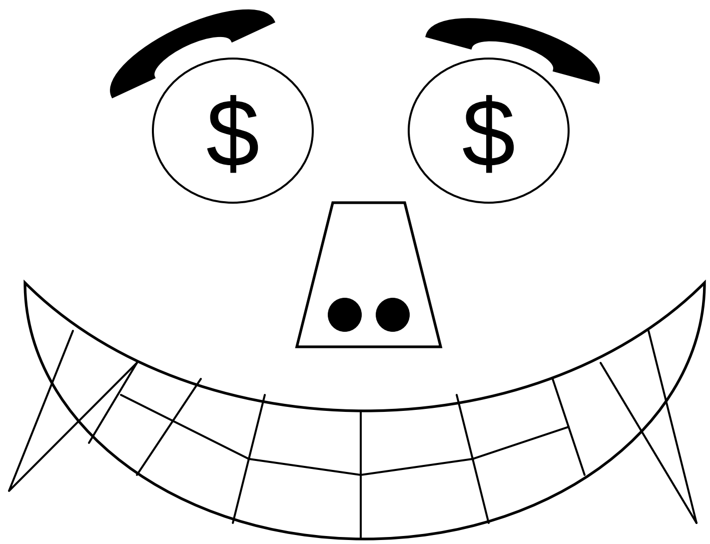
How much, in total, will the A.P. earn in their life?

20 + 20 + 20 + 20 + 20 + … = Infinity

if inflation is

$\frac{1}{\gamma} = 1.03$

20k next year is worth only $\gamma$20 this year




What's wrong with this argument?

# Discounted Rewards

"A reward (payment) in the future is not worth quite as much as a reward now."

- Because of chance of obliteration
- Because of inflation

Example:

Being promised $10,000 next year is worth only 90% as much as receiving $10,000 right now.

Assuming payment *n* years in future is worth only $(0.9)^n$ $\gamma^n$ of payment now, what is the AP's Future Discounted Sum of Rewards ?

$$20 + \gamma 20 + \gamma^2 20 \ldots$$

$$= \frac{20}{1-\gamma}$$

# Discount Factors

People in economics and probabilistic decision-making do this all the time.

The "Discounted sum of future rewards" using discount factor $\gamma$" is

(reward now) +

$\gamma$ (reward in 1 time step) +

$\gamma^2$ (reward in 2 time steps) +

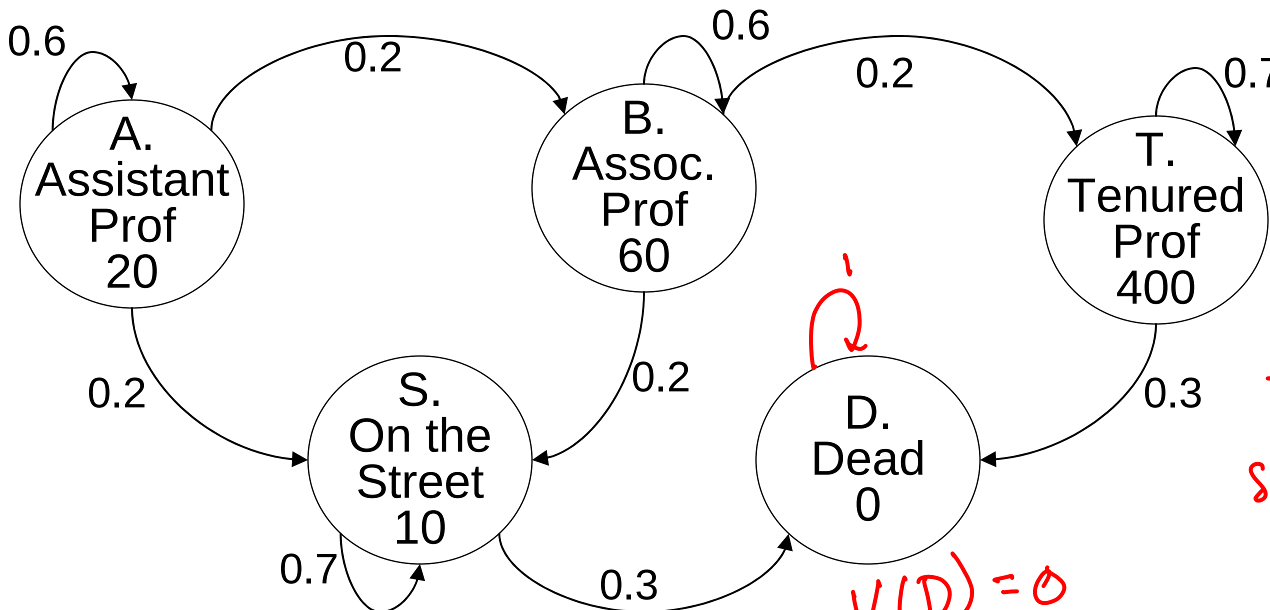
$\gamma^3$ (reward in 3 time steps) +

:

