

# Approximating Center Points with Iterated Radon Points

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## Abstract

We describe a practical and provably good algorithm for approximating center points in any number of dimensions. Here  $c$  is a center point of a point set  $P$  in  $\mathbb{R}^d$  if every closed halfspace containing  $c$  contains at least  $|P|/(d+1)$  points of  $P$ . Our algorithm has a small constant factor and is the first approximate center point algorithm whose complexity is subexponential in  $d$ . Moreover, it can be optimally parallelized to require  $O(\log^2 d \log \log n)$  time. Our algorithm has been used in mesh partitioning methods, and has the potential to improve results in practice for constructing weak  $\epsilon$ -nets and other geometric algorithms. We derive a variant of our algorithm with a time bound fully polynomial in  $d$ , and show how to combine our approach with previous techniques to compute high quality center points more quickly.

## 1 Introduction

A *center point* of a set  $P$  of  $n$  points in  $\mathbb{R}^d$  is a point  $c$  of  $\mathbb{R}^d$  such that every hyperplane passing through  $c$  partitions  $P$  into two subsets each of size at most  $nd/(d+1)$  [8, 24]. This balanced separation property makes the center useful for efficient divide-and-conquer algorithms in geometric computing [1, 17, 19, 21, 24] and large-scale scientific computing [13, 17, 19, 21]. Note that we are not referring here to the center of mass, or centroid. For brevity hereafter we'll call a center point just a *center*.

The existence of a center of any point set follows from a classical theorem due to Helly [6]. However, finding an exact center seems to be a difficult task. It is possible to compute centers by solving a set of  $\Theta(n^d)$  linear inequalities, using linear programming. The only improved results are that a center in two dimensions can be computed in  $O(n \log^5 n)$  time, and in three dimensions in  $O(n^2 \log^7 n)$  time [5]; the two-dimensional result has been very recently improved to linear time [12].

For most applications, it suffices to have an *approximate* center, one for which every hyperplane through  $x$  partitions  $P$  into subsets of size at least  $n(\frac{1}{d+1} - \epsilon)$ . Such a center may be found with high probability by taking a random sample of  $P$  and computing an exact center of that sample. In order to achieve probability  $1 - \delta$  of computing a correct approximate center, a sample of size  $\Theta(d/\epsilon^2 \log d/\epsilon + \log 1/\delta)$  is required [23, 11, 17, 21], and the time to compute the center is  $O(d^2(d/\epsilon^2 \log d/\epsilon + \log 1/\delta)^d)$  [3]. This bound is constant in that it does not depend on  $n$ , but it has a constant factor exponential in  $d$ . Alternatively, a deterministic linear-time sampling algorithm can be used in place of random sampling [14, 21], but one must again compute a center of the sample using linear programming in time exponential in  $d$ .

This exponential dependence on  $d$  is a problem even when  $d$  is as small as three or four. For example, the experimental results show that the sampling algorithm must choose a sample of about five hundred to eight hundred points in three dimensions. The sampling algorithm thus solves a system of  $\binom{500}{3} \approx 20$  million linear inequalities. Many of our applications, e.g., in mesh partitioning [18], require an approximate center in four

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or more dimensions, for which the number of sample points and inequalities required is even larger. The seemingly efficient sampling algorithm is too expensive for practical applications.

In this paper, we give a practical and provably good method for constructing centers, from which we derive several center approximation algorithms. A version of this method was originally proposed as a heuristic to replace the linear programming based sampling algorithm by Miller and Teng [16] and has been implemented as a subroutine in their geometric mesh partitioning algorithm [10, 18]. The experimental results are encouraging. Our algorithm can also be used as part of a method for quickly computing weak  $\epsilon$ -nets with respect to convex sets [1] and in other geometric applications of centers.

The simplest form of our algorithm runs in  $O((d \log 1/\delta)^{\log d})$  time and uses randomization to find an  $\Omega(1/d^2)$ -center with probability  $1 - \delta$ . It does not use linear programming and has a small constant factor, making it suitable for practical applications. It can be efficiently parallelized in  $O(\log^2 d \log \log n)$  time on distributed- and shared-memory parallel machines. To the best of our knowledge, it is the first approximate center algorithm whose complexity is subexponential in  $d$ . We next describe a slightly more complicated form of the algorithm which takes time polynomial in both  $d$  and  $\log 1/\delta$  and again computes  $\Omega(1/d^2)$ -centers. Finally, we show how to combine our algorithm with the linear programming sampling method to compute  $(\frac{1}{d+1} - \epsilon)$ -centers with probability  $1 - \delta$ , in time  $(d/\epsilon)^{O(d)} \log 1/\delta$ .

