Church and Curry: Combining Intrinsic and Extrinsic Typing

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Dedicated to Peter Andrews
on the occasion of his retirement

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Church’s Simple Theory of Types

- Language and logic for the formalization of mathematics
  - (Church 1940)
  - Stood the test of time (72 years!)
  - HOL, Isabelle/HOL, TPS, LEO, Satallax, ...
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  - Russell and Whitehead’s ramified theory of types
  - Church and Rosser’s (untyped) λ-calculus
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- Components
  - Simply typed $\lambda$-calculus (this talk)
  - Logical axioms and inference rules
Church’s definitions
Simply Typed $\lambda$-Calculus

- Church’s definitions
- Types
  1. $\iota$ and $\sigma$ are types.
  2. If $\alpha$ and $\beta$ are types, then $\alpha \rightarrow \beta$ is a type.
     (Church wrote $\beta\alpha$)
Simply Typed $\lambda$-Calculus

- Church’s definitions
- Types
  1. $\iota$ and $\sigma$ are types.
  2. If $\alpha$ and $\beta$ are types, then $\alpha \to \beta$ is a type. (Church wrote $\beta\alpha$)
- Well-formed terms $M^\alpha$ of type $\alpha$
  1. Any variable $x^\alpha$ or constant $c^\alpha$ is a term.
  2. If $x^\alpha$ is a variable and $M^\beta$ a term then $(\lambda x. M)^{\alpha\to\beta}$ is a term.
  3. If $M_1^{\alpha\to\beta}$ and $M_2^\alpha$ are terms, then $(M_1 \ M_2)^\beta$ is a term.
Every well-formed term has an intrinsic type, including variables.
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Supports conventions, such as

> In the remainder of this [talk] we assume that all terms are well-formed according to the above definition.
Intrinsic Typing

- Every well-formed term has an intrinsic type, including variables.
- This kind of intrinsic formulation has become rare, but it has a number of advantages.
- Supports conventions, such as:
  
  In the remainder of this [talk] we assume that all terms are well-formed according to the above definition.

- In a logical framework:
  
  \[
  \begin{align*}
  tp & : \text{type}. \\
  \text{arrow} & : tp \to tp \to tp. \\
  \text{tm} & : tp \to \text{type}. \\
  \text{lam} & : (\text{tm} A \to \text{tm} B) \to \text{tm} (\text{arrow} A B). \\
  \text{app} & : \text{tm} (\text{arrow} A B) \to \text{tm} A \to \text{tm} B.
  \end{align*}
  \]
Untyped \(\lambda\)-Calculus

- (Church 1932) (Church and Rosser 1936)
- Terms
  1. Any variable \(x\) or constant \(c\) is a term.
  2. If \(x\) is a variable and \(M\) a term then \((\lambda x. M)\) is a term.
  3. If \(M_1\) and \(M_2\) are terms, then \((M_1 M_2)\) is a term.
Extrinsic Typing

- (Curry 1934) [for combinators, not \( \lambda \)-terms]
- Types as properties of terms
Extrinsic Typing

- (Curry 1934) [for combinators, not λ-terms]
- Types as properties of terms
- Typing judgments defined by rules

\[
\begin{align*}
\frac{x : \alpha \in \Gamma}{\Gamma \vdash x : \alpha} & \quad \frac{c : \alpha \in \Sigma}{\Gamma \vdash c : \alpha} \\
\Gamma, x : \alpha \vdash M : \beta \quad (x \not\in \text{dom}(\Gamma)) & \quad \Gamma \vdash \lambda x. M : \alpha \rightarrow \beta \\
\Gamma \vdash M : \alpha \rightarrow \beta & \quad \Gamma \vdash N : \alpha \\
& \quad \frac{\Gamma \vdash M \; N : \beta}{\Gamma \vdash M \; N : \beta}
\end{align*}
\]
A term can have multiple types

\[
\begin{align*}
&\vdash \lambda x. x : \tau \rightarrow \tau \\
&\vdash \lambda x. x : (\tau \rightarrow \tau) \rightarrow (\tau \rightarrow \tau)
\end{align*}
\]
Types as Properties