: (infinite sum)

# The Academic Life

Assume Discount Factor $\gamma = 0.9$




$V(D) = 0$

$V(T) =$
$400 +$
$0.3\gamma \cancel{V(D)}$ 0.6
$+ 0.7\gamma V(T)$
solve for
$V(T)$
$= \dfrac{400}{1 - 0.7\gamma}$

Define:

$V_A$ = Expected discounted future rewards starting in state A

$V_B$ = Expected discounted future rewards starting in state B

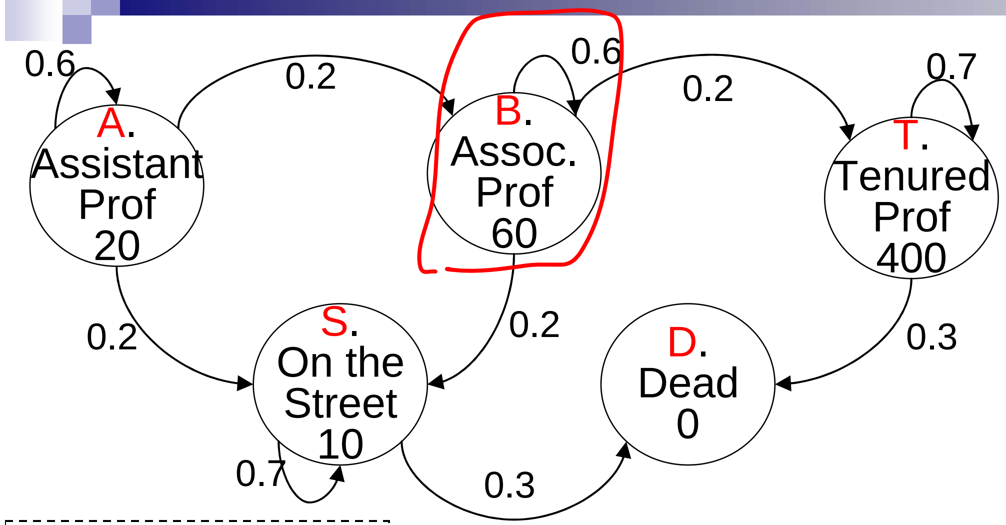
$V_T$ = " " " " " " " T

$V_S$ = " " " " " " " S

$V_D$ = " " " " " " " D

How do we compute $V_A, V_B, V_T, V_S, V_D$ ?

# Computing the Future Rewards of an Academic




Assume Discount Factor $\gamma = 0.9$

$$V(B) = 60 + \gamma \cdot 0.6\, V(B) + \gamma\, 0.2\, V(T) + \gamma\, 0.2\, V(S)$$

$$V(S) = 10 + \gamma\, 0.7\, V(S) + \gamma \cdot 0.3\, V(D)$$

# Joint Decision Space

Markov Decision Process (MDP) Representation:

- State space:
  - Joint state **x** of entire system
- Action space:
  - Joint action **a**= {$a_1$,…, $a_n$} for all agents
- Reward function:
  - Total reward R(**x**,**a**)
    - sometimes reward can depend on action
- Transition model:
  - Dynamics of the entire system P(**x**'|**x**,**a**)