## 1.1 Outline

The following section reviews some fundamental geometrical facts, and introduces the *Radon point* of a set of points. Section 3 gives our basic algorithm, based on iterated computation of Radon points. This algorithm is analyzed first in one dimension, in §4, then in general in §5. Section 6 discusses the use of random sampling to eliminate dependence on the number of input points. Section 7 gives our polynomial-time variant. Finally, section 8 shows how to combine our approach with the linear programming algorithm.

## 2 Centers and Their Relatives

Let  $P$  be a finite set of points in  $\mathbb{R}^d$ . A hyperplane  $h$  in  $\mathbb{R}^d$  divides  $P$  into three subsets:  $P^+ = h^+ \cap P$ ,  $P^- = h^- \cap P$ , and  $P \cap h$ . The *splitting ratio* of  $h$  over

$P$ , denoted by  $\phi_h(P)$ , is defined as

$$\phi_h(P) = \max \left( \frac{|P^+|}{|P|}, \frac{|P^-|}{|P|} \right)$$

For each  $0 < \beta \leq 1/2$ , a point  $c \in \mathbb{R}^d$  is a  $\beta$ -center of  $P$  if every hyperplane containing  $c$   $(1 - \beta)$ -splits  $P$ . A  $(\frac{1}{d+1})$ -center is simply called a *center*.

**Proposition 2.1 (Centers)** *Each point set  $P \subset \mathbb{R}^d$  has a center.*

This fact, first observed by Danzer *et al.*[6], is a corollary of Helly's Theorem. (See also [18, 21]; Edelsbrunner's text[8] gives proofs for the results in this section.)

**Theorem 2.2 (Helly)** *Suppose  $\mathcal{K}$  is a family of at least  $d + 1$  convex sets in  $\mathbb{R}^d$ , and  $\mathcal{K}$  is finite or each member of  $\mathcal{K}$  is compact. Then if each  $d + 1$  members of  $\mathcal{K}$  have a common point, there is a point common to all members of  $\mathcal{K}$ .*

The proof that centers exist is roughly as follows: consider a set of  $d + 1$  halfspaces each containing fewer than  $\beta n$  points of  $P$ , for  $\beta \leq 1/(d + 1)$ . There must be points of  $P$  not in any of these halfspaces; hence any set of  $d + 1$  halfspaces containing more than  $n(1 - \beta)$  points of  $P$  must have a common point. It follows from Helly's Theorem that the family of halfspaces each containing more than  $n(1 - \beta)$  points of  $P$  has nonempty intersection. Any point in that intersection is a  $\beta$ -center.

This proof sketch implies that  $\beta$ -centers form a convex region, the intersection of a family of halfspaces; it is not too hard to show that this region is the intersection of a *finite* family of halfspaces.

**Theorem 2.3 ([6])** *If  $P \subset \mathbb{R}^d$  has  $n$  points, then its set of  $\beta$ -centers is the intersection of a family of closed halfspaces whose members each contain at least  $n(1 - \beta)$  points of  $P$ , and have  $d$  points of  $P$  in their bounding hyperplanes.*

The *linear programming algorithm*, based on this result, seeks to find a point in the common intersection of a family of no more than  $\binom{n}{d} < n^d$  halfspaces. This problem is linear programming in  $d$  dimensions, with less than  $n^d$  inequality constraints; it can be solved in  $O(n^d)$  time using a deterministic algorithm[15, 4] or a randomized one[3]; the latter algorithm gives a much smaller constant factor.

Another consequence of this theorem, discussed in §5, is that a candidate center need only be verified with respect to the orderings on the points induced by the

normals to  $n^d$  hyperplanes; if each such ordering is satisfied with probability  $p < 1/n^d$ , then the candidate is a center with non-zero probability.

Helly's Theorem can be proven using another result important for this paper, Radon's Theorem.

**Theorem 2.4 (Radon)** *If  $P \subset \mathbb{R}^d$  with  $|P| \geq d + 2$ , then there is a partition  $(P_1, P_2)$  of  $P$  such that the convex hull of  $P_1$  has a point common with the convex hull of  $P_2$ .*

**Proof:** Suppose  $P = \{p_1, \dots, p_n\}$  with  $n \geq d + 2$ . Consider the system of  $d + 1$  homogeneous linear equations

$$\sum_{i=1}^n \alpha_i = 0 = \sum_{i=1}^n \alpha_i p_i^j \quad (i \leq j \leq d),$$

where  $p_i = (p_i^1, \dots, p_i^d)$  in the usual coordinates of  $\mathbb{R}^d$ . Since  $n \geq d + 2$ , the system has a nontrivial solution  $(\alpha_1, \dots, \alpha_n)$ . Let  $U$  be the set of all  $i$  for which  $\alpha_i \geq 0$ , and  $V$  the set for which  $\alpha_i < 0$ , and  $c = \sum_{i \in U} \alpha_i > 0$ . Then  $\sum_{i \in V} \alpha_i = -c$  and  $\sum_{i \in U} (\alpha_i/c) p_i = \sum_{i \in V} (\alpha_i/c) p_i$ .