- A term can have multiple types
  
  \[ \vdash \lambda x. x : \iota \rightarrow \iota \]
  
  \[ \vdash \lambda x. x : (\iota \rightarrow \iota) \rightarrow (\iota \rightarrow \iota) \]

- Express explicitly in the form of a single type
Types as Properties

- A term can have multiple types
  
  \[ \vdash \lambda x. x : \iota \rightarrow \iota \]
  
  \[ \vdash \lambda x. x : (\iota \rightarrow \iota) \rightarrow (\iota \rightarrow \iota) \]
  
- Express explicitly in the form of a single type

- Parametric polymorphism (universal types)
  
  \[ \vdash \lambda x. x : \forall t. t \rightarrow t \]
A term can have multiple types

\[ \vdash \lambda x. x : \iota \rightarrow \iota \]
\[ \vdash \lambda x. x : (\iota \rightarrow \iota) \rightarrow (\iota \rightarrow \iota) \]

Express explicitly in the form of a single type

Parametric polymorphism (universal types)

\[ \vdash \lambda x. x : \forall t. t \rightarrow t \]

Ad hoc polymorphism (intersection types)

\[ \vdash \lambda x. x : (\iota \rightarrow \iota) \land ((\iota \rightarrow \iota) \rightarrow (\iota \rightarrow \iota)) \]
Extrinsic Rules

- Parametric polymorphism

\[
\begin{align*}
\Gamma \vdash M : \beta & \quad (t \not\in \text{ftv}(\Gamma)) \\
\Gamma \vdash M : \forall t. \beta & \quad \forall I \\
\Gamma \vdash M : [\beta/t] \alpha & \quad \forall E
\end{align*}
\]
Extrinsic Rules

- Parametric polymorphism

\[ \Gamma \vdash M : \beta \quad (t \not\in \text{ftv}(\Gamma)) \]
\[ \frac{}{\Gamma \vdash M : \forall t. \beta} \] \( \forall I \)
\[ \frac{}{\Gamma \vdash M : \forall t. \alpha} \] \( \forall E \)

\[ \Gamma \vdash M : \forall t. \alpha \]
\[ \frac{}{\Gamma \vdash M : [\beta / t] \alpha} \]

- Intersection polymorphism

\[ \Gamma \vdash M : A \quad \Gamma \vdash M : B \]
\[ \frac{}{\Gamma \vdash M : A \land B} \] \( \land I \)

\[ \frac{}{\Gamma \vdash M : A \land B} \] \( \land E_1 \)
\[ \frac{}{\Gamma \vdash M : A \land B} \] \( \land E_2 \)
\[ \frac{}{\Gamma \vdash M : A} \] \( \land E_1 \)
\[ \frac{}{\Gamma \vdash M : B} \] \( \land E_2 \)
Can type terms more generally

\[ \vdash \lambda x. x \ x : (\forall \alpha. \alpha \rightarrow \alpha) \rightarrow (\forall \beta. \beta \rightarrow \beta) \]
\[ \vdash \lambda x. x \ x : ((\iota \rightarrow \iota) \land \iota) \rightarrow \iota \]
Can type terms more generally

\[ \vdash \lambda x. x \ x : (\forall \alpha. \alpha \to \alpha) \to (\forall \beta. \beta \to \beta) \]
\[ \vdash \lambda x. x \ x : ((\iota \to \iota) \land \iota) \to \iota \]

Can type type terms more accurately

\[ \vdash \lambda x. s(s(s\ x)) : (\text{even} \to \text{odd}) \land (\text{odd} \to \text{even}) \]
The typing judgment is \textbf{undecidable}.
The typing judgment is **undecidable**

Challenge: generalize to a complete language

- Practical
- Useful
- Philosophically justified
- Easy to reason about
Layering Type Systems

- Intrinsic simple types for basic consistency
  - Avoiding Russell’s paradox
Layering Type Systems

- Intrinsic **simple types** for basic consistency
  - Avoiding Russell’s paradox
- Extrinsic **sorts** on well-formed terms for precision
  - Circumvent problems with general extrinsic types
  - Achieve pragmatic goals
Intrinsic simple types for basic consistency
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Extrinsic sorts on well-formed terms for precision
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Instances
  - Intersection types to datasort refinement (this talk)
  - Dependent types to index refinements
Layering Type Systems