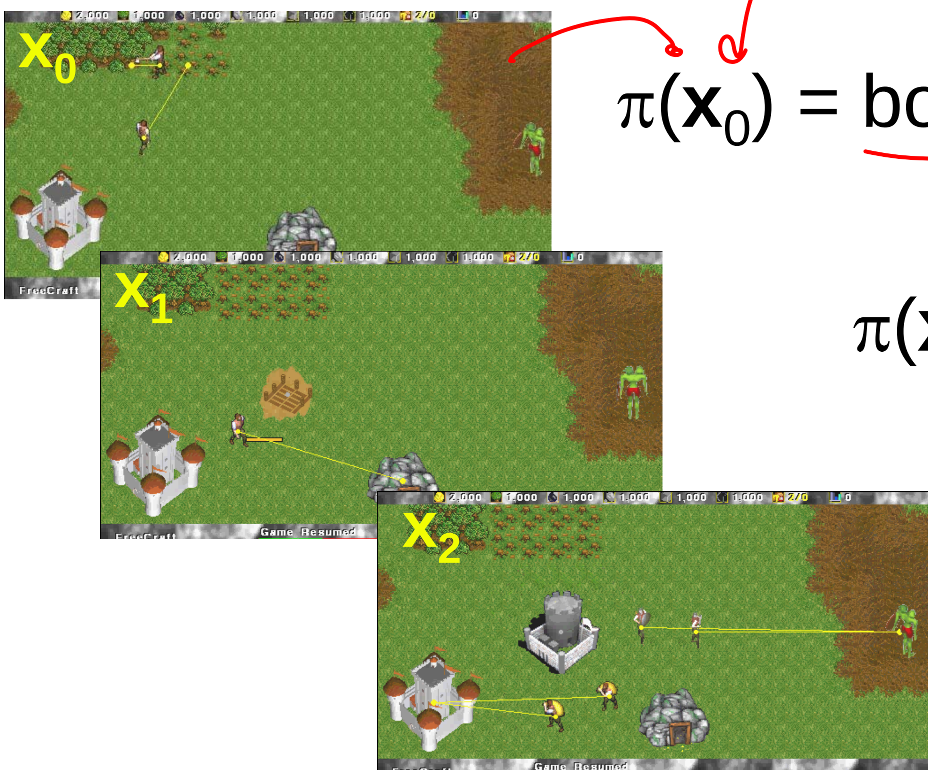
# Policy

Policy: $\pi(\mathbf{x}) = \mathbf{a}$

At state **x**, action **a** for all agents

state

action




$\pi(\mathbf{x}_0)$ = both peasants get wood

$\pi(\mathbf{x}_1)$ = one peasant builds barrack, other gets gold

$\pi(\mathbf{x}_2)$ = peasants get gold, footmen attack

# Value of Policy

Value: $V_\pi(\mathbf{x})$

Expected long-term reward starting from **x**

$V(x_0) = E_\pi[R(x_0) + \gamma R(x_1) + \gamma^2 R(x_2)] \ldots$

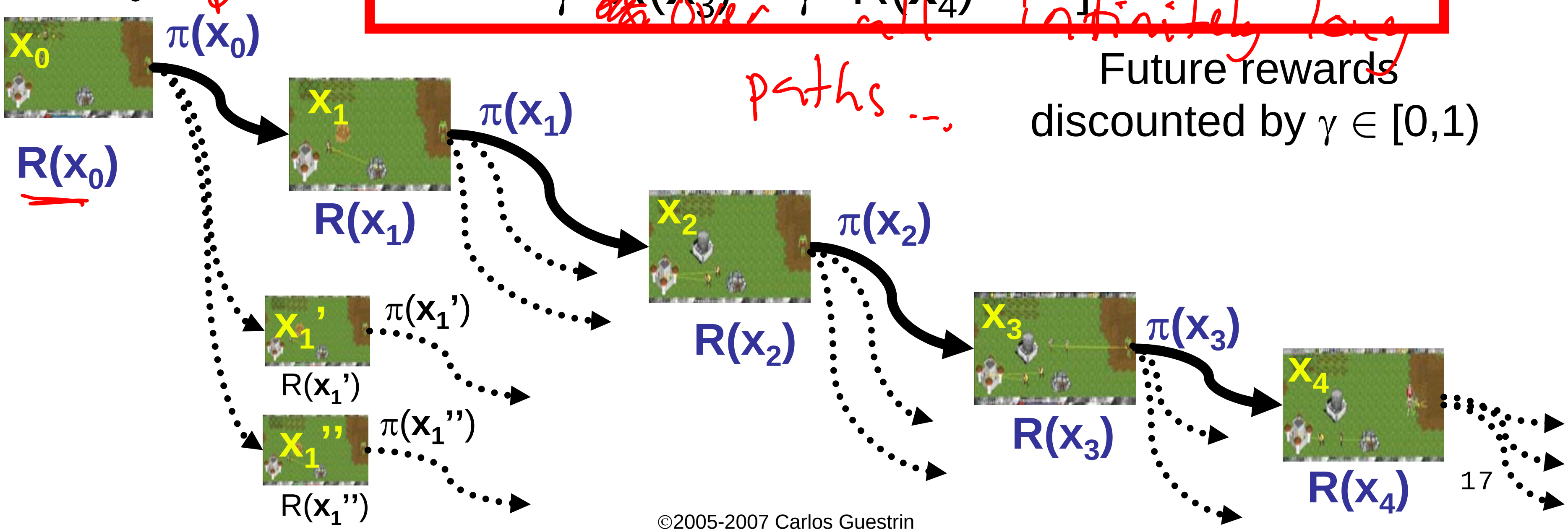
action

$$V_\pi(\mathbf{x}_0) = E_\pi[R(\mathbf{x}_0) + \gamma R(\mathbf{x}_1) + \gamma^2 R(\mathbf{x}_2) + \gamma^3 R(\mathbf{x}_3) + \gamma^4 R(\mathbf{x}_4) + \quad ]$$

expected value over all infinitely long paths ...

