# Automated Theorem Proving in a Simple Meta Logic for LF

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**Abstract.** Higher-order representation techniques allow elegant encodings of logics and programming languages in the logical framework LF, but unfortunately they are fundamentally incompatible with induction principles needed to reason about them. In this paper we develop a meta-logic $\mathcal{M}_2$ which allows inductive reasoning over such LF encodings, and describe its implementation in Twelf, a special-purpose automated theorem prover for properties of logics and programming languages. We have used Twelf to automatically prove a number of non-trivial theorems, including type preservation for Mini-ML and the deduction theorem for intuitionistic propositional logic.

## 1 Introduction

The logical framework LF [HHP93] has been designed as a meta-language for representing deductive systems which are common in the study of logics and programming languages. It allows concise encodings of many common inference systems, such as natural deduction and sequent calculi, type systems, operational semantics, compilers, abstract machines, etc. (see [Pfe96] for a survey). These representations often lead directly to implementations, either via the constraint logic programming paradigm [Pfe94] or via general search using tactics and tacticals.

The logical framework derives its expressive power from the use of dependent types together with "higher-order" representation techniques which directly support common concepts in deductive systems, such as variable binding and capture-avoiding substitution, parametric and hypothetical judgments, and substitution properties. The fact that these notions are an integral part of the logical framework would seem to make it an ideal candidate not only for reasoning *within* various inference systems, but for reasoning *about* properties of such systems.

Unfortunately, higher-order representation techniques are fundamentally incompatible with the induction principles needed to reason about such encodings.[1] In the literature three approaches have been studied in order to overcome these problems, while retaining the advantages a logical framework can offer. The first called *schema-checking* [Roh94,RP96] implements meta-theoretic proofs as relations whose

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[1] See [DPS97] for a detailed analysis.

operational reading as logic programs realizes the informal proofs. This has been applied successfully in many case studies (see [Pfe96]), but lacks automation. The second is based on reflection via a modal provability operator. At present it is unclear how this idea, developed for simple types in [DPS97], interacts with dependent types, and if it is flexible enough for many of the theorems that can be treated with schema-checking. The third is to devise an explicit (meta-)meta-logic for reasoning about logical framework encodings. For the simpler logical framework of hereditary Harrop formulas this approach has been followed by McDowell and Miller [MM97,McD97]. However, their system suffers from the lack of dependent types and is not directly amenable to automated deduction, since it offers only induction over the natural numbers.[2]

In this paper we develop a simple meta-logic $\mathcal{M}_2$ for LF and sketch its implementation in the Twelf system. $\mathcal{M}_2$ was designed explicitly to support automated inductive theorem proving and has been applied successfully to prove, for example, value soundness and type preservation for Mini-ML, completeness of a continuation stack machine with respect to a natural semantics for Mini-ML, soundness and completeness of uniform derivations with respect to resolution (which is a critical step in the correctness of compilers for logic programming languages), the deduction theorem for intuitionistic propositional logic using Hilbert's axiomatization, and the existence of an embedding of Cartesian closed categories into the simply-typed $\lambda$-calculus. In each case we specified only the theorem and the induction variable, the proof was completely automatic in every other respect.

We view Twelf as a special-purpose automated theorem prover for the theory of programming languages and logics. It owes its success to the expressive power of the logical framework combined with the simplicity of the meta-logic which nonetheless allows direct expression of informal mathematical arguments. Its main current limitations are the lack of facilities for incorporating lemmas and for proving properties which require reasoning about *open* LF objects. We plan to address the former by adapting standard techniques from inductive and resolution theorem proving and the latter by borrowing successful ideas from schema-checking.

This paper is organized as follows: In Section 2 we briefly describe the logical framework LF and introduce a programming language Mini-ML and a type preservation result as running example. The meta logic $\mathcal{M}_2$ is introduced in Section 3 which is implemented in the proof assistant Twelf which we discuss in Section 4. Section 5 compares the most closely related work before we assess the results and discuss future work.

## 2 The Logical Framework

The type theory underlying the logical framework LF is an extension of the simply-typed $\lambda$-calculus by dependent types. It also appears under the name $\lambda P$ in Barendregt's $\lambda$-cube [Bar92]. It is defined by three syntactic categories of objects, type families, and kinds [HHP93].

[2] See Section 5 for a more detailed comparison.

$$
\begin{array}{lll}
\text{Kinds:} & K & ::= \text{type} \mid \Pi x\!:\!A.\, K \\
\text{Type Families:} & A & ::= a \mid A\ M \mid \Pi x\!:\!A_1.\, A_2 \\
\text{Objects:} & M & ::= c \mid x \mid \lambda x\!:\!A.\, M \mid M_1\ M_2 \\
\text{Signatures:} & \Sigma & ::= \cdot \mid \Sigma, a : K \mid \Sigma, c : A \\
\text{Contexts:} & \Gamma & ::= \cdot \mid \Gamma, x : A
\end{array}
$$

We use $a$ for type family constants, $c$ for object constants, and $x$ for variables. Atomic types have the form $a\ M_1 \ldots M_n$ and function types $\Pi x : A_1.\, A_2$, which we may write $A_1 \to A_2$ if $x$ does not occur free in $A_2$. We assume that constants and variables are declared at most once in a signature and context, respectively. As usual we apply tacit renaming of bound variables to maintain this assumption and to guarantee capture-avoiding substitution.

