Simplifying rational functions

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This document describes how to use synthetic division and partial fraction expansion to reduce a rational function to its canonical form. Synthetic division and partial fraction expansion are implemented in Matlab’s \texttt{residue} function, which is a good way to experiment with them.

1 Partial fractions

Suppose we have a rational function

\[
\frac{B(s)}{A(s)} = \frac{b_n s^n + b_{n-1} s^{n-1} + \ldots + b_0}{a_m s^m + a_{m-1} s^{m-1} + \ldots + a_0}
\]

We would like to represent it in a simpler form. It turns out that any rational function can be decomposed in the partial fraction expansion

\[
\frac{B(s)}{A(s)} = K(s) + \frac{C_1(s)}{P_1(s)} + \frac{C_2(s)}{P_2(s)} + \ldots
\]

Here \(K(s)\) is a polynomial, and \(C_i(s)\) and \(P_i(s)\) are “simple” polynomials. \(K(s)\) is called the direct term; it is necessary only if \(n < m\), and if it exists it has degree \((n-m)\).

The denominator polynomials \(P_i\) depend on the roots of \(A\), which are also called poles of the rational function. (Roots of \(B\) are called zeros of the rational function.) Each isolated root \(x_i\) of \(A\) results in a denominator polynomial of the form \(P_i(s) = s - x_i\); each complex conjugate pair of roots \(x_i \pm y_i i\) gives a denominator of the form \(P_i(s) = s^2 + 2x_i s + x_i^2 - y_i^2\). Multiple roots result in terms of higher degree; for example a real root \(x_i\) with multiplicity \(k\) gives a denominator \(P_i(s) = (s - x_i)^k\).

To determine the direct term we can use \textit{synthetic division} (see below). So for now let us assume \(n < m\). In this case we can use the Heaviside method (also called the cover-up method) to determine the coefficient polynomials \(C_i\).

The simplest case is an isolated root \(x_i\) of \(A(s)\). In this case, \(C_i\) is a constant, and we have

\[
(s - x_i) \frac{B(S)}{A(S)} = (s - x_i) \left( K(s) + \frac{C_1(s)}{P_1(s)} + \frac{C_2(s)}{P_2(s)} + \ldots \right)
\]
\[ \left( s - x_i \right) \frac{B(S)}{A(S)} \right|_{s=x_i} = C_i \]

where the second equation holds because every term on the right-hand side contains a factor \((s - x_i)\) except for the term \((s - x_i)C_i/(s - x_i)\). So, we can determine \(C_i\) by deleting one of the factors \((s - x_i)\) of \(A_i\) from our rational function, and evaluating the result at \(x_i\).

For example, suppose we have

\[
\frac{A(s)}{B(s)} = \frac{1}{(s^2 + 1)(s - 2)}
\]

We will then have a term in our expansion

\[
\frac{a}{s - 2}
\]

To determine \(a\), we evaluate \(1/(s^2 + 1)\) at \(s = 2\). This tells us that \(a = 1/5\), so our term is

\[
\frac{1/5}{s - 2}
\]

This way of determining coefficients gives the method its name: we “covered up” the factor \(1/(s - 2)\) of \(B(s)/A(s)\) and evaluated the remaining expression at \(s = 2\).

If our denominator has a repeated root or a complex conjugate pair of roots (or even a repeated conjugate pair), then we will have a factor \(P_i(s)\) in the denominator which has degree \(d > 1\). This factor will result in a term \(C_i(s)/P_i(s)\) in our expansion, where \(\deg(C_i) < d\). In this case we can determine the coefficients of \(C_i\) by evaluating \(P_i(s)B(s)/A(s)\) at the \(d\) points where \(P_i\) is zero; this will result in \(d\) equations in the \(d\) unknown coefficients.

For example, consider again the rational function

\[
\frac{A(s)}{B(s)} = \frac{1}{(s^2 + 1)(s - 2)}
\]

The factor \((s^2 + 1)\) leads to a term in our expansion

\[
\frac{as + b}{s^2 + 1}
\]

To determine \(a\) and \(b\), we evaluate \(1/(s - 2)\) at the two points at which \((s^2 + 1)\) is 0, namely \(\pm i\). This gets us two equations,

\[
a\mathbf{i} + b = \frac{1}{\mathbf{i} - 2} = -\frac{\mathbf{i} + 2}{5} \quad -a\mathbf{i} + b = \frac{1}{-\mathbf{i} - 2} = \frac{\mathbf{i} - 2}{5}
\]

Solving these equations gives \(a = -1/5\) and \(b = -2/5\); combining the new term with our previous result tells us that our final expansion is

\[
\frac{B(s)}{A(s)} = \frac{1/5}{s - 2} - \frac{s/5 + 2/5}{s^2 + 1}
\]
2 Synthetic division

We are given a rational function $B(s)/A(s)$ with numerator degree $n$ and denominator degree $m$. If $n \geq m$, we can pull out a quotient term $K(s)$, leaving a remainder term $R(s)$ with degree($R$) < $m$, so that

$$\frac{B(s)}{A(s)} = K(s) + \frac{R(s)}{A(s)}$$

The process is analogous to long division, and is called synthetic division. We will illustrate it by example: suppose we start with

$$\frac{B(s)}{A(s)} = \frac{s^3 - s^2 + s + 1}{s^2 - 4s + 3}$$

We are looking for $K(s)$ and $R(s)$, with degree($R$) < 2, so that

$$B(s) = K(s)A(s) + R(s)$$

To get the highest-order term of $B(s)$ (namely $s^3$) right, we can see that we have to multiply $A(s)$ by $s$. If we set $K_1(s) = s$, we have

$$R_1(s) = B(s) - K_1(s)A(s) = (s^3 - s^2 + s + 1) - (s^3 - 4s^2 + 3s) = 3s^2 - 2s + 1$$

This gets us partway to our goal: $R_1(s)$ has a smaller degree than $B(s)$ did, but not small enough. But, we can repeat the process: to get rid of the leading term of $R_1(s)$ (namely $3s^2$), we can multiply $A(s)$ by 3. Setting $K_2(s) = s + 3$, we have

$$R_2(s) = B(s) - K_2(s)A(s) = (s^3 - s^2 + s + 1) - (s^3 - 4s^2 + 3s) - (3s^2 - 12s + 9)$$

Cancelling terms gives $R_2(s) = 10s - 8$, which has sufficiently low degree, so we can take $R = R_2$ and $K = K_2$. 

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