Future rewards discounted by $\gamma \in [0,1)$

Start from $\mathbf{x}_0$



# Computing the value of a policy

$$V_{\pi}(\mathbf{x_0}) = \mathbf{E}_{\pi}[R(\mathbf{x}_0) + \gamma\, R(\mathbf{x}_1) + \gamma^2\, R(\mathbf{x}_2) + \gamma^3\, R(\mathbf{x}_3) + \gamma^4\, R(\mathbf{x}_4) + \quad ]$$

- Discounted value of a state:
  - value of starting from $x_0$ and continuing with policy $\pi$ from then on

$$\begin{aligned} V_\pi(x_0) &= E_\pi[R(x_0) + \gamma R(x_1) + \gamma^2 R(x_2) + \gamma^3 R(x_3) + \cdots] \\ &= E_\pi[\sum_{t=0}^{\infty} \gamma^t R(x_t)] \end{aligned}$$

- A recursion!

$$V_\pi(x_0) = E_\pi[\sum_{t=0}^{\infty} \gamma^t R(x_t)] = E_\pi[R(x_0) + \gamma \sum_{t=1}^{\infty} \gamma^{t-1} R(x_t)]$$

$$= \underbrace{E_\pi[R(x_0)]} + \gamma\, \underbrace{E_\pi[\sum_{t=1}^{\infty} \gamma^{t-1} R(x_t)]}$$

$$V_\pi(x_0) = R(x_0) + \gamma\, E_{x_1}[V_\pi(x_1)] = R(x_0) + \gamma \sum_{x_1} P(x_1 \mid x_0, a=\pi(x_0))\, V_\pi(x_1)$$

# Computing the value of a policy 1 – the matrix inversion approach

n possible states

$$\underbrace{V_\pi(x) \;=\; R(x) + \gamma \sum_{x'} P(x' \mid x, a = \pi(x)) V_\pi(x')}$$

- Solve by simple matrix inversion:

$V_\pi = R + \gamma P_\pi V_\pi$

$(I - \gamma P_\pi) V_\pi = R$

$\underbrace{V_\pi = (I - \gamma P_\pi)^{-1} R}$

simple matrix inversion

$V_\pi = \begin{pmatrix} \vdots \\ V_\pi(x_i) \\ \vdots \end{pmatrix}$ n

$R = \begin{pmatrix} \vdots \\ R(x_i) \\ \vdots \end{pmatrix}$ n, $R(x_i, a = \pi(x_i))$

$x' = x^{(t)}$ next time step, n

$P_\pi = \begin{pmatrix} & & \\ & P(x^{(t)} \mid x^{(t-1)} = i, a = \pi(x^{(t-1)} = i)) & \\ & & \end{pmatrix}$

$x = x^{(t-1)} = i$

current n time step

# Computing the value of a policy 2 – iteratively

$V_\pi = R + \gamma P_\pi V_\pi$

$$V_\pi(x) \;=\; R(x) + \gamma \sum_{x'} P(x' \mid x, a = \pi(x)) V_\pi(x')$$

- If you have 1000,000 states, inverting a 1000,000x1000,000 matrix is hard!
- Can solve using a simple convergent iterative approach: (a.k.a. dynamic programming)
  - Start with some guess $V_0$
  - Iteratively say:
    - $V_{t+1} = R + \gamma\, P_\pi\, V_t$
  - Stop when $||V_{t+1}-V_t||_\infty \leq \varepsilon$
    - means that $||V_\pi-V_{t+1}||_\infty \leq \varepsilon/(1-\gamma)$

good guess: $V_0 = R$

converges to $V_\pi = (I-\gamma P_\pi)^{-1} R$

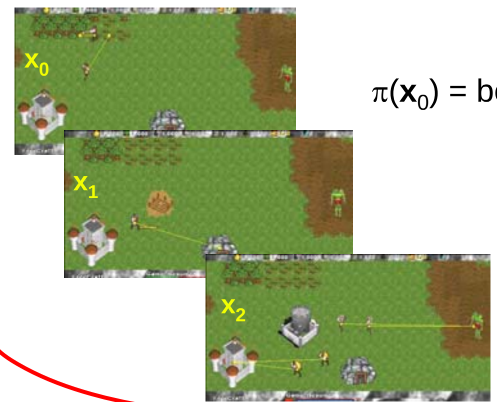
(because, $R + \gamma P_\pi V_\pi$ is a contraction)

# But we want to learn a **Policy**

- So far, told you how good a policy is…
- But how can we choose the best policy???

