

I5-780: Grad AI

Lec. 9: Linear programs, Duality

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Admin

- Have you tested your handin directories?
 - ▶ `/afs/cs/user/aothman/dropbox/USERID/`
 - ▶ where USERID is your Andrew ID

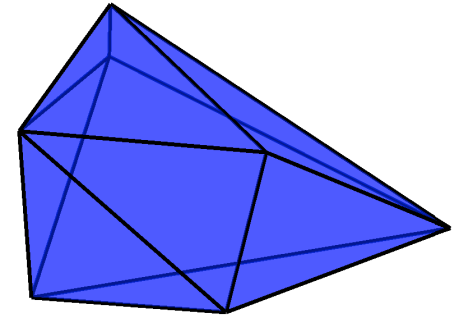
- Poster session:

- ▶ ~~???~~ May 9 1:30-4:30 7th floor atrium
GHC

Review

- LPs, ILPs, MILPs
 - ▶ \mathbb{R} or \mathbb{Z} variables
 - ▶ linear $\leq \geq =$
 - ▶ linear objective
 - ▶ LP relaxations, integrality gap
 - ▶ relation to SAT, MAXSAT, PBI
 - ▶ complexity (LP: P; ILP: NP & no approx)
 - ▶ (in)feasible, (sub)optimal, (in)active

Review



- Standard form: all vars ≥ 0 , all = constraints
- Nonsingular: n vars $\geq m$ constraints, rank m
- Basis
 - ▶ spans $\text{Rng}(A)$ ($m \times m$ invertible submatrix)
 - ▶ corresponds to “corner”
 - ▶ using row ops to make basic variables into “slacks” \rightarrow **tableau** notation
- Degeneracy: distinct bases yield same corner
- Naïve algorithm: check all bases

Finding corners

x	y	u	v	w	RHS
1	1	1	0	0	4
2	5	0	1	0	12
1	2	0	0	1	5

set $x, y = 0$

$u = 4 \quad v = 12 \quad w = 5$

1	1	1	0	0	4
2	5	0	1	0	12
1	2	0	0	1	5

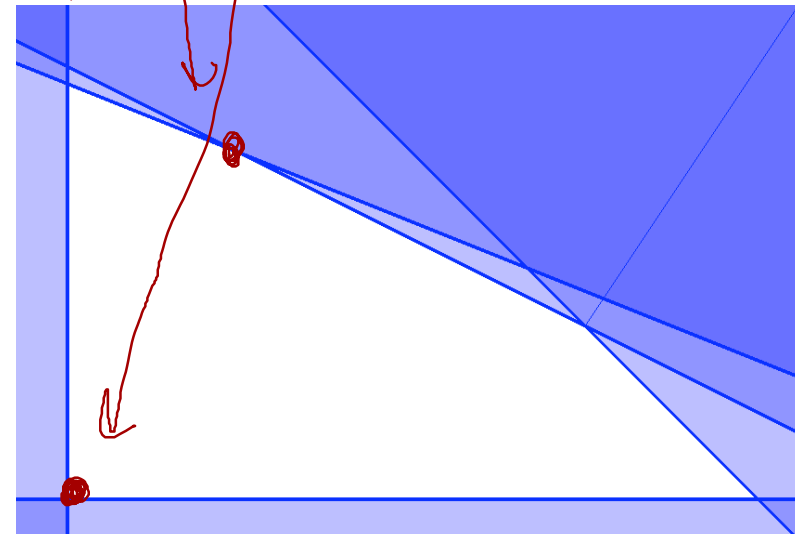
set $v, w = 0$

$x = 1 \quad y = 2 \quad u = 1$

1	1	1	0	0	4
2	5	0	1	0	12
1	2	0	0	1	5

set $x, u = 0$

$y = 4$
 $v = -8$
 $w = -3$



Simplex in one slide

(ignoring degeneracy, which is actually important)

- Given a nonsingular standard-form LP
 - ▶ make it nonsingular if needed
- Start from a feasible basis and its tableau
 - ▶ big-M if needed
- Pick non-basic variable w/ objective > 0 (max)
- Pivot it into basis, getting neighboring basis
 - ▶ select exiting variable to keep feasibility
- Repeat until all non-basic variables have objective < 0 (max)

Example

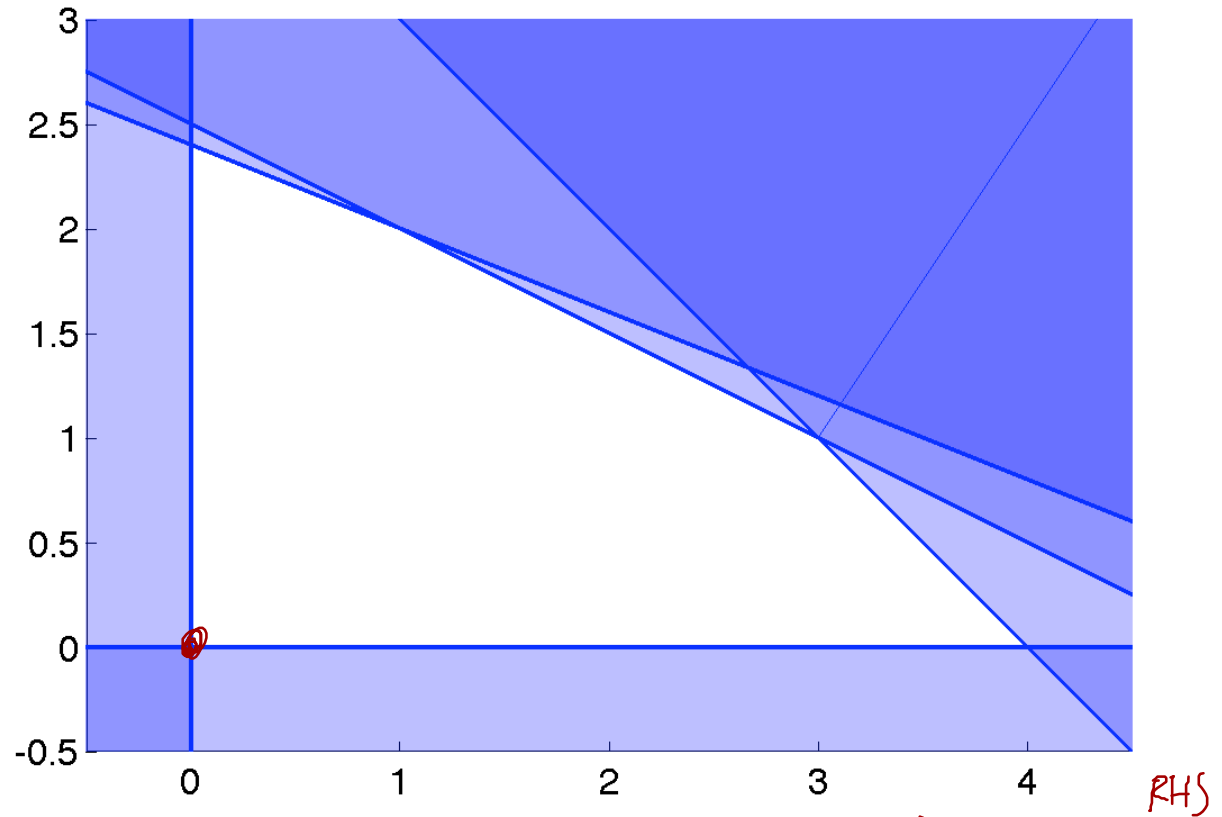
$$\max 2x + 3y \text{ s.t.}$$

$$x + y \leq 4$$

$$2x + 5y \leq 12$$

$$x + 2y \leq 5$$

$$x, y \geq 0$$



Handwritten notes:
 pivot in (y)
 pivot out (t)