Therefore, there is a partition  $(P_1, P_2)$  of  $P$  such that the convex hull of  $P_1$  has a point common with the convex hull of  $P_2$ .  $\square$

Call the partition  $(P_1, P_2)$  of the theorem a *Radon partition*. We'll call the point common to the hulls of  $P_1$  and  $P_2$  a *Radon point* of  $P$ . These points are the basis of our algorithm.

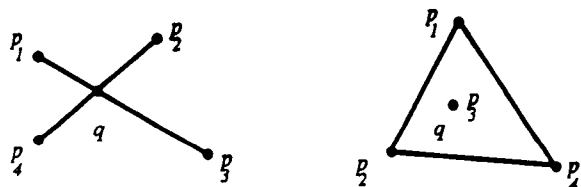


Figure 1: The Radon point of four points in  $\mathbb{R}^2$ . When no point is in the convex hull of the other three (the left figure), then the Radon point is the unique cross of two linear segments. Otherwise (the right figure), the point that is in the convex hull of the other three is a Radon point.

**Definition 2.5 (Radon points)** *Let  $P$  be a set of points in  $\mathbb{R}^d$ . A point  $q \in \mathbb{R}^d$  is a Radon point [6] if  $P$  can be partitioned into 2 disjoint subsets  $P_1$  and  $P_2$  such that  $q$  is a common point of the convex hull of  $P_1$  and the convex hull of  $P_2$ .*

Radon's Theorem implies that any set  $P$  of more than  $d + 1$  points has a Radon point. To compute a Radon

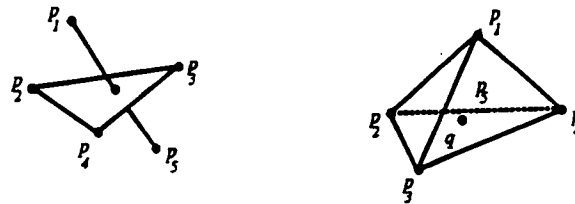


Figure 2: The Radon point of five points in  $\mathbb{R}^2$ . Two cases are similar to these of two dimensions.

point, we need only to compute a Radon point for any  $d + 2$  points of  $P$ . As in the proof above, this requires only the solution of a linear system, and so takes  $O(d^3)$  time.

Why are Radon points useful in computing centers? A Radon point of a set of  $d + 2$  points is a  $2/(d + 2)$ -center of that set: any closed halfspace containing a Radon point  $r$  must contain a point of  $P_1$  and a point of  $P_2$ . Hence the splitting ratio of a hyperplane containing  $r$  is at most  $d$ .

### 3 The Basic Algorithm

We now describe our algorithm for approximate centers. The algorithm iteratively reduces the point set by replacing groups of  $(d + 2)$  points by their Radon points. Such a reduction is guided by a complete  $(d + 2)$ -way tree. We will show that the final point of this reduction process is an approximate center with high probability.

**Algorithm 1 (Iterated Radon Points):**

**Input:** a set of points  $P \subset \mathbb{R}^d$

1. Construct a complete balanced  $(d + 2)$ -way tree  $T$  of  $L$  leaves (for an integer  $L$ ).
2. For each leaf of  $T$ , choose a point from  $P$  uniformly at random, independent of other leaves.
3. Evaluate tree  $T$  in a bottom-up fashion to assign a point in  $\mathbb{R}^d$  to each internal node of  $T$  such that the point of each internal node is a Radon point of the points with its  $(d + 2)$  children.
4. Output the point associated with the root of  $T$ .

A complete  $(d + 2)$ -way tree of  $L$  leaves has at most  $L/(d + 2)$  internal nodes. The above algorithm takes  $O(d^2 L)$  time, with a small constant factor. Clearly, our algorithm can be implemented in  $O(\log^2 d \log L)$  time using  $O(d^2 L / (\log^2 d \log L))$  processors in parallel. Our experimental results suggest that, independent of the

size of original point set,  $L = 800$  is sufficient for three dimensions and  $L = 1000$  for four dimensions.

## 4 Analysis: One Dimension

We wish to show that Algorithm 1 above finds a  $(1/d^2)$ -center with small probability of error. We first give a proof for one dimension and then extend it to higher dimensions.

In one dimension, the center of a point set is essentially the median. If the point set has an odd number of points, then its median is the only center. Otherwise, every point in the closed interval between the two medians is a center. Algorithm 1, when restricted to one dimension, gives the following algorithm for approximating the median of a linearly ordered set.