- Intrinsic simple types for basic consistency
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- Extrinsic sorts on well-formed terms for precision
  - Circumvent problems with general extrinsic types
  - Achieve pragmatic goals
- Instances
  - Intersection types to datasets refinement (this talk)
  - Dependent types to index refinements
- Parametric polymorphism is a different story
Multiple techniques in Church’s Type Theory
- Church numerals
- Constants and axioms

We have only basic type $\text{ι}$ ($\text{o}$ is for truth values)

Example: natural numbers
- Constants $z^{\text{ι}}$ and $s^{\text{ι} \rightarrow \text{ι}}$ (constructors)
- Constant $\text{nat}^{\text{ι} \rightarrow \text{o}}$ (predicate)

Axioms
- $\text{nat}(z)$
- $\forall x^{\text{ι}}. \text{nat}(x) \supset \text{nat}(s(x))$

Lists, trees, etc. all have type $\text{ι}$

Sort out using sorts
Representing Data

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  - Church numerals
  - Constants and axioms
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Representing Data

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  - Church numerals
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- We have only basic type $\iota$ ($\sigma$ is for truth values)
- Example: natural numbers
  - Constants $z^{\iota}$ and $s^{\iota \rightarrow \iota}$ (constructors)
  - Constant $n a t^{\iota \rightarrow \sigma}$ (predicate)
  - Axioms
    
    $\text{nat}(z)$
    $\forall x^{\iota}. \text{nat}(x) \supset \text{nat}(s(x))$
Representing Data

- Multiple techniques in Church’s Type Theory
  - Church numerals
  - Constants and axioms
- We have only basic type \( \iota \) (\( \sigma \) is for truth values)
- Example: natural numbers
  - Constants \( z^\iota \) and \( s^{\iota \rightarrow \iota} \) (constructors)
  - Constant \( \text{nat}^{\iota \rightarrow \sigma} \) (predicate)
  - Axioms
    \[
    \text{nat}(z) \\
    \forall x^\iota. \text{nat}(x) \supset \text{nat}(s(x))
    \]
- Lists, trees, etc. all have type \( \iota \)
  - Sort out using sorts
Example

\[ \text{nat}\mathbin{\alpha} \text{ sort} \]
\[ z\mathbin{\alpha} : \text{nat} \]
\[ s\mathbin{\alpha\rightarrow\alpha} : \text{nat} \rightarrow \text{nat} \]
Example

\[
\text{nat}^\ell \quad \text{sort}
\]
\[
z^\ell : \text{nat}
\]
\[
s^{\ell \rightarrow \ell} : \text{nat} \rightarrow \text{nat}
\]

Define sorts \( S^\alpha \) refining type \( \alpha \) under signature \( \Sigma \)

1. A base sort \( Q^\ell \) declared in \( \Sigma \) is a simple sort.
2. If \( S^\alpha \) and \( T^\beta \) are simple sorts, then \((S \rightarrow T)^{\alpha \rightarrow \beta}\) is a simple sort.
Sorting Judgment

- Context $\Gamma$ consisting of declarations $x^\alpha : S^\alpha$
- Sorting judgment $\Gamma \vdash M^\alpha : S^\alpha$
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- Sorting judgment $\Gamma \vdash M^\alpha : S^\alpha$
- Defined only for terms of intrinsic type $\alpha$ and sort refining the same type $\alpha$!
Sorting Judgment

- Context $\Gamma$ consisting of declarations $x^\alpha : S^\alpha$
- Sorting judgment $\Gamma \vdash M^\alpha : S^\alpha$
- Defined only for terms of intrinsic type $\alpha$ and sort refining the same type $\alpha$!
- Rules

\[
\frac{x : S \in \Gamma}{\Gamma \vdash x : S} \quad \frac{c : S \in \Sigma}{\Gamma \vdash c : S}
\]

\[
\frac{\Gamma, x : S \vdash M : T \quad (x \notin \text{dom}(\Gamma))}{\Gamma \vdash \lambda x. M : S \rightarrow T}
\]

\[
\frac{\Gamma \vdash M : S \rightarrow T \quad \Gamma \vdash N : S}{\Gamma \vdash MN : T}
\]
Subsort declarations $Q_1^\iota \leq Q_2^\iota$