Policy: $\pi(\mathbf{x}) = \mathbf{a}$

At state $\mathbf{x}$, action $\mathbf{a}$ for all agents




$\pi(\mathbf{x}_0)$ = both peasants get wood

$\pi(\mathbf{x}_1)$ = one peasant builds barrack, other gets gold

$\pi(\mathbf{x}_2)$ = peasants get gold, footmen attack

- Suppose there was only one time step:
  - world is about to end!!!
  - select action that maximizes reward!

$$\underset{\text{one timestep to go}}{\pi(x_0)} = \arg\max_a R(x_0) + \gamma \sum_{x'} P(x' \mid a, x_0)\, R(x')$$

# Another recursion!

- Two time steps: address tradeoff
  - good reward now
  - better reward in the future

$V(x_0) =$

$\max_a R(x_0)$

$+ \gamma \sum_{x'} P(x' \mid x_0, a)\, V(x')$

going to Vegas & spending all my savings

v.

saving for retirement

include a bunch of max's

# Unrolling the recursion

- Choose actions that lead to best value in the long run
  - Optimal value policy achieves optimal value V*

optimal

$$V^*(x_0) = \max_{a_0} R(x_0, a_0) + \gamma E_{a_0}[\underbrace{\max_{a_1} R(x_1, a_1) + \gamma^2 E_{a_1}[\max_{a_2} R(x_2, a_2) + \cdots]]}_{V^*(x_1)}$$

$$V^*(x_0) = \max_{a_0} R(x_0, a_0) + \gamma \sum_{x'} P(x' \mid x_0, a) \, V^*(x')$$

# Bellman equation [Bellman 60]

- Evaluating policy $\pi$:

$$V_\pi(x) \;=\; R(x) + \gamma \sum_{x'} P(x' \mid x, a = \pi(x)) V_\pi(x')$$

- Computing the optimal value $V^*$ - Bellman equation

$$\boxed{V^*(\mathbf{x}) = \max_{\mathbf{a}} R(\mathbf{x},\mathbf{a}) + \gamma \sum_{\mathbf{x}'} P(\mathbf{x}' \mid \mathbf{x},\mathbf{a}) V^*(\mathbf{x}')}$$

# Optimal Long-term Plan

Optimal value function $V^*(\mathbf{x})$ → Optimal Policy: $\pi^*(\mathbf{x})$

$$Q^*(\mathbf{x},\mathbf{a}) = R(\mathbf{x},\mathbf{a}) + \gamma \sum_{\mathbf{x}'} P(\mathbf{x}' \mid \mathbf{x},\mathbf{a}) V^*(\mathbf{x}')$$

**Optimal policy:**

$$\pi^*(\mathbf{x}) = \arg\max_{\mathbf{a}} Q^*(\mathbf{x},\mathbf{a})$$

$$\pi^*(x) = \arg\max_{a} R(x,a) + \gamma \sum_{x'} P(x' \mid x,a)\, V^*(x')$$

# Interesting fact – Unique value

$$V^*(\mathbf{x}) = \max_{\mathbf{a}} R(\mathbf{x},\mathbf{a}) + \gamma \sum_{\mathbf{x}'} P(\mathbf{x}'|\mathbf{x},\mathbf{a}) V^*(\mathbf{x}')$$

- *Slightly surprising fact*: There is only one $V^*$ that solves Bellman equation!
  - there may be many optimal policies that achieve $V^*$
- *Surprising fact*: optimal policies are good everywhere!!!

$$V^*(x) = V_{\pi^*}(x) \geq V_{\pi}(x), \;\; \forall x, \;\; \forall \pi$$

no worse

everywhere

any other policy

# Solving an MDP

Solve Bellman equation → Optimal value $V^*(\mathbf{x})$ → Optimal policy $\pi^*(\mathbf{x})$

$$V^*(\mathbf{x}) = \max_{\mathbf{a}} R(\mathbf{x},\mathbf{a}) + \gamma \sum_{\mathbf{x}'} P(\mathbf{x}' \mid \mathbf{x},\mathbf{a}) V^*(\mathbf{x}')$$

**Bellman equation is non-linear!!!**

Many algorithms solve the Bellman equations:

- Policy iteration [Howard '60, Bellman '57]
- Value iteration [Bellman '57]
- Linear programming [Manne '60]
- …

# Value iteration (a.k.a. dynamic programming) – the simplest of all

$$V^*(\mathbf{x}) = \max_{\mathbf{a}} R(\mathbf{x},\mathbf{a}) + \gamma \sum_{\mathbf{x}'} P(\mathbf{x}' \mid \mathbf{x},\mathbf{a}) V^*(\mathbf{x}')$$

- Start with some guess $V_0$ = R
- Iteratively say:
  - $$V_{t+1}(\mathbf{x}) = \max_{\mathbf{a}} R(\mathbf{x},\mathbf{a}) + \gamma \sum_{\mathbf{x}'} P(\mathbf{x}' \mid \mathbf{x},\mathbf{a}) V_t(\mathbf{x}')$$

argmax a

converges to $V^*$

- Stop when $\|V_{t+1}-V_t\|_\infty \leq \varepsilon$
  - means that $\|V^*-V_{t+1}\|_\infty \leq \varepsilon/(1-\gamma)$