**Algorithm 2: (Fast Approximate Median)**

**Input:** a set of real numbers  $P = \{p_1, \dots, p_n\}$

1. Construct a complete balanced 3-way tree  $T$  of  $L$  leaves (for an integer  $L$ ).
2. For each leaf of  $T$ , choose an element from  $P$  uniformly at random, independent of other leaves.
3. Evaluate tree  $T$  in a bottom-up fashion: at each internal node, keep the median of the numbers of its three children.
4. Output the number associated with the root of  $T$ .

The *rank* of a number  $p_i$  is the position that  $p_i$  would take in the sorted list of values in  $P$ . By induction, it can be shown that number associated with each node of the tree  $T$  belongs to  $P$ . The operation of internal nodes is **comparison based**, **only the relative ranks** (not the values) matter. Without loss of generality we assume that  $P$  is a permutation of the set  $\{1/n, 2/n, \dots, 1\}$ .

We first note that the expected rank of the output of Algorithm 2 is  $n/2$ . This is because the output of Algorithm 2 is always from  $P$ , and because the operation in each internal node of the tree is symmetric with respect to the ranks. We now show that Algorithm 2 finds an approximate median with high probability.

Let  $f_h(x)$  be the probability that the output of Algorithm 2, when using a tree  $T$  of height  $h$ , is no larger than  $x$ . Because the number of a leaf of the tree is chosen uniformly at random from the set  $P$ ,  $f_0(x) \leq x$ . We now express  $f_h(x)$  in terms of  $f_{h-1}(x)$ .

Let  $r$  be the root of  $T$  and  $c_1, c_2$ , and  $c_3$  be its three children. Let  $I(v)$  be the number chosen by the node  $v$  in  $T$ . We have that  $I(r)$  is the median of  $I(c_1), I(c_2)$ ,

and  $I(c_3)$ . Thus,  $I(r) \leq x$  iff at least two of  $I(c_1), I(c_2)$ , and  $I(c_3)$  are less than  $x$ . Notice that each value  $I(c_i)$  ( $i \in \{1, 2, 3\}$ ) is chosen as the output of Algorithm 2 on a tree of height  $h - 1$ , and that each value  $I(c_i)$  is independent of the other two such values. Hence,

$$\begin{aligned} f_h(x) &= \binom{3}{2} (f_{h-1}(x))^2 (1 - f_{h-1}(x)) + (f_{h-1}(x))^3 \\ &= 3(f_{h-1}(x))^2 - 2(f_{h-1}(x))^3 \\ &\leq 3(f_{h-1}(x))^2. \end{aligned}$$

By induction, we have

$$\begin{aligned} f_h &\leq 3(f_{h-1}(x))^2 \\ &\leq 3 \cdot 3^2 \dots 3^{2^h} (f_0(x))^{2^{h+1}} \\ &= \frac{1}{3} (3x)^{2^{h+1}}. \end{aligned}$$

A number is a  $\beta$ -median of a set  $P = \{p_1, \dots, p_n\} \subset \mathbb{R}$  if both  $|\{p_i < q\}| \leq (1 - \beta)n$  and  $|\{p_i > q\}| \leq (1 - \beta)n$ .

**Theorem 4.1** For any  $\beta < 1/3$ , Algorithm 2 on a tree of height  $h$ , i.e., of sample size  $L = 3^h$ , outputs a  $\beta$ -median with probability of error at most  $(3\beta)^{2^{h+1}}$ .

For example, when  $\beta = 1/4$ , we have the following corollary.

**Corollary 4.2** Algorithm 2 finds a  $1/4$ -median in random  $O((\log 1/\delta)^{\log_2 3})$  time with probability of error at most  $\delta$ .

A better analysis can be used to show that Algorithm 2 finds a  $(1/2 - \epsilon)$ -median with very high probability for all constant  $0 < \epsilon < 1/2$ . Michelangelo Grigni observed that a method of Valiant [22] on constructing short monotone formulae for the majority function is very close to our construction (Algorithm 2) of approximated median, and Valiant's analysis can be adapted to give a proof that Algorithm 2 finds a  $(1/2 - \epsilon)$ -median with very high probability for all constant  $0 < \epsilon < 1/2$ .

## 5 Analysis: Higher Dimensions

Theorem 4.1 can be extended to higher dimensions. We start with some structural properties of  $\beta$ -centers.

Let  $l$  be a line in  $\mathbb{R}^d$ . The *projection* of a point  $p \in \mathbb{R}^d$  onto  $l$  is a point  $q \in l$  such that the line passing through  $p$  and  $q$  is **perpendicular** to  $l$ . By assigning a direction to  $l$ , we can introduce a linear ordering among points

on  $l$ . For a point set  $P = \{p_1, \dots, p_n\}$ , let  $\text{rank}_l(p_i)$  be the rank of the projection of  $p_i$  among all projections of  $P$ . If two lines  $l$  and  $l'$  are parallel to each other and have the same direction (in vector sense), then for all  $i: 1 \leq i \leq n$ ,  $\text{rank}_l(p_i) = \text{rank}_{l'}(p_i)$ .

