Using Simple Physical Models for Image Segmentation

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What is Image Segmentation?

Input

Output
Three Talks in One

• Image Segmentation Experimental
• Image Segmentation Theory
• Solving needed linear systems
What is Hard for Computers!

In the spaces below, type three (3) different English words appearing in the picture above.
Image Processing is everywhere

✓ Medical Image Analysis
✓ Matting & Manipulation
✓ Data Mining & Information Retrieval
✓ Intelligent Surveillance Systems
CS Reduction

• Convert the image segmentation problem into a well studied computer science problem.

• Hopefully, use an off-the-shelf solution to the CS problem.

• First attempt, Shi and Malik (2000)
Image Segmentation as Graph Partitioning
Basic Approach

Generate an affinity graph

- Each pixel a vertex
- Neighboring pixels are connected with an edge
- The weight $w_{ij}$ corresponds to their similarity
Ways to View edges

- Max-flow min-cut models
  - Each edge is a “pipe” that can carry a flow up to $W_{ij}$
  - Pick a source and sink and find mincut

- Electrical conductor models
  - Each edge is a conductor of size $W_{ij}$
  - Set voltage on some nodes to +1 and some to -1
Siemens Assisted Segmentation 2005

User assisted segmentation of the heart
no prior knowledge of hearts
More Ways to View edges

• Shortest Path Models
  • The distance between neighbors is $1/w_{ij}$

• Random Walk Models
  • The probability we walk on an edge is proportional to $w_{ij}$
  • Distance between node is the expect commute time
Even More Ways to View edges

- Spring Models (This talk)
  - The spring constant is $w_{ij}$
  - Separate based on modes of vibration (eigenvectors).
- Classic Spring Solution.
  - Compute a few low frequency eigenvectors say 2.
  - Map the vertices into 2D using the eigenvectors.
  - Apply a geometric cut the graph.
Airfoil Graph and its Spectral Embedding
Image Segmentation as Graph Partitioning

Goal: segment into 4 pieces
Output of the Classic Spring Model

Spring model using 4 eigenvectors
Output for Our Spectral Rounding Algorithm
What is new

- Spectral Rounding:
  - A better method to use eigenvectors for graph partitioning.

- Fast Planar Solvers:
  - Optimal time linear solvers for planar systems.
Inside the CS Reduction - Graph Partitioning

Data Graph

Standard Algorithm
Spectral Rounding
The Standard Spring Model Algorithm

Result - Standard Algorithm

Representation used by the Standard Algorithm
Our Technology

Result - Spectral Rounding

SR leverages the physical intuition!
Mathematical Formulation

- Ohm’s Law and Graph Laplacians

- Let $A_{ij} = w_{ij}$ and $D_{ii} = \sum_{j} w_{ij}$

- The Laplacian $L = D - A$

- Simple fact $LV = I$ where $V$ is voltage $I$ current
Mathematical Formulation

- Solving conductor model problems reduces to solving Laplacians
- Here the Graphs are in fact planar.
Mathematical Formulation for spring models

• We consider the case where the node has mass equal to its weighted degree. The Normalized Laplacian!

• Thus our eigenvalues and vectors satisfy  \( Lf = \lambda Df \)
Mathematical Formulation for spring models

- Zero eigenvalue \( L1 = 0D1 \)
- Rayleigh quotient
  \[
  \lambda_2 = \inf_{f \perp D_1} \frac{f^T L f}{f^T D f}
  \]
- Goal: reweight graph to reduce \( \lambda_2 \)
Spectral Rounding
Edge reweighting

• Algorithm
  • Solve \( Lf = \lambda_2 Df \)
  • Reweight graph getting \( L' \) and \( D' \)
  • Solve \( L'f = \lambda_2 D'f \)
  • repeat while Lambda not zero
  • repeat while best threshold cut is changing
Spectral Rounding
Finding a good reweighting