$\Rightarrow$ $\pi_{t+1}$ = Greedy Policy $V_{t+1}$

$\|V^* - V_{\pi_{t+1}}\|_\infty \leq \frac{\varepsilon\gamma}{(1-\gamma)^2}$

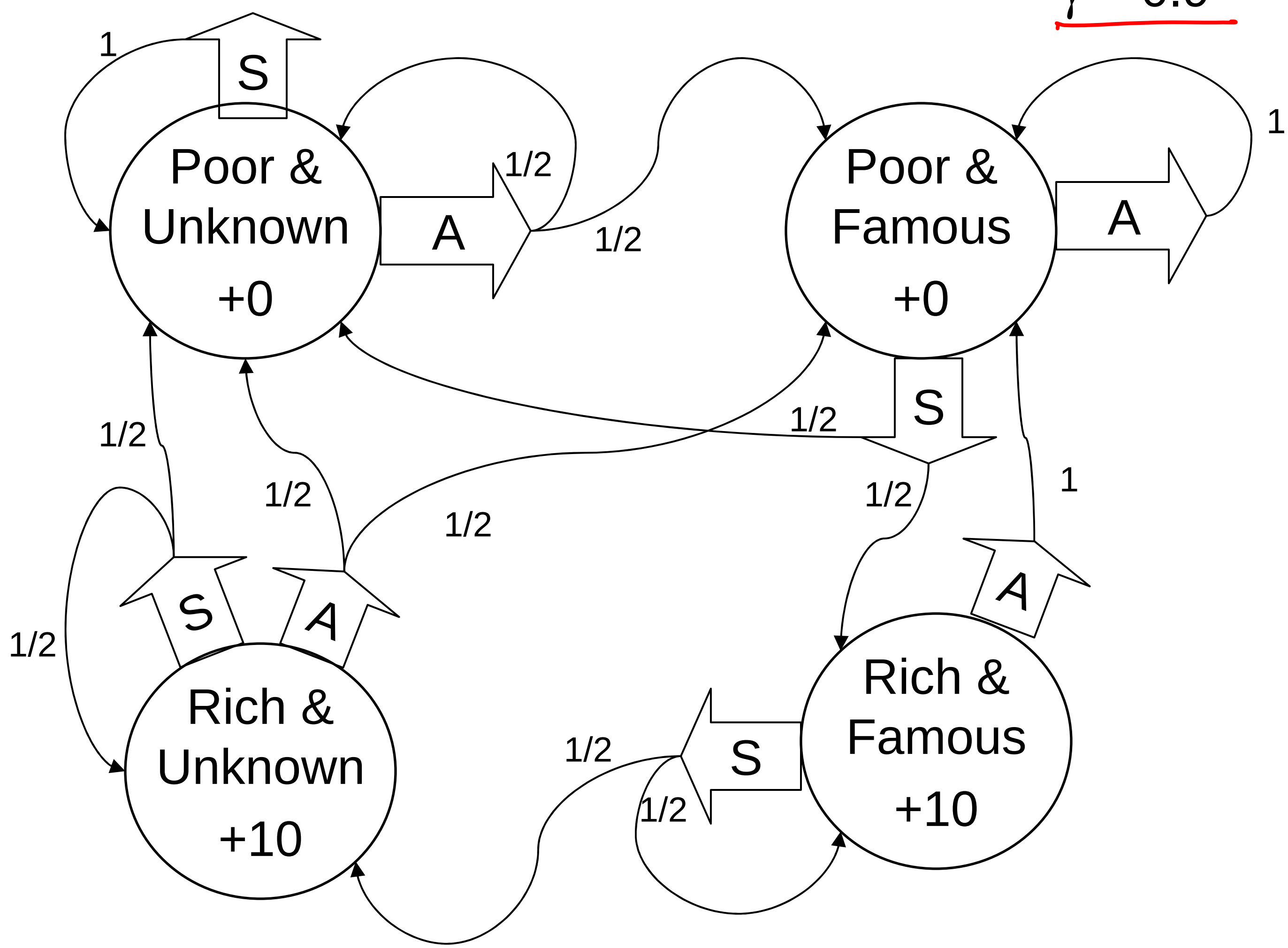
+ can get $\pi^*$

# A simple example

You run a startup company.