**Lemma 5.1** *Let  $P = \{p_1, \dots, p_n\}$  be the point set. Then a point  $c$  is a  $\beta$ -center of  $P$  if and only if for all lines  $l$ , the projection of  $c$  onto  $l$  is a  $\beta$ -median of the projections of  $P$  onto  $l$ .*

**Proof:** Suppose  $c$  is a  $\beta$ -center of  $P$ . Let  $H$  be the hyperplane passing through  $c$  normal to  $l$ . Clearly, the projection  $c'$  of  $c$  onto  $l$  is the intersection of  $H$  and  $l$ . Notice that the projections of two points is on the same side of  $c'$  (on line  $l$ ) iff they belongs to the same halfspace defined by  $H$ . Therefore  $c'$  is a  $\beta$ -median of the projections of  $P$ . The other direction of the lemma can be proved similarly.  $\square$

In order to check whether a point  $c$  is a  $\beta$ -center of  $P$ , we need only check the splitting ratio of the  $O(n^{d-1})$  combinatorially distinct hyperplanes through  $c$ . Equivalently by Lemma 5.1, it is sufficient to check  $O(n^{d-1})$  lines (normal to the set of hyperplanes above) to see whether the projection of  $c$  is a  $\beta$ -median of the projections of  $P$ .

If  $c$  is unknown, then Theorem 2.3 implies that  $O(n^d)$  possible hyperplanes or normal lines need to be checked.

**Corollary 5.2** *For each point set  $P$  in  $\mathbb{R}^d$ , there is a set of  $O(n^d)$  lines such that a point  $c$  is a  $\beta$ -center of  $P$  iff for each line  $l$  from this line set, the projection of  $c$  is a  $\beta$ -median of the projections of  $P$  onto  $l$ .*

We now study the projection of Algorithm 1 onto a given line. Suppose we have  $d+2$  points  $p_1, \dots, p_{d+2}$ . Let  $r$  be the Radon point of  $p_1, \dots, p_{d+2}$  and  $(P_1, P_2)$  be a corresponding Radon partition. For each hyperplane  $H$  passing through  $r$ , each (open) halfspace of  $H$  contains at most  $d$  points from  $\{p_1, \dots, p_{d+2}\}$ , because  $r$  belongs to the convex hull of both  $P_1$  and  $P_2$ . Let  $l$  be the line passing through the origin that is normal to  $H$ . From the discussion above, the projection of  $r$  is between (inclusively) the second smallest and the second largest projections of  $P$  onto  $l$ . In our analysis we will forget the higher-dimensional constraints on the problem and assume that any point between the second-smallest and second-largest projections could be chosen in a worst-case pattern of such choices. Therefore, with respect to a given line, Algorithm 1 can be emulated by the following process in one dimension.

**Algorithm 3:** (Projection of Algorithm 1)

**Input:** a set of real numbers  $Q = \{q_1, \dots, q_n\}$

1. Construct a complete balanced  $(d+2)$ -way tree  $T$  of  $L$  leaves (for an integer  $L$ ).
2. For each leaf of  $T$ , choose an element from  $P$  uniformly at random, independent of other leaves.
3. Evaluate tree  $T$  in a bottom-up fashion: at each internal node, choose a number arbitrarily between the second smallest and the second largest numbers of its  $d+2$  children.
4. Output the number associated with the root of  $T$ .

We would like to prove the following lemma (which is parallel to Theorem 4.1).

**Lemma 5.3** *Let  $\beta_d = 1/\binom{d+2}{2}$ . For any  $\beta < \beta_d$ , Algorithm 3 above on a tree of height  $h$ , (i.e.,  $L = (d+2)^h$ ), outputs a  $\beta$ -median with probability of error at most  $(\beta/\beta_d)^{2^h}$ .*

**Proof:** Because we only concern the relative ranks of the input set  $Q$ , without loss of generality, we assume that  $Q$  is a permutation of the set  $\{1/n, 2/n, \dots, 1\}$ .

Let  $f_h(x)$  be the probability that the output of Algorithm 3, when using a tree  $T$  of height  $h$ , is no larger than  $x$ . Because the number of a leaf of the tree is chosen uniformly at random from the set  $Q$ ,  $f_0(x) \leq x$ . We now express  $f_h(x)$  in terms of  $f_{h-1}(x)$ .