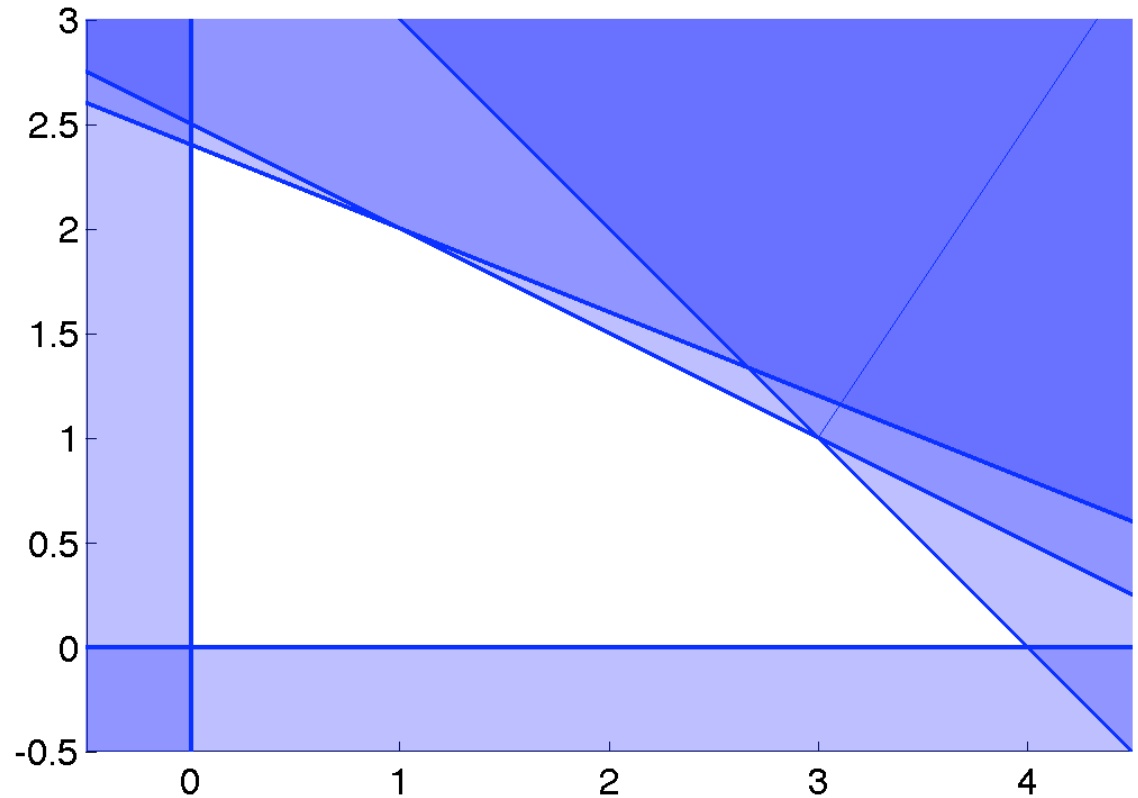
x	(y)	s	(t)	u	RHS
1	1	1	0	0	4
2	5	0	1	0	12
1	2	0	0	1	5
2	3	0	0	0	↑

x	y	s	t	u	RHS
3/5	0	1	-1/5	0	8/5

Handwritten notes:
 $y \leq 4$
 $y \leq 12/5$ ←
 $y \leq 5/2$

Example

$$\begin{aligned} \max \quad & 2x + 3y \text{ s.t.} \\ & x + y \leq 4 \\ & 2x + 5y \leq 12 \\ & x + 2y \leq 5 \end{aligned}$$



x	y	s	t	u	RHS
0.4	1	0	0.2	0	2.4
0.6	0	1	-0.2	0	1.6
0.2	0	0	-0.4	1	0.2
0.8	0	0	-0.6	0	↑

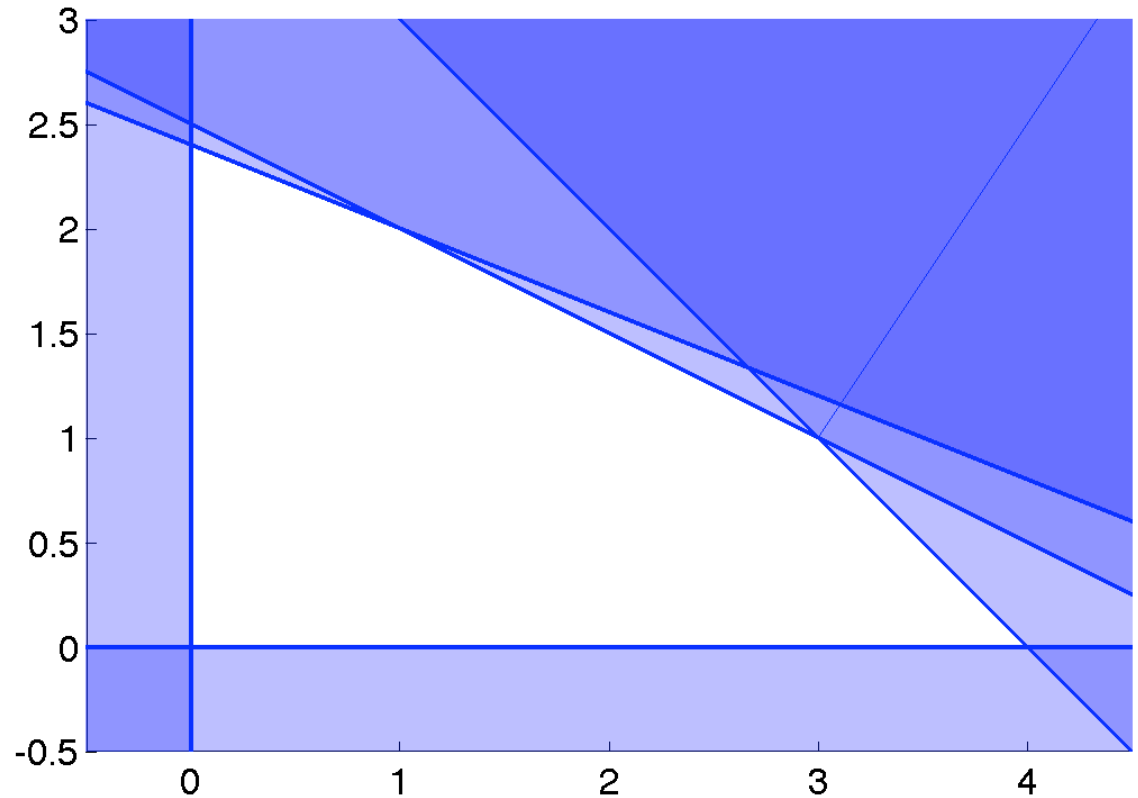
Example

$$\max 2x + 3y \text{ s.t.}$$

$$x + y \leq 4$$

$$2x + 5y \leq 12$$

$$x + 2y \leq 5$$



<u>x</u>	<u>y</u>	<u>s</u>	<u>t</u>	<u>u</u>	<u>RHS</u>
1	0	0	-2	5	1
0	1	0	1	-2	2
0	0	1	1	-3	1
0	0	0	1	-4	↑