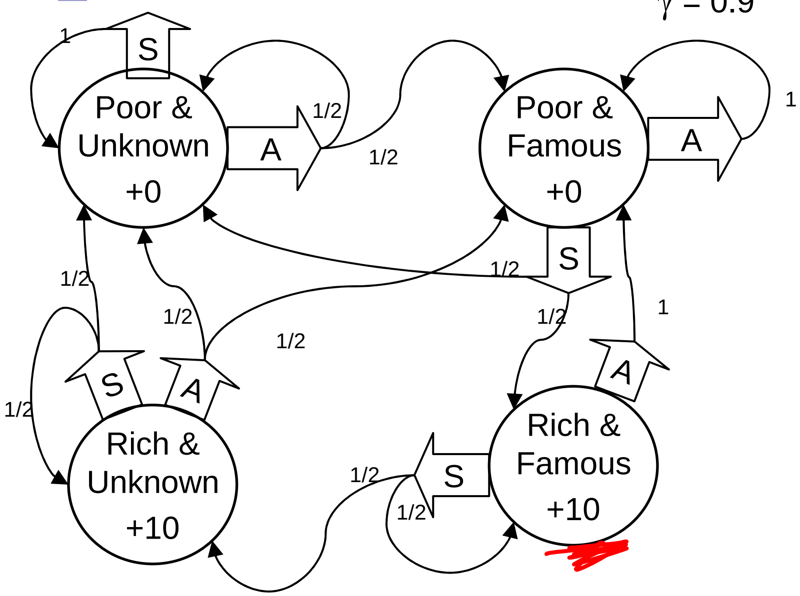
In every state you must choose between Saving money or Advertising.

$\gamma = 0.9$



# Let's compute $V_t(x)$ for our example

$\gamma = 0.9$



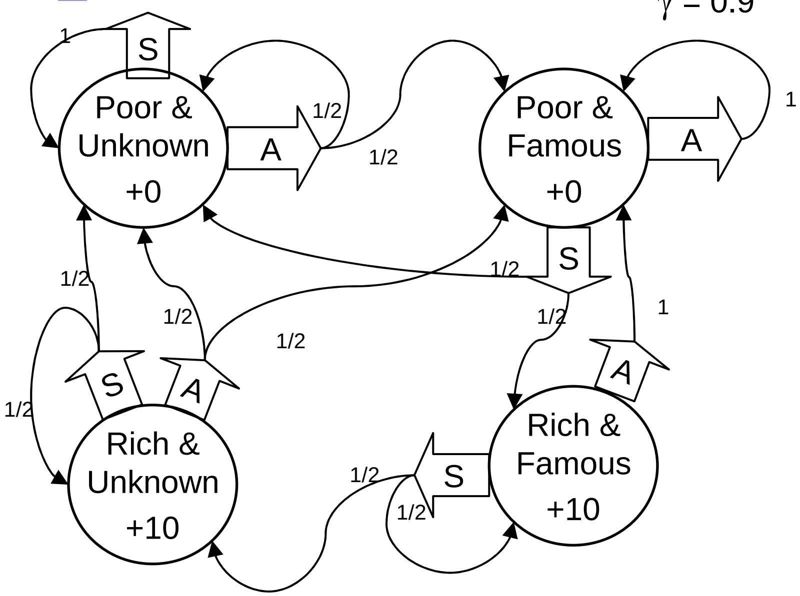

| t | $V_t(PU)$ | $V_t(PF)$ | $V_t(RU)$ | $V_t(RF)$ |
|---|---|---|---|---|
| 1 | 0 | 0 | 10 | 10 |
| 2 | | | | 19 |
| 3 | | | | |
| 4 | | | | |
| 5 | | | | |
| 6 | | | | |

$$\underset{=2}{V_{t+1}}(RF) = 10 + \gamma \max \begin{cases} a=A: & 1 \cdot V_{t=1}(\cancel{PF}) \quad 9 \quad 0 \\ a=S: & 0.5\, V_1(RF) + 0.5\, V_1(RU) \end{cases}$$

$$V_{t+1}(\mathbf{x}) = \max_{\mathbf{a}} \underline{R(\mathbf{x},\mathbf{a}) + \gamma \sum_{\mathbf{x}'} P(\mathbf{x}' \mid \mathbf{x},\mathbf{a}) V_t(\mathbf{x}')}$$

# Let's compute $V_t(x)$ for our example

$\gamma = 0.9$




| t | $V_t$(PU) | $V_t$(PF) | $V_t$(RU) | $V_t$(RF) |
|---|---|---|---|---|
| 1 | 0 | 0 | 10 | 10 |
| 2 | 0 | 4.5 | 14.5 | 19 |
| 3 | 2.03 | 6.53 | 25.08 | 18.55 |
| 4 | 3.852 | 12.20 | 29.63 | 19.26 |
| 5 | 7.22 | 15.07 | 32.00 | 20.40 |
| 6 | 10.03 | 17.65 | 33.58 | 22.43 |

$$V_{t+1}(\mathbf{x}) = \max_{\mathbf{a}} R(\mathbf{x},\mathbf{a}) + \gamma \sum_{\mathbf{x}'} P(\mathbf{x}' \mid \mathbf{x},\mathbf{a}) V_t(\mathbf{x}')$$