The inputs to the root  $r$  of a tree of height  $h$  are from the outputs of  $d+2$  trees of height  $h-1$ . Let  $I(r)$  be the number chosen by the root. We have  $I(r) \leq x$  only if at least two of its  $d+2$  inputs are less than  $x$ . Therefore,

$$f_h(x) = 1 - (1 - f_{h-1}(x))^{d+2} - (d+2)f_{h-1}(x)(1 - f_{h-1}(x))^{d+1}.$$

We can write the precise inclusion and exclusion form of the left hand side of the equation above. But the following upper bound is good enough for our analysis. There are in total  $\binom{d+2}{2}$  different pairs from the  $(d+2)$  distinct inputs. We call a pair  $(a, b)$  good if both  $a \leq x$  and  $b \leq x$ . The probability that a pair is good is equal to  $(f_{h-1}(x))^2$ . Therefore,  $f_h(x)$ , the probability that there exists at least such a pair is bounded above by  $\binom{d+2}{2}(f_{h-1}(x))^2$ .

We have

$$\begin{aligned} f_h(x) &\leq \binom{d+2}{2} (f_{h-1}(x))^2 \\ &\leq \binom{d+2}{2} \cdot \binom{d+2}{2}^2 \dots \binom{d+2}{2}^{2^h} f_0(x)^{2^{h+1}} \end{aligned}$$

$$= \beta_d(x/\beta_d)^{2^{h+1}}.$$

□

Therefore, for any  $\beta < \beta_d$ , with very high probability (the error probability is doubly exponentially small with respect to the height of the tree), the projection of the output of Algorithm onto a line is a  $\beta$ -median of the projections of the input point set  $P$  onto the line. By Corollary 5.2, the probability that the output of Algorithm 1 is not a  $\beta$ -center of  $P$  is at most  $n^d(\beta/\beta_d)^{2^h}$ .

**Theorem 5.4** *Algorithm 1 finds an  $\Omega(1/d^2)$ -center in  $O((d \log n)^{\log_2 d})$  time, with probability of error at most  $1/n$ .*

**Proof:** Suppose Algorithm 1 uses a tree of  $L$  leaves. From the analysis above, there is a constant  $c$  such that Algorithm 1 finds an  $O(1/d^2)$ -center in  $O(d^2 L)$  time with probability of error at most  $n^d c^{-2^h}$ . So,  $L = O((d \log n)^{\log_2 d})$  is sufficiently large to ensure that this probability of error is at most  $\delta$ . □

## 6 Random Sampling

Both the linear programming algorithm and Algorithm 1 have running times dependent on  $n$ , the number of input points. It is possible to eliminate this dependence by computing the center of a random subset of  $P$ ; applied to the LP algorithm, the resulting *sampling algorithm* is only Monte Carlo, yielding an approximate center only with high probability. However, the reduction in running time is quite substantial. The basis of the sampling approach is the following theorem, a corollary of results on  $\epsilon$ -approximations.[23]

**Theorem 6.1** *If  $S \subset P$  of size  $s$  is chosen uniformly at random, with all subsets of size  $s$  equally likely, then a center of  $S$  is a  $(\frac{1}{d+1} - \epsilon)$ -center of  $P$  with probability at least  $1 - \delta$ , if*

$$s = O\left(\frac{d}{\epsilon^2} \log \frac{d}{\epsilon} + \log \frac{1}{\delta}\right)$$

but sufficiently large.

To reduce the probability of error below  $1/n$ , the sampling algorithm requires  $s = \Omega(\frac{d}{\epsilon^2} \log \frac{d}{\epsilon} + \log n)$  and takes  $O(d^2 s^d)$  time.

To apply sampling to our algorithm, we need to generalize this theorem somewhat to show that with high probability any approximate center of the random sample is an approximate center of the original point set.

**Lemma 6.2** *Let  $m = O(d/\epsilon^2 \log d/\epsilon + \log 1/\delta)$  points be chosen randomly from our input set. Then with probability  $\delta$ , any  $\beta$ -center of the random sample will be a  $(\beta - \epsilon)$ -center for the original point set.*

**Proof:** This fact can be proved in a very similar manner to the original sampling theorem. With probability  $\delta$ , any randomly chosen set of  $m = O(d/\epsilon^2 \log d/\epsilon + \log 1/\delta)$  points will be an  $\epsilon$ -sample, having the following property: any halfspace containing  $cm$  points of the input set will also contain between  $(c - \epsilon)m$  and  $(c + \epsilon)m$  points of the sample. Suppose halfspace  $H$  contains fewer than  $(\beta - \epsilon)n$  points of our input. Then if our sample is an  $\epsilon$ -sample,  $H$  must contain fewer than  $\beta m$  points of the sample, and hence cannot contain any  $\beta$ -center of the sample. So with probability  $1 - \delta$  any  $\beta$ -center of the sample is also a  $(\beta - \epsilon)$ -center of the input set. □

We can now modify Algorithm 1 somewhat to avoid any dependence on  $n$  in the time bound. If we wish a failure probability at most  $\delta$ , we simply choose a random sample of the input consisting of  $O(d^2 \log d + \log 1/\delta)$  points, so that with probability at least  $1 - \delta/2$  any  $O(1/d^2)$ -center of the sample is also an  $O(1/d^2)$ -center of the input, and then apply Algorithm 1 with a tree size chosen so that the failure probability is at most  $\delta/2$ .