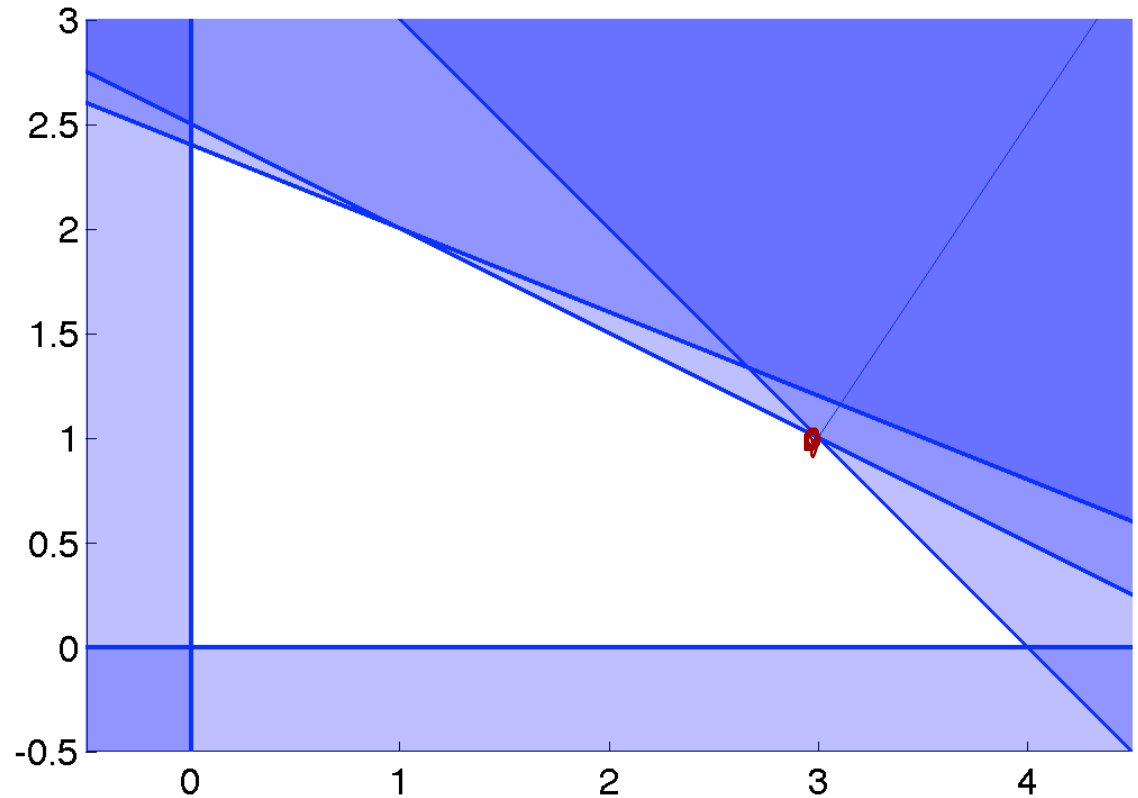
Example

$$\max 2x + 3y \text{ s.t.}$$

$$x + y \leq 4$$

$$2x + 5y \leq 12$$

$$x + 2y \leq 5$$



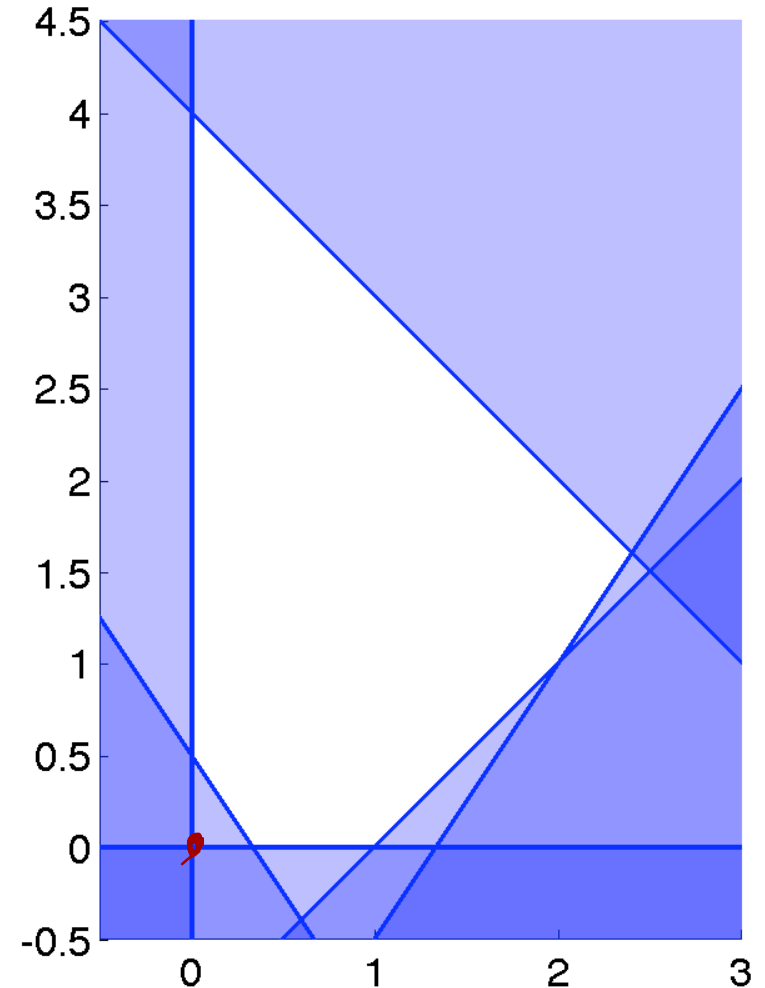
<u>x</u>	<u>y</u>	<u>s</u>	<u>t</u>	<u>u</u>	<u>RHS</u>
1	0	2	0	-1	3
0	1	-1	0	1	1
0	0	1	1	-3	1
0	0	-1	0	-1	↑

multiple constraints violated ==> multiple extra variables, each with $-M$ in objective

Big M

x	y	z	t	u	v	w	RHS
1	1	0	1	0	0	0	4
3	-2	0	0	1	0	0	4
1	-1	0	0	0	1	0	1
-3	-2	1	0	0	0	1	-1
1	-2	$-M$	0	0	0	0	\uparrow

- So far, assumed we started w/ initial feasible basis
- How do we get one?
 - ▶ for each violated constraint, add var w/ coeff -1
 - ▶ penalize in objective, include in initial basis



Ex: combinatorial auctions

- Goods: Newspaper, Magazine, L shoe, R shoe
- Bids (note use of bidding language: 7 rt 16 numbers for B_1 and 1 rt 16 for B_2):
 - ▶ N: +5; M: +4
 - ▶ N, M: -3
 - ▶ L, R: +10
 - ▶ N, L, R: -5;
M, L, R: -4;
N, M, L, R: +3