New rules

$Q \leq Q$

$Q_1 \leq Q_2$, $Q_2 \leq Q_3$ 

$Q_1 \leq Q_3$

$\Gamma \vdash M : Q$, $Q \leq Q'$

$\Gamma \vdash M : Q'$

Defined on base types only

Can derive principles for higher types
Subsorting Example

- Refining natural numbers

\[
\begin{align*}
\text{zero} & \leq \text{nat} \\
\text{pos} & \leq \text{nat} \\
\text{z} & : \text{zero} \\
\text{s} & : \text{nat} \rightarrow \text{pos}
\end{align*}
\]
Refining natural numbers

zero $\leq$ nat
pos $\leq$ nat

$z ~:~$ zero
$s ~:~$ nat $\rightarrow$ pos

Examples

\[ \vdash \lambda x. x : \text{nat} \rightarrow \text{nat} \]
\[ \vdash \lambda x. x : \text{zero} \rightarrow \text{nat} \]
\[ \vdash \lambda x. \lambda y. x ~y : (\text{nat} \rightarrow \text{zero}) \rightarrow (\text{zero} \rightarrow \text{nat}) \]
\[ \vdash \lambda x. s ~x : \text{nat} \rightarrow \text{pos} \]
Want to express and exploit multiple properties of terms
Combining Properties

- Want to express and exploit multiple properties of terms
- Example: even and odd numbers

\[
\begin{align*}
\text{even} & \leq \text{nat} \\
\text{odd} & \leq \text{nat} \\
z & : \text{even} \\
s & : \text{even} \rightarrow \text{odd} \\
s & : \text{odd} \rightarrow \text{even}
\end{align*}
\]
Combining Properties

- Want to express and exploit multiple properties of terms
- Example: even and odd numbers

\[
\begin{align*}
\text{even} & \leq \text{nat} \\
\text{odd} & \leq \text{nat} \\
z & : \text{even} \\
s & : \text{even} \rightarrow \text{odd} \\
s & : \text{odd} \rightarrow \text{even}
\end{align*}
\]

- Have no way to express in one sort:

\[
\begin{align*}
\Gamma & \vdash \lambda x^\text{even}. s(s(s x)) : \text{even} \rightarrow \text{odd} \\
\Gamma & \vdash \lambda x^\text{odd}. s(s(s x)) : \text{odd} \rightarrow \text{even}
\end{align*}
\]
Intersecting Sorts

Define sorts $S^{\alpha}$ refining types $\alpha$ (in intrinsic style)

1. A base sort $Q^\iota$ declared in $\Sigma$ is a sort.
2. If $S^{\alpha}$ and $T^{\beta}$ are sorts, then $(S \rightarrow T)^{\alpha \rightarrow \beta}$ is a sort.
3. If $S^{\alpha}$ and $T^{\alpha}$ are sorts then $(S \land T)^{\alpha}$ is a sort.
4. $\top^{\alpha}$ is a sort for each type $\alpha$. 
Extended Sorting Judgments

- Recall $\Gamma \vdash M^\alpha : S^\alpha$
Extended Sorting Judgments

- Recall $\Gamma \vdash M^\alpha : S^\alpha$
- New typing rules

\[
\begin{align*}
\Gamma \vdash M : S_1 & \quad \Gamma \vdash M : S_2 \\
\quad & \\
\therefore \quad \Gamma \vdash M : S_1 \land S_2 \\
\end{align*}
\]

\[
\begin{align*}
\Gamma \vdash M : S_1 \land S_2 \\
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\Gamma \vdash M : S_1 \land S_2 \\
\quad & \\
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\]

\[
\begin{align*}
\Gamma \vdash M : S_1 \land S_2 \\
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\]

Philosophically justified in the sense of Dummett/Martin-Löf
Extended Sorting Judgments