**Theorem 6.3** *For any  $\delta$ , the above modification to Algorithm 1 finds an  $O(1/d^2)$ -center in random  $O((d \log d + \log 1/\delta)^{\log_2 d})$  time, with probability of error at most  $\delta$ .*

## 7 A Polynomial Algorithm

Algorithm 1 is subexponentially dependent on  $d$ , but not polynomial. Moreover, the dependence on  $\delta$  involves an exponent of  $\log d$ , showing that increasing the reliability of the algorithm is costly in time. The problem is in the tree-like structure of the algorithm and in the branching factor of  $d + 2$  in that tree. If  $n$  is small (e.g. after the sampling modification of the previous section is applied) the number of leaves in the tree may end up being much larger than  $n$  itself. We now give the polynomial-time Algorithm 4 without this excess. The structure of Algorithm 4 is a layered DAG rather than a tree, with greater height, but with much smaller width.

Algorithm 4 applies the following scheme  $z = \Theta(d + \log \log n)$  times to a set  $T$ , which is  $P$  initially: independently choose  $n$  random subsets of  $T$  each of size  $d + 2$ ; replace  $T$  by the Radon points of these subsets. After this loop, choose any point of the final  $T$  as a center.

With sufficiently large  $n$  ( $n = \Omega(d^4 \log^2 d)$ ), this algorithm returns a  $\beta_d/3e$ -center with probability  $1 - 1/n$ ; it takes  $O(n(d^4 + \log \log n))$  time. (Recall that  $\beta_d \equiv 1/\binom{d+2}{2}$ .)

Our analysis begins with a tail estimate for binomial distributions.

**Lemma 7.1** ([2], I.1) *Let  $x$  be a binomial random variate with  $n$  trials and success probability  $p$ ; for  $u > 1$ , let  $m = \lceil upn \rceil$ , with  $1 \leq m < n$ . Then  $\text{Prob}\{x \geq upn\}$  is less than*

$$u^{-upn} \frac{1}{\sqrt{2\pi}} \frac{u}{u-1} \left( \frac{n}{m(n-m)} \right)^{1/2} \left( \frac{1-p}{1-up} \right)^{(1-up)n}$$

**Proof:** The proof, omitted here, uses the inequality form of Stirling's approximation.  $\square$

**Theorem 7.2** *After  $z = \Theta(d + \log \log n + \log_n 1/\delta)$  iterations, Algorithm 4 returns an  $O(1/d^2)$ -center with probability at least  $1 - \delta$ , for any  $n = \Omega(d^{4+\epsilon} + d^{2+\epsilon} \log 1/\delta)$ . Algorithm 4 requires  $O(nd^3(d + \log \log n + \log_n 1/\delta))$  time.*

**Proof:** As in Theorem 5.4, it's enough to analyze a similar algorithm, where  $P$  is the set of values  $\{1/n, 2/n, \dots, 1\}$ , and a subset of size  $d+2$  yields its second-smallest element for the new version of  $T$ . (In fact, for the algorithm here,  $T$  is a multiset: two subsets may yield the same value, which will then have multiplicity greater than one.) We will find an upper bound for the probability that the final  $T$  has any members less than  $k/n$ , for  $k = \beta_d n/3e$ .

Let  $T_i$  denote  $T$  after iteration  $i$ , let  $U_i \equiv T_i \cap \{1/n \dots k/n\}$ , and let  $t_i \equiv |U_i|$ . Here  $T_0 = S$  and  $t_0 = k$ . We want to bound the probability that  $t_z > 0$ .

The key observation is that for a given value of  $t_{i-1}$ , the probabilities that two random subsets yield numbers in  $1/n \dots k/n$  are independent; thus  $t_i$  given  $t_{i-1}$  is bounded by a binomial random variable with  $n$  independent trials, each with success probability bounded above by  $p_{i-1} \equiv (t_{i-1}/n)^2 \binom{d+2}{2}$ . We can use Lemma 7.1 to bound the probability that  $t_i > \gamma t_{i-1}$ , for sufficiently large  $\gamma$ : let  $\alpha \equiv \binom{d+2}{2}/n$ , and put  $u = \gamma/\alpha t_{i-1}$ , so that if  $\gamma \geq 2\alpha t_{i-1}$ , then  $u \geq 2$ . With  $u \geq 2$ , we have  $u/(u-1) \leq 2$ , and with  $n > 2$ , we have  $n/m(n-m) < 2$  for  $1 \leq m < n$ . With these bounds, and using  $1+x \leq e^x$ ,