▶ M: +10

Bidder 1

Bidder 2

Winner determination

- Goods: Newspaper, Magazine

- Bids:

$$(n_1, m_1) = (\overline{n_1}, \overline{m_1})$$

▶ N: +5; M: +4

▶ N, M: +4

▶ N, M: -3

Bidder 1

Bidder 2

$$0 \leq n_1, n_2, m_1, m_2 \leq 1$$

$$n_1 + n_2 \leq 1$$

$$m_1 + m_2 \leq 1$$

$$y \leq n_2$$

$$y \leq m_2$$

$$(1-x) \leq (1-n_1) + (1-m_1)$$

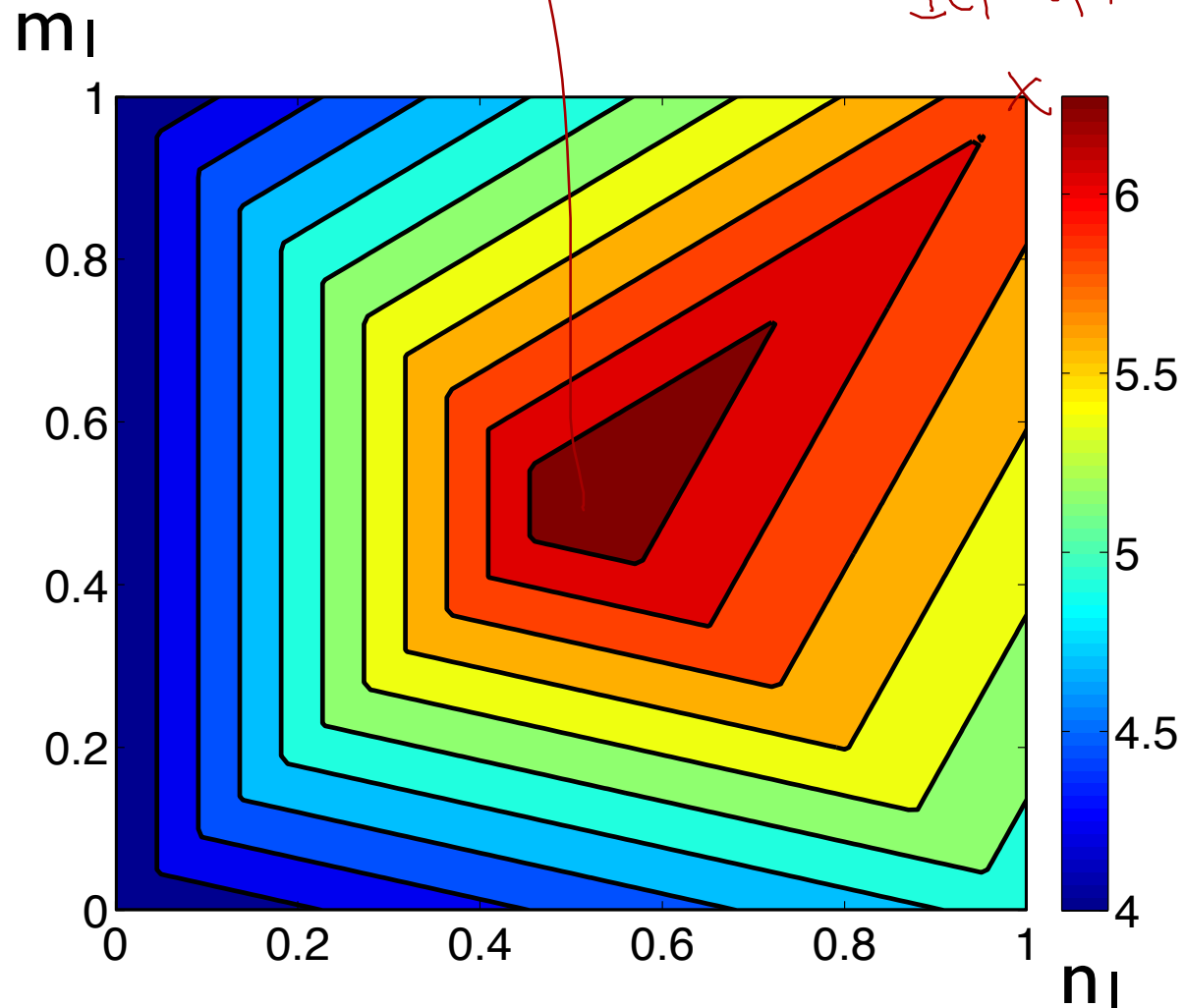
$$0 \leq x, y \leq 1$$

$$\max 5n_1 + 4m_1 + 4y - 3x$$

(all $\in \mathbb{Z}$) \rightarrow delete for LP relax

Bounds

- Any feasible point yields lower bd: (N to B_I , keep M) $\rightarrow 5$
- Upper bound: solve LP relaxation
 - ▶ a bit expensive
 - ▶ can we be lazier?



Being lazy

- A “hard” LP:

$$\max x + y \text{ s.t.}$$

$$x + y \leq 3$$

$$x \leq 1$$

$$y \leq 1$$

OK, we got lucky

- What if it were:

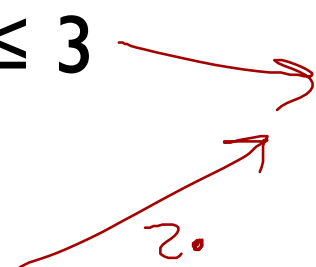
$$\max x + 3y \text{ s.t.}$$

$$x + y \leq 3$$

$$x \leq 1$$

$$y \leq 1$$

$$x + 3y \leq 5$$



How general is this?

- What if it were:

$$\max px + qy \text{ s.t.}$$

$$a (x + y \leq 3)$$

$$b (x \leq 1)$$

$$c (y \leq 1)$$

$$a(x+y-3) + b(x-1) + c(y-1) \leq 0$$

$$(a+b)x + (a+c)y \leq 3a+b+c$$

$$\begin{aligned} a, b, c &\geq 0 \\ a+b &= p \\ a+c &= q \\ px + qy &\leq 3a+b+c \\ \min 3a+b+c & \end{aligned}$$

Dual

Let's do it again

- Note $\geq, \leq, =$ constraints, min obj

$$\min x - 2y \text{ s.t.}$$

$$a (x + y \geq 2)$$

$$b (y \leq 3)$$

$$c (2x - y = 0)$$

$$d (x \geq 0)$$

$$e (y \geq 0)$$

$$a(x+y-2) + b(y-3) + c(2x-y)$$

$$+ dx + ey \geq 0$$

$$(a+2c+d)x = x - 2y$$

$$+ (a+b-c+e)y$$

$$\geq 2a + 3b$$

$$a + 2c + d = 1$$

$$a + b - c + e = -2$$

$$\max 2a + 3b \quad \text{Dual}$$

$$\begin{array}{l} a \geq 0 \\ b \leq 0 \\ c \in \mathbb{R} \\ d \geq 0 \\ e \geq 0 \end{array}$$