# Policy iteration – Another approach for computing $\pi^*$

- Start with some guess for a policy $\pi_0$
- Iteratively say:
  - evaluate policy: $$V_t(\mathbf{x}) = R(\mathbf{x}, \mathbf{a} = \pi_t(\mathbf{x})) + \gamma \sum_{\mathbf{x}'} P(\mathbf{x}' | \mathbf{x}, \mathbf{a} = \pi_t(\mathbf{x})) V_t(\mathbf{x}')$$
    $$V_t = (I - \gamma P_{\pi_t})^{-1} R$$
  - improve policy: $$\pi_{t+1}(\mathbf{x}) = \underset{\mathbf{a}}{\max}{}_{\text{arg}}\, R(\mathbf{x}, \mathbf{a}) + \gamma \sum_{\mathbf{x}'} P(\mathbf{x}' | \mathbf{x}, \mathbf{a}) V_t(\mathbf{x}')$$
- Stop when
  - policy stops changing
    - usually happens in about 10 iterations
  - or $||V_{t+1} - V_t||_\infty \leq \varepsilon$
    - means that $||V^* - V_{t+1}||_\infty \leq \varepsilon/(1-\gamma)$

open problem:
PI converges in polynomial time?

# Policy Iteration & Value Iteration: Which is best ???

It depends.

- Lots of actions? Choose Policy Iteration
- Already got a fair policy? Policy Iteration
- Few actions, acyclic? Value Iteration

Best of Both Worlds:

- Modified Policy Iteration [Puterman]
  - …a simple mix of value iteration and policy iteration

3rd Approach

Linear Programming

# LP Solution to MDP

[Manne '60]

Value computed by linear programming:

$$\text{minimize}: \sum_{\mathbf{x}} V(\mathbf{x})$$

$$\text{subject to}: \begin{cases} V(\mathbf{x}) \geq R(\mathbf{x},\mathbf{a}) + \gamma \sum_{\mathbf{x}'} P(\mathbf{x}' \mid \mathbf{x},\mathbf{a}) V(\mathbf{x}') \\ \forall \mathbf{x}, \mathbf{a} \end{cases}$$

- One variable $V(\mathbf{x})$ for each state
- One constraint for each state **x** and action **a**
- **Polynomial time solution**

# What you need to know

- What's a Markov decision process
  - state, actions, transitions, rewards
  - a policy
  - value function for a policy
    - computing $V_\pi$
- Optimal value function and optimal policy
  - Bellman equation
- Solving Bellman equation
  - with value iteration, policy iteration and linear programming

# Acknowledgment

- This lecture contains some material from Andrew Moore's excellent collection of ML tutorials:
  - http://www.cs.cmu.edu/~awm/tutorials

# Reinforcement Learning

Machine Learning – 10701/15781
Carlos Guestrin
Carnegie Mellon University

April 30th, 2007

# The Reinforcement Learning task

**World**: You are in state 34.
Your immediate reward is 3. You have possible 3 actions.

**Robot**: I'll take action 2.
**World**: You are in state 77.
Your immediate reward is -7. You have possible 2 actions.

**Robot**: I'll take action 1.
**World**: You're in state 34 (again).
Your immediate reward is 3. You have possible 3 actions.

# Formalizing the (online) reinforcement learning problem

- Given a set of states **X** and actions **A**
  - in some versions of the problem size of **X** and **A** unknown

- Interact with world at each time step *t*:
  - world gives state $\mathbf{x}_t$ and reward $r_t$
  - you give next action $\mathbf{a}_t$

- **Goal**: (quickly) learn policy that (approximately) maximizes long-term expected discounted reward

# The "Credit Assignment" Problem

| | | |
|---|---|---|
| I'm in state 43, | reward = 0, | action = 2 |
| " " " 39, | " = 0, | " = 4 |
| " " " 22, | " = 0, | " = 1 |
| " " " 21, | " = 0, | " = 1 |
| " " " 21, | " = 0, | " = 1 |
| " " " 13, | " = 0, | " = 2 |
| " " " 54, | " = 0, | " = 2 |
| " " " 26, | " = 100, | |

Yippee! I got to a state with a big reward! But which of my actions along the way actually helped me get there??

This is the Credit Assignment problem.

# Exploration-Exploitation tradeoff

- You have visited part of the state space and found a reward of 100
  - is this the best I can hope for???

- **Exploitation**: should I stick with what I know and find a good policy w.r.t. this knowledge?
  - at the risk of missing out on some large reward somewhere
- **Exploration**: should I look for a region with more reward?
  - at the risk of wasting my time or collecting a lot of negative reward

# Two main reinforcement learning approaches

- Model-based approaches:
  - explore environment $\rightarrow$ learn model ($P(\mathbf{x}'|\mathbf{x},\mathbf{a})$ and $R(\mathbf{x},\mathbf{a})$) (almost) everywhere
  - use model to plan policy, MDP-style
  - approach leads to strongest theoretical results
  - works quite well in practice when state space is manageable
- Model-free approach:
  - don't learn a model $\rightarrow$ learn value function or policy directly
  - leads to weaker theoretical results
  - often works well when state space is large