- Recall $\Gamma \vdash M^\alpha : S^\alpha$
- New typing rules

\[
\begin{align*}
\Gamma \vdash M : S_1 \quad & \Gamma \vdash M : S_2 \\
\hline
\Gamma \vdash M : S_1 \land S_2
\end{align*}
\]

\[
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\Gamma \vdash M : S_1 \land S_2 \quad & \Gamma \vdash M : S_1 \land S_2 \\
\hline
\Gamma \vdash M : S_1 \\
\Gamma \vdash M : S_2
\end{align*}
\]

\[
\Gamma \vdash M : \top
\]

- Philosophically justified in the sense of Dummett/Martin-Löf
Example Revisited

- Can now conjoin properties

\[ \vdash \lambda x. s(s(s x)) : \text{(even} \rightarrow \text{odd)} \]
\[ \land \text{(odd} \rightarrow \text{even)} \]
\[ \land \text{(nat} \rightarrow \text{pos)} \]
\[ \land \ldots \]
Example Revisited

- Can now conjoin properties

\[ \vdash \lambda x. s(s(s\ x)) : \text{(even} \rightarrow \text{odd)} \]
\[ \wedge \text{(odd} \rightarrow \text{even)} \]
\[ \wedge \text{(nat} \rightarrow \text{pos)} \]
\[ \wedge \ldots \]

- Every (well-formed) term has a principal sort
Some Results

- Sort checking is decidable
  - “Proof:” there are effectively only finitely many refinements of a given type
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  - “Proof:” there are effectively only finitely many refinements of a given type
- Sorting is closed under $\beta$-reduction
  - “Proof:” standard substitution property
- Sorting is closed under $\eta$-expansion
  - “Proof:” induction over sorts
Define $\eta^\alpha(M)$ as $\eta$-long form of $M^\alpha$
More Results

- Define $\eta^\alpha(M)$ as $\eta$-long form of $M^\alpha$
- Define subsorting $S^\alpha \leq T^\alpha$ at higher types

$$S \leq T \iff x:S \vdash \eta^\alpha(x) : T$$
More Results

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- Extend subsumption rule
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- Sorting is closed under $\beta$-expansion
  - “Proof:” intersect all sorts the abstracted term is used at
More Results

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- Sorting is closed under $\beta$-expansion
  - “Proof:” intersect all sorts the abstracted term is used at
- Sorting is closed under $\eta$-reduction
  - “Proof:” by subsorting at higher types
More Results

- Define $\eta^\alpha(M)$ as $\eta$-long form of $M^\alpha$
- Define subsorting $S^\alpha \leq T^\alpha$ at higher types
  
  \[
  S \leq T :\iff x : S \vdash \eta^\alpha(x) : T
  \]

- Extend subsumption rule
- Sorting is closed under $\beta$-expansion
  - “Proof:” intersect all sorts the abstracted term is used at
- Sorting is closed under $\eta$-reduction
  - “Proof:” by subsorting at higher types

- Conclusion

  *Extrinsic (Curry) sorting with intersections refining intrinsic (Church) typing is closed under $\lambda$-conversion!*
Expressive power extends tree automata to higher types

Canonical ($= \beta$-normal, $\eta$-long) terms can be typed bidirectionally

- In: Festschrift in Honor of Peter B. Andrews on his 70th Birthday, C. Benzmüller, C. Brown, J. Siekmann, and R. Statman, editors
- Crucial for logical frameworks (Lovas 2010)

- Refinement type inference for ML (Freeman 1994)
- Practical refinements type for SML (Davies 2005)
- Dependent refinements over decidable domains (Xi 1998)
- Unifying sort and dependent refinements (Dunfield 2007)
Church’s original intrinsic formulation of the simply-typed λ-calculus has fallen into disfavor, perhaps unjustly.
Church’s original intrinsic formulation of the simply-typed \( \lambda \)-calculus has fallen into disfavor, perhaps unjustly.

It suggests an elegant layering with Curry’s extrinsic typing judgment (translated to \( \lambda \)-calculus).
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It suggests an elegant layering with Curry’s extrinsic typing judgment (translated to $\lambda$-calculus).

Can be usefully combined with Coppo et al.’s intersection types for high expressiveness, precision, and surprisingly strong metatheoretic results.