Summary of LP duality

- Use multipliers to write combined constraints

max problem
min

$\geq \rightarrow +ve$

$\leq \rightarrow -ve$

$= \rightarrow \mathbb{R}$

} multipliers

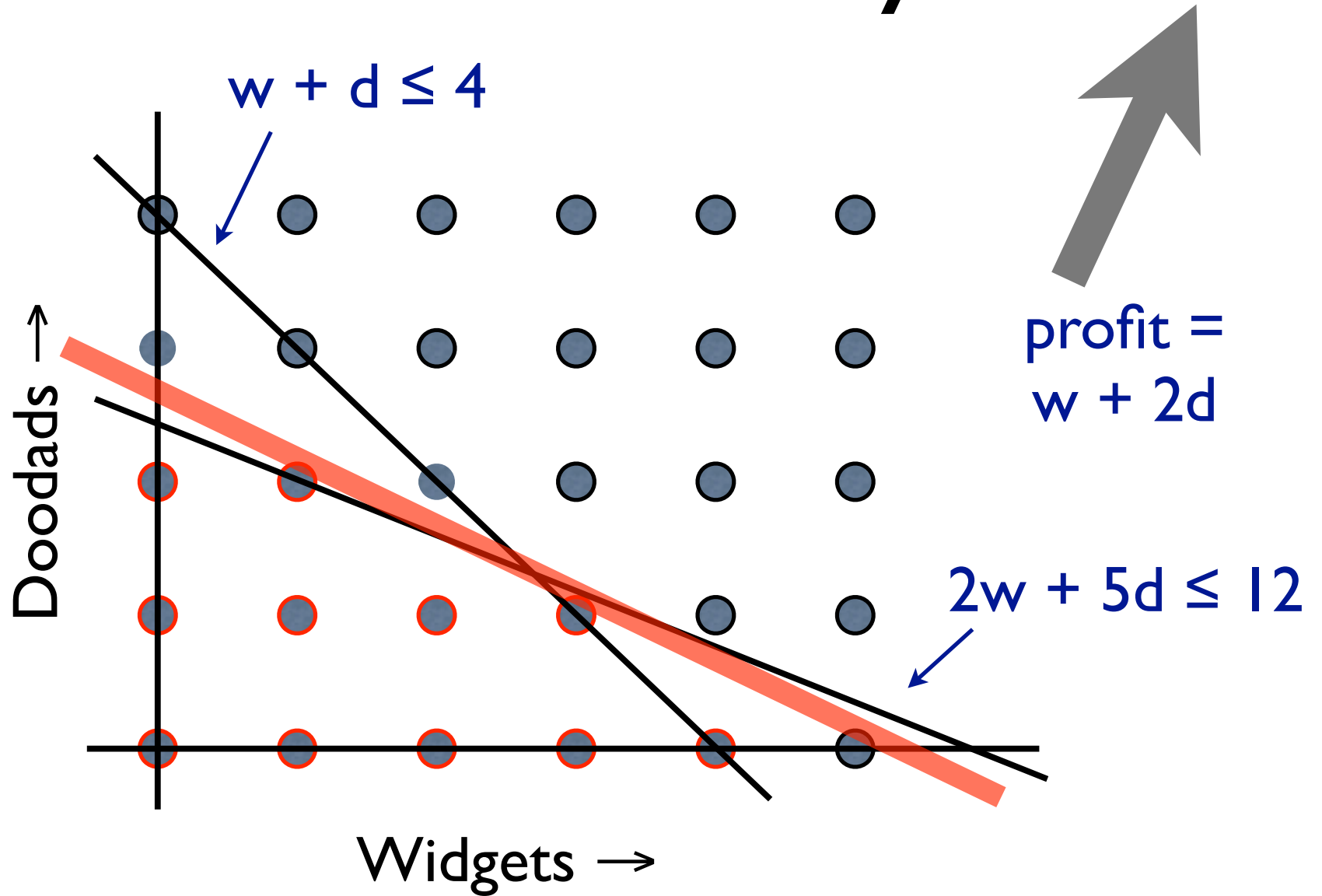
- Constrain multipliers to give us a bound on objective (by matching coefficients)
- Optimize to get tightest bound
- Q: what happens if we take dual of dual?

Ordering

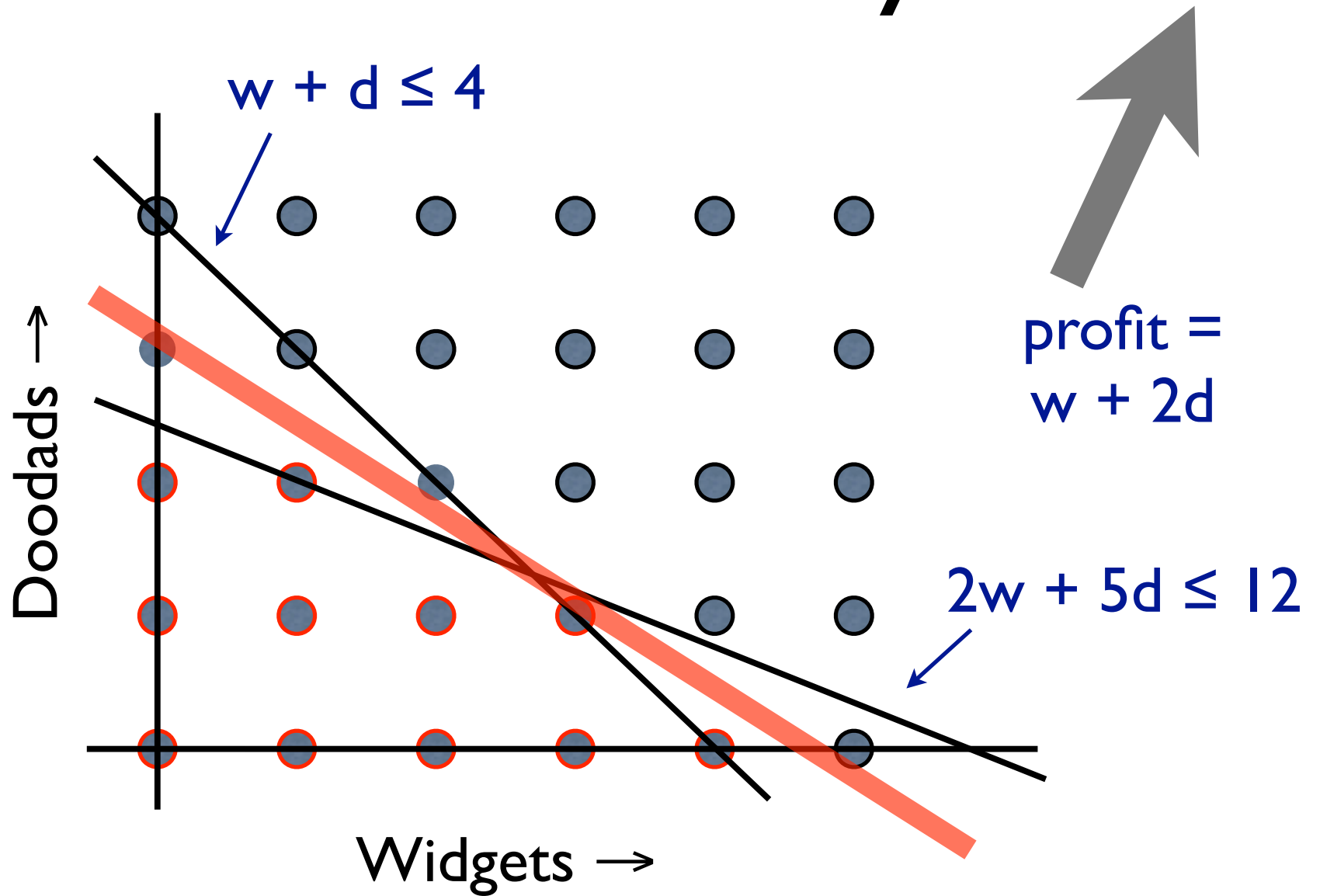
- For primal max problem (dual min):
 - ▶ primal feas \leq primal opt \leq dual opt \leq dual feas
- For primal min problem (dual max):
 - ▶ primal feas \geq primal opt \geq dual opt \geq dual feas