Lemma 1. Given a weighted symmetric graph $G = (V, E, w)$ then the normalized Rayleigh quotient can be written as

$$\frac{f^T L f}{f^T D f} = \frac{\sum_{(i,j) \in E, i < j} (f_i - f_j)^2 w_{ij}}{\sum_{(i,j) \in E, i < j} (f_i^2 + f_j^2) w_{ij}}$$  \hspace{1cm} (3.3)

where $f_i = f(v_i)$
Finding a good reweighting
Mediant of fractions

Definition 1. Given formal fractions

\[
\frac{a_1}{b_1}, \ldots, \frac{a_n}{b_n}
\]

the **fractional average** is the formal fraction

\[
\frac{\sum_{i=1}^{n} a_i}{\sum_{i=1}^{n} b_i}
\]

where the \(a_i\)'s and \(b_i\)'s are reals.
Finding a good reweighting using Mediant of fractions

Lemma 2. If $\frac{a_1}{b_1} \leq \cdots \leq \frac{a_n}{b_n}$ and $w_1 \geq \cdots \geq w_n$ then

$$\frac{\sum_{i=1}^{n} a_i}{\sum_{i=1}^{n} b_i} \geq \frac{\sum_{i=1}^{n} a_i w_i}{\sum_{i=1}^{n} b_i w_i}$$

The inequality is strict if for some pair $1 \geq i < j \leq n$ we have that $\frac{a_i}{b_i} < \frac{a_j}{b_j}$ and $w_i > w_j$. 
Inverse Fractional Reweighting

- Given $L$ and $D$ we get $L'$ and $D'$

- where 
  $$w_{ij}' = \frac{f_i^2 + f_j^2}{(f_i - f_j)^2} w_{ij}$$

- Gives 
  $$\frac{f^T L f}{f^T D f} \geq \frac{f^T L' f}{f^T D' f}$$

- Problem: in general 
  $$\lambda_2 \not\geq \lambda_2'$$
1D Family of Matrices

- 1D family
  \[ W(t) = W + tW' \]

- Theorem:
  \[ \lambda = \frac{f^T L f}{f^T D f} > \frac{f^T L' f}{f^T D' f} \quad \text{implies} \quad \frac{d\lambda(t)}{dt} < 0 \]
Defining Segmentation Quality

- Two Measures of Quality
  - A Mathematical Quantity e.g. Normalized Cut (NC)
  - Human hand segmentation

We do well with respect to both measures
Normalized Cut

- Definition:

\[ nc(G) = \min_{V_1, \ldots, V_k} \frac{1}{k} \sum_{i=1}^{k} \frac{\text{cut}(V_i, V \setminus V_i)}{\text{vol}(V_i)} \]

Where \( V_1 \cdots V_k \) is a Partition of \( V \).
Comparison with Human Segmentation

Input Image | Human | EIG | SR
--- | --- | --- | ---
![Image](image1.png) | ![Image](image2.png) | ![Image](image3.png) | ![Image](image4.png)

$k = 4$

![Image](image5.png) | ![Image](image6.png) | ![Image](image7.png) | ![Image](image8.png)

$k = 5$

![Image](image9.png) | ![Image](image10.png) | ![Image](image11.png) | ![Image](image12.png)

$k = 11$

![Image](image13.png) | ![Image](image14.png) | ![Image](image15.png) | ![Image](image16.png)

$nc = .0051$

$nc = .0017$

$nc = .0076$

$nc = .0060$

$nc = .0068$

$nc = .0033$
Comparison with Human Segmentation

Figure 5.7: Example segmentations from the Berkely Hand Segmentation Database. Image results comparing the $k$-way cut generated from hand segmentation (column 2), the standard spectral algorithm (column 3), and spectral rounding with expansion edges and the derivative heuristic (column 4). For each image, the number of segments was fixed for both the spectral rounding algorithm SR and the standard algorithm Eig. Each method was initialized with the same weight matrix, and the reported cut costs are given on the original weighted graph (i.e., affinity matrix).
Results: Medical Images

MRI data of left ventricle

nc(SR)=0.019  nc(EIG)=0.061
nc(SR)=0.024  nc(EIG)=0.057
nc(SR)=0.048  nc(EIG)=0.068
nc(SR)=0.021  nc(EIG)=0.021