$$\begin{aligned} \text{Prob}\{t_i \geq \gamma t_{i-1}\} &= \text{Prob}\{t_i \geq up_{i-1}n\} \\ &< u^{-upn} \frac{1}{\sqrt{2\pi}} \frac{u}{u-1} \\ &\quad \left( \frac{n}{m(n-m)} \right)^{1/2} \end{aligned}$$

$$\begin{aligned} &\left( \frac{1-p_{i-1}}{1-up_{i-1}} \right)^{(1-up_{i-1})n} \\ &< (u/e)^{-up_{i-1}n} 2/\sqrt{\pi} \\ &= (e\alpha t_{i-1}/\gamma)^{\gamma t_{i-1}} 2/\sqrt{\pi}. \quad (1) \end{aligned}$$

Now put  $\gamma = 2e\alpha t_{i-1}$ . If  $\gamma t_{i-1} > \hat{d}$ , where  $\hat{d} \equiv 1 + (d+3) \lg n + \log 1/\delta$ , then the probability that  $t_i \geq \gamma t_{i-1} = 2e\alpha t_{i-1}^2$  is no more than  $\delta/n^{d+3}$ . Hence in  $z' = O(\log \log n)$  steps, we have  $t_{z'} \leq (2e\alpha k)^{2^{z'}-1} k \leq \hat{d}$ , with probability at least  $(1-\delta/n^{d+3})^{z'} \geq 1-2z'\delta/n^{d+3}$ . Moreover, with  $\gamma = \hat{d}/t_{i-1}$  and applying (1), it follows inductively that  $t_i \leq \hat{d}$  for  $i \geq z'$ , also with high probability at each step. The probability that any of these bounds fail for  $t_i$ , for  $1 \leq i \leq z$ , is at most  $2z\delta/n^{d+3}$ .

The probability that  $t_i = 0$  is  $(1-p_{i-1})^n$ , and so the probability that  $t_i > 0$  is  $1 - (1-p_{i-1})^n < 2np_{i-1}$ ; hence

$$\text{Prob}\{t_{z'+q} > 0\} \leq (2\alpha \hat{d}^2)^q,$$

which is less than  $\delta/n^{d+3}$  for  $n/\log^2 n = \Omega(d^{4+\epsilon} + d^{2+\epsilon} \log 1/\delta)$  and  $q = \Theta(d + \log_n 1/\delta)$ . Hence after  $z = O(d + \log_n 1/\delta)$  iterations, every point in  $T$  is a  $\beta_d/3e$ -median point for a given line with probability at least  $1 - 2d\delta/n^{d+3}$ . By Corollary 5.2, every such point is a  $\beta_d/3e$ -center with probability at least  $1 - \delta$ .

The time bound for Algorithm 4 follows from the bound on the number of iterations, and the  $O(d^3)$  work needed to find a Radon point.  $\square$

**Corollary 7.3** *Algorithm 4 together with random sampling can be used to compute an  $1/3e \binom{d+2}{2}$ -center with probability at least  $1 - \delta$ , in time  $O(d^9 \log d + d^8 \log 1/\delta + d^{5+\epsilon} \log^2 1/\delta)$ .*

## 8 High Quality Centerpoints

Our algorithms are very efficient, but only produce  $O(1/d^2)$ -centers. The linear programming algorithm can produce better centers, but much more slowly; in particular not only is there a constant factor that depends exponentially on  $d$ , but there is also a nonconstant term of the form  $O(\log 1/\delta)^d$ . We now show how to combine our algorithm with linear programming to eliminate this term.

Suppose we wish to compute a  $(\frac{1}{d+1} - \epsilon)$ -center, for some  $\epsilon < 1/(d+1)$ . We take a collection of  $k$  random samples, each of size  $O(d/\epsilon^2 \log d/\epsilon)$ . The linear programming algorithm gives us a center in each, which is a center of our original set with probability at least  $1 - \exp(O(-d^3 \log d))$ . If we choose  $k = \Theta(\log 1/\delta)$ , we can show using Lemma 7.1 that with probability  $1 - \delta/2$ , all but  $O(k/d^2)$  of the linear programming solutions are centers of the original set. We now find an



$O(1/d^2)$ -center of these  $k$  LP solutions with probability  $1 - \delta/2$ ; this must then also be an approximate center of our original set.

**Theorem 8.1** *In time  $O((d/\epsilon^2 \log d/\epsilon)^{d+O(1)} \log 1/\delta)$  we can find a  $(\frac{1}{d+1} - \epsilon)$ -center, with probability  $1 - \delta$ .*

**Acknowledgments** We would like to thank Mic Grigni pointing out that Valiant's method in [22] adapts to our approximate median algorithm, and Dan Spielman for helpful discussions.

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