"usually" =

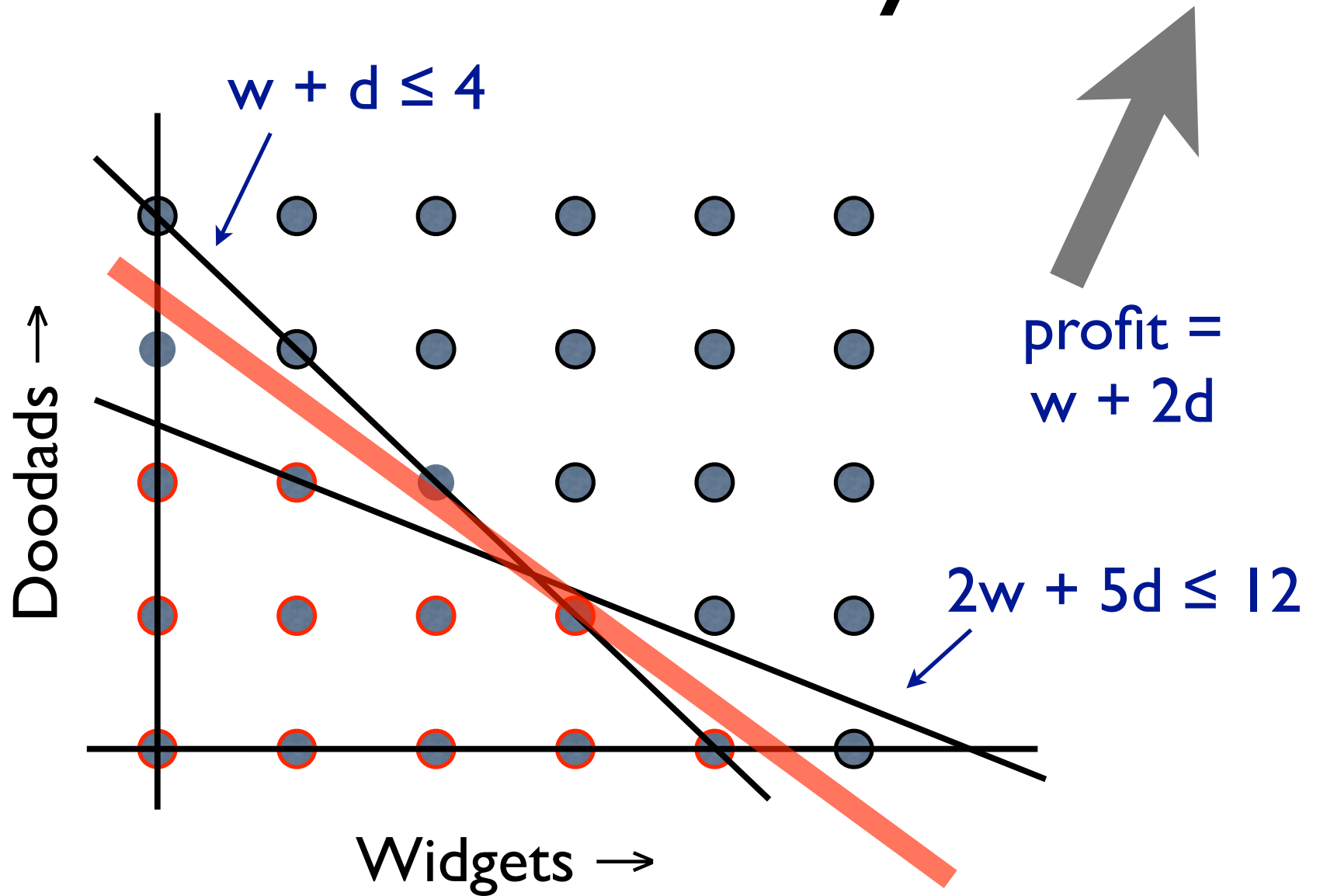
Geometrically



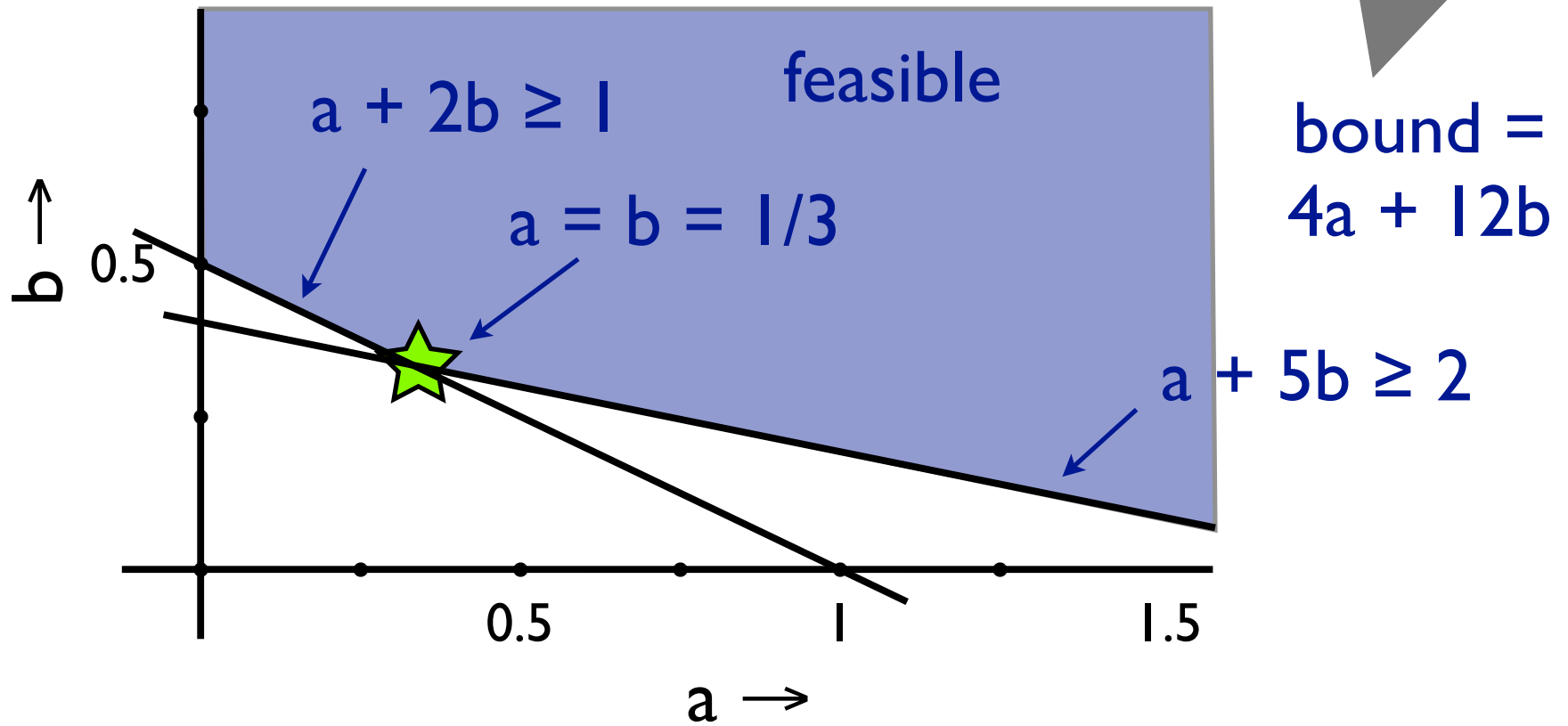
Geometrically



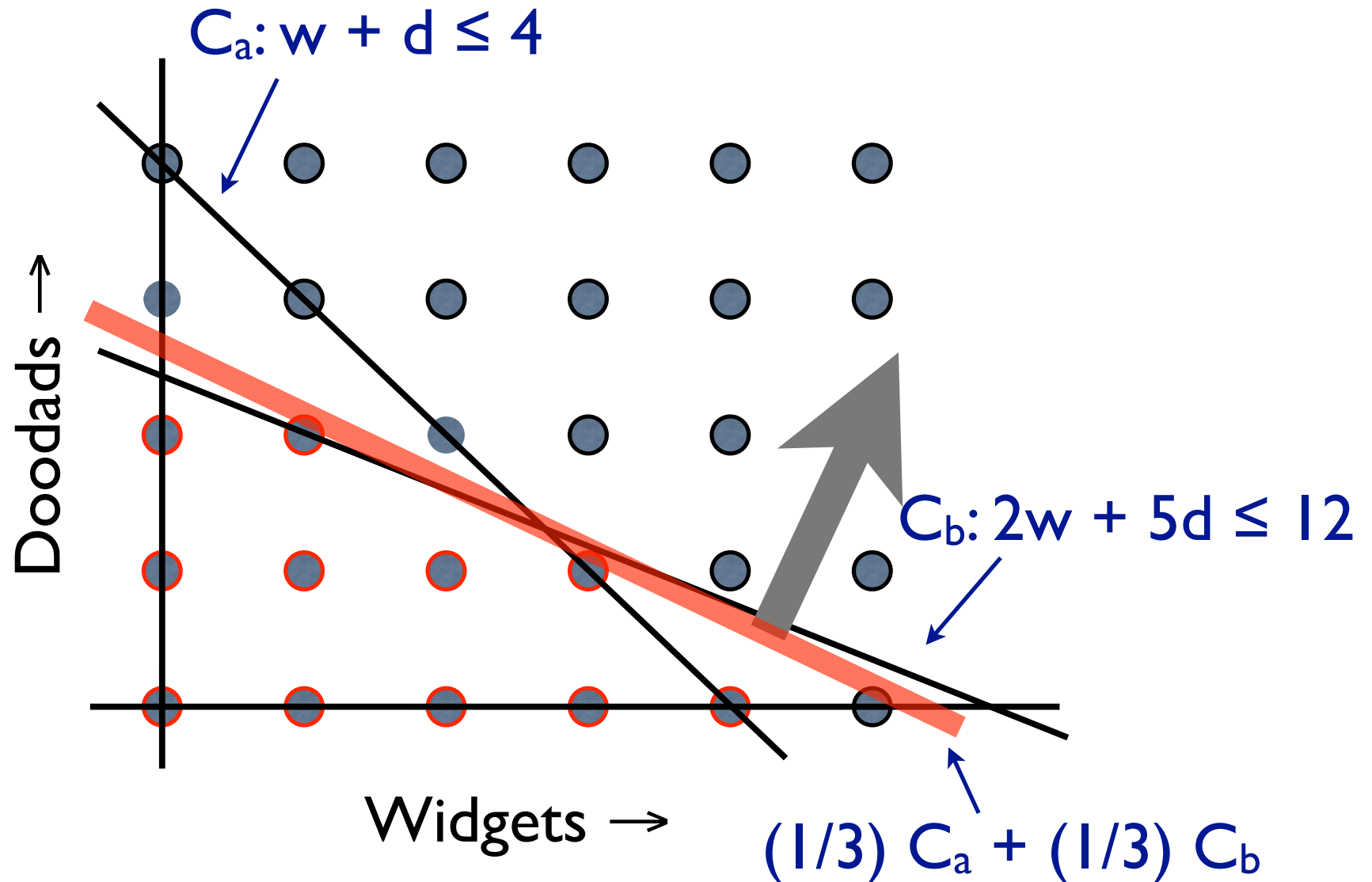
Geometrically



Dual widgets



Dual variables as multipliers



So why bother?

- Reason 1: any feasible solution to dual yields upper bound (compared with only optimal solution to primal)
- Reason 2: dual might be easier to work with
- Reason 3: solvers can often work w/ primal and dual at the same time for no extra cost

Interpreting the dual variables

- Primal variables in the factory LP were how many widgets and doodads to produce
- Interpreted dual variables as multipliers for primal constraints—not much intuition
- Often possible to interpret dual variables as ***prices*** for primal constraints

Dual variables as prices

- Suppose someone offered us a quantity ε of wood, loosening constraint to

$$w + d \leq 4 + \varepsilon$$

- How much should we be willing to pay for this wood?