35
Medical Segmentation

retinal volume processing

assisted tumor extraction
Spectral OCT
Spectral Rounding: Global vs. Local

Threshold - common in MIP
Spectral Rounding: **Global** vs. **Local**
S.R. in action...
Segmentation Results
Segmentation Results
Fly Through
NFL Extraction:
Detection of the Nerve Fiber Layer Contour
NFL Extraction:
Intensity proportional to probability-of-a-cut under the eigen-space
Another view of SR
Mammogram Segmentations
Numerical Algorithms

- Solving Laplacian $Lx = b$
- Finding eigenvectors $Lf = \lambda_2 Df$
- Spielman and Teng
  - $O(n \log^k n)$ time for some $k$. 
Iterative Solvers $Ax = b$

- Richardson: $x^{(i+1)} \leftarrow (I - A)x^{(i)} + b$
- Preconditioned: $B^{-1}Ax = B^{-1}b = b'$
  
  $$x^{(i+1)} \leftarrow (I - B^{-1}A)x^{(i)} + b'$$
- Computing $z = B^{-1}Ax^{(i)}$
  - $y \leftarrow Ax^{(i)}$
  - solve $Bz = y$
Combinatorial Preconditioners

- Recall: A graph G, B graph H
- Vaidya: Max Weight Spanning Tree.
- EEST: Low Stretch Spanning Tree.
- Gremban-M: Steiner Tree

All these generate one for all of G
Planar Solvers

1. The speed of planar solvers has been dramatically improving over the last 50 years.
2. We have an optimal sequential time algorithm.
3. It also can be used on in parallel.
Dealing with larger images

1950-1980: Handling benchmark images

1950 - $n^2$ algorithm

1980 - $n^{1.5}$ algorithm

Image sizes up to 100K pixels
Dealing with larger images

1980-1992: Handling digital cam pictures

1980 - $n^{1.5}$ algorithm

1992 - $n^{1.2}$ algorithm

Images up to 1 mega-pixel
Dealing with larger images

1992-2006: Handling large medical images

1992 - $n^{1.2}$ algorithm

This is us: $n$ algorithm

Images up to 1 giga-pixel
Dealing with larger images

The parallel era: as fast as it gets

Giga-pixel images and beyond...
Our Preconditioner

• Partition G into small pieces with small boundary.

• Use one of the known preconditioners for each piece.
Our Partitioner

Planar $G = (V,E)$

- Partition $P_1, ..., P_m$ of $E$
- $|P_i| \leq k$
- sum over boundaries $\leq O(n/ \sqrt{k})$
- Work: $O(n)$
- Time: $O(k \log n)$
Dealing with larger images

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Image sizes up to 100K pixels
Dealing with larger images

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Time

Size

x 10^7

x 10^5
Dealing with larger images

1992-2006: Handling large medical images

1992 - $n^{1.2}$ algorithm

This is us: $n$ algorithm

Images up to 1 giga-pixel
Dealing with larger images

The parallel era: as fast as it gets

Giga-pixel images and beyond...
Image Segmentation in Surveillance
Thanks

Any image software can be improved by adding good image segmentation code.
Input Data

EigSR

nc = 0.0093
nc = 0.0048

nc = 0.0085
nc = 0.0080

nc = 0.0081
nc = 0.0055

nc = 0.0047
nc = 0.0006
Major Types of Image Segmentation

- Assisted Segmentation
  - Input from the consumer
  - Prior Knowledge (e.g. model of the heart)
- Unassisted Segmentation
  - No Prior Knowledge – No User Input
This Talk addresses a harder problem!

- Unassisted Segmentation without prior knowledge of the scene (image contents)

- Our methods can be used with prior information as well.
Image Segmentation

**Probabilistic:**
- Besag ’74
- Geman & Geman ’84
- Vesklner, Zabih, Boykov ’98
- ..., Zhu, ... ’01+
- Tu et al. ’05

**Statistical:**
- Diday & Simon ’80
- Comaniciu & Meer ’99+

**Graph Partitioning:**
- Vesklner, Zabih, Boykov ’97+
- Freeman & Perona ’97
- Shi & Malik ’98+
- Yu & Shi ’03
- Sharon et al. ’00

**Variational/Contour:**
- Kass, Witkin, Terzopoulos ’88
- Mumford & Shah ’89+
- Sethian ’96+
- Zhu, Lee, Yuille ’95

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