Dual variables as prices

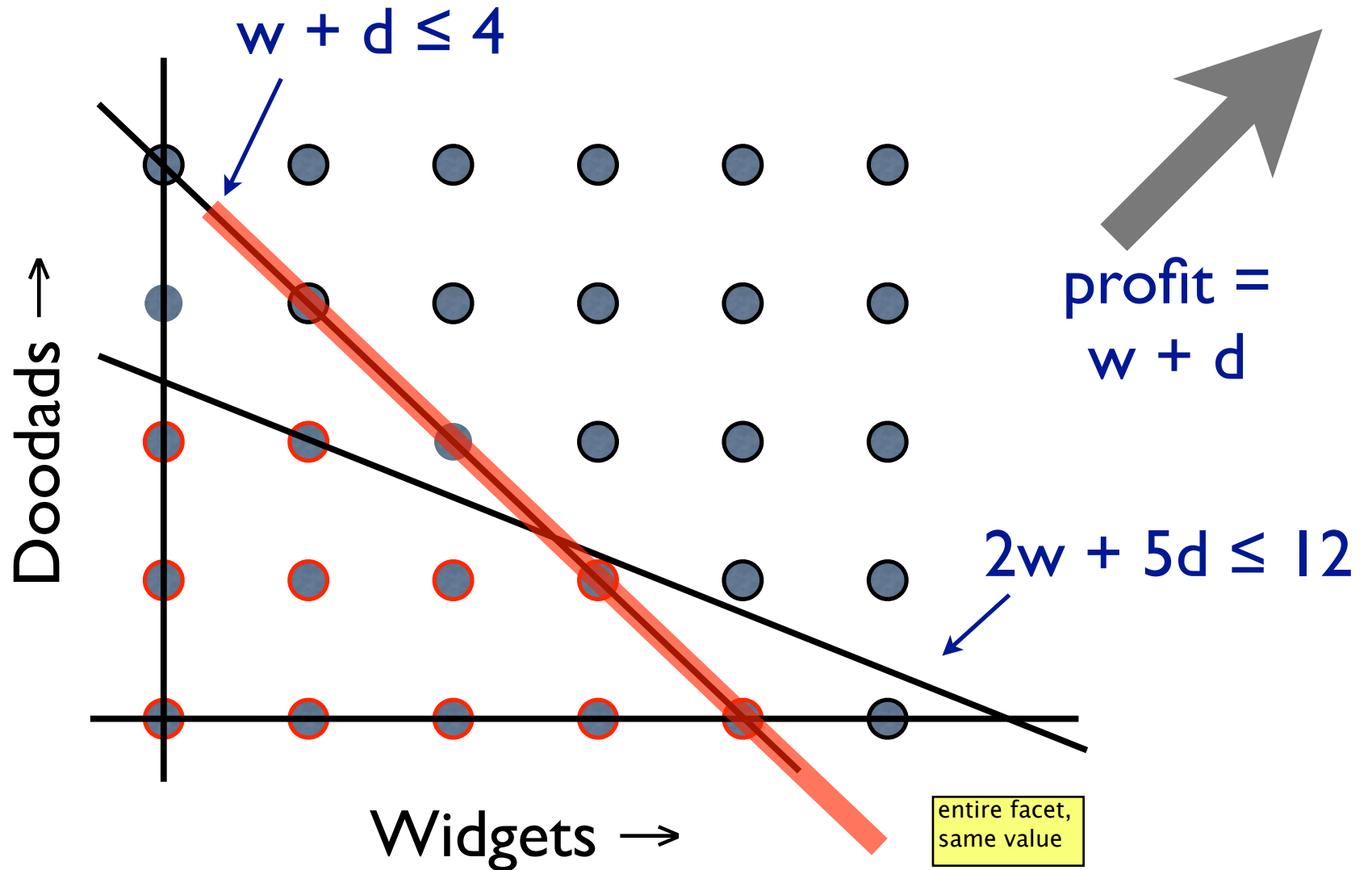
- Dual constrs stay same: $a + 2b \geq 1$, $a + 5b \geq 2$
- Dual objective becomes: $\min (4+\varepsilon)a + 12b$
- Previous solution $a = b = 1/3$ still feasible
 - ▶ still optimal if ε small enough
- Bound changes to $(4+\varepsilon)a + 12b$, increase by $\varepsilon/3$
- So we should pay up to \$1/3 per unit of wood (in small quantities)

Same is true for steel: since $b=1/3$ at opt, we would pay \$1/3 per unit in small quantities

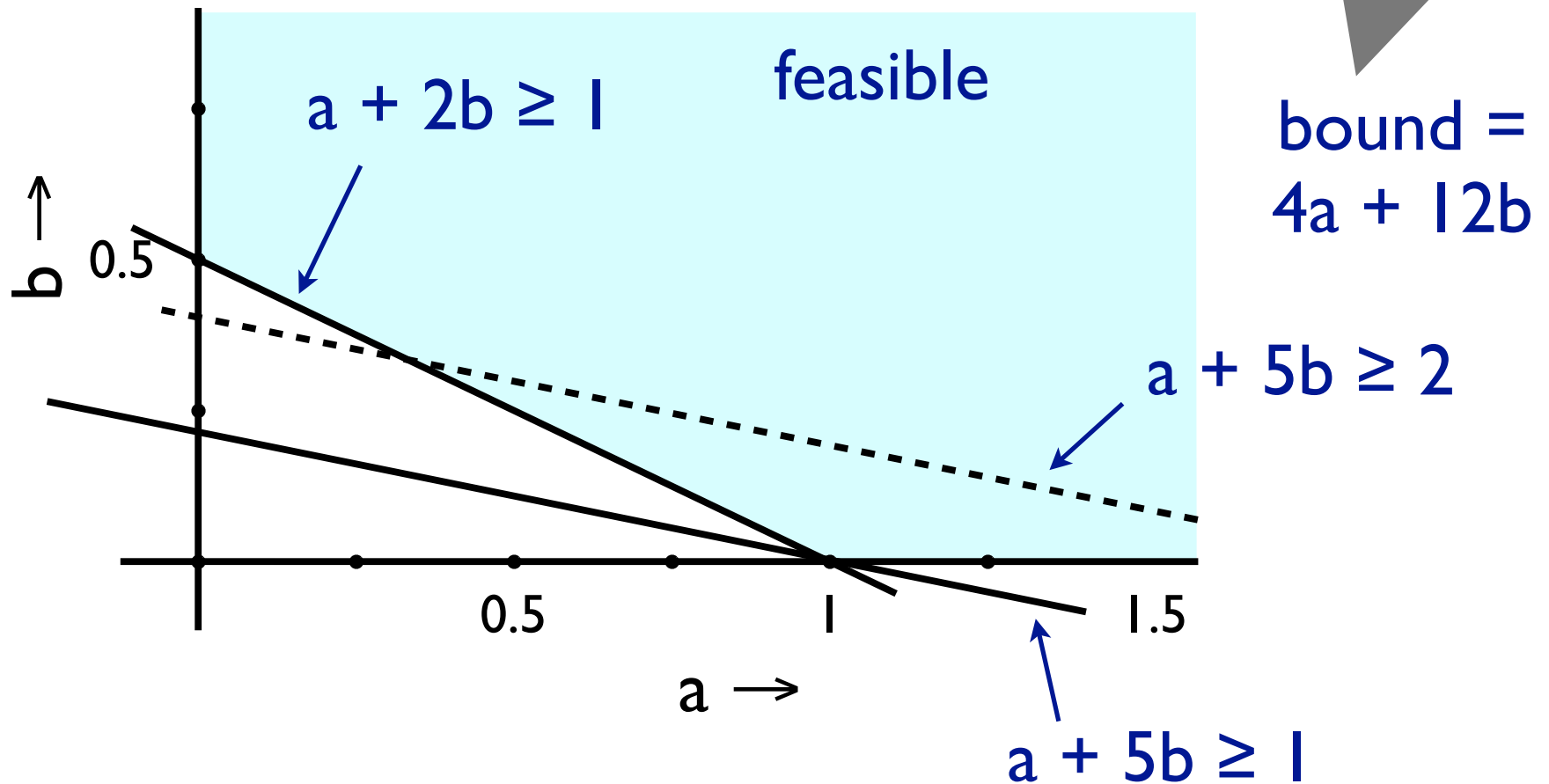
Dual degeneracy

- Primal degenerate = two bases, same corner
- Dual can be degenerate too
 - ▶ so, 4 possibilities for degeneracy
- E.g., what if objective were $w+d$ (not $w+2d$)?

Dual degeneracy



Dual degeneracy



Complementary slackness

- Suppose a constraint is inactive. Would we pay anything to have it relaxed?
- Write $s_j \geq 0$ for slack in primal constraint j
- Write $d_j \geq 0$ for dual variable (multiplier, price) for constraint j
- CS: at optimal primal and dual solutions,

$$\sum_j s_j d_j = 0$$

- Uses: certificate of optimality, proving that optimal solution satisfies some property