The LF type theory is defined by a number of mutually dependent judgments which we will not re-iterate here. The main typing judgment has the form $\Gamma \vdash_\Sigma M : A$ and expresses that object $M$ has type $A$ in context $\Gamma$ with respect to signature $\Sigma$. We generally assume that signature $\Sigma$ is valid and fixed and therefore omit it from the typing and other related judgments introduced later on. We will also need to explicitly require the validity of contexts, written as $\vdash \Gamma$ ctx. In a slight departure from [HHP93] we take $\beta\eta$-conversion as our notion of definitional equality, since this guarantees that every well-typed object has an equivalent *canonical form*, that is, a long $\beta\eta$-normal form. The requisite theory may, for example, be found in [Geu92].

As a running example we will use Mini-ML in the formulation of [Pfe92] which goes back to [MP91], culminating in an automatic proof of type preservation. While space only permits showing the fragment including abstraction, application, and recursion, our automatic proof also treats the remaining features of Mini-ML including polymorphism and an inductively defined type.

Mini-ML expressions are represented by canonical LF objects of type exp. The representation function $\ulcorner e \urcorner$ shown below is *adequate*, that is, it is a compositional bijection between Mini-ML expressions and canonical objects of type exp. Compositionality in this context means that $\ulcorner e[e'/x] \urcorner = \ulcorner e \urcorner [\ulcorner e' \urcorner / x]$ and allows us to represent substitution in the object language by $\beta$-reduction in the logical framework, since $\ulcorner e \urcorner [\ulcorner e' \urcorner / x]$ is definitionally equal to $(\lambda x\!:\!\text{exp}.\, \ulcorner e \urcorner)\ \ulcorner e' \urcorner$.

$$\text{Mini-ML Expressions: } e ::= x \mid \mathbf{lam}\ x.\, e \mid e_1 \mathbf{@} e_2 \mid \mathbf{fix}\ x.\, e$$

$$
\begin{array}{rl}
 & \text{exp} : \text{type} \\
\ulcorner \mathbf{lam}\ x.\, e \urcorner = \text{lam}\ (\lambda x\!:\!\text{exp}.\, \ulcorner e \urcorner) & \text{lam} : (\text{exp} \to \text{exp}) \to \text{exp} \\
\ulcorner e_1 \mathbf{@} e_2 \urcorner = \text{app}\ \ulcorner e_1 \urcorner\ \ulcorner e_2 \urcorner & \text{app} : \text{exp} \to \text{exp} \to \text{exp} \\
\ulcorner \mathbf{fix}\ x.\, e \urcorner = \text{fix}\ (\lambda x\!:\!\text{exp}.\, \ulcorner e \urcorner) & \text{fix} \ : (\text{exp} \to \text{exp}) \to \text{exp} \\
\ulcorner x \urcorner = x &
\end{array}
$$

Expressions form the basic syntactic structure for which we define an operational semantics and a typing discipline. The evaluation judgment $e \hookrightarrow v$ for expressions $e$ and $v$ is defined by the following rules of inference.

$$\frac{}{\mathbf{lam}\ x.\, e \hookrightarrow \mathbf{lam}\ x.\, e}\ \mathsf{evLam} \qquad \frac{e_1 \hookrightarrow \mathbf{lam}\ x.\, e_1' \qquad e_2 \hookrightarrow v_2 \qquad e_1'[v_2/x] \hookrightarrow v}{e_1 @ e_2 \hookrightarrow v}\ \mathsf{evApp}$$

$$\frac{e[\mathbf{fix}\ x.\, e/x] \hookrightarrow v}{\mathbf{fix}\ x.\, e \hookrightarrow v}\ \mathsf{evFix}$$

Derivations of a judgment $e \hookrightarrow v$ are represented by canonical LF objects of type ev $\ulcorner e \urcorner\ \ulcorner v \urcorner$ over the following signature, which means that ev is a constant type family indexed by two expressions.

$$\begin{array}{ll} \text{ev} & : \text{exp} \to \text{exp} \to \text{type} \\ \text{ev\_lam} & : \text{ev}\ (\text{lam}\ \mathbf{E})\ (\text{lam}\ \mathbf{E}) \\ \text{ev\_app} & : \text{ev}\ (\text{app}\ \mathbf{E_1}\ \mathbf{E_2})\ \mathbf{V} \\ & \quad \leftarrow\ \text{ev}\ \mathbf{E_1}\ (\text{lam}\ \mathbf{E_1'}) \\ & \quad \leftarrow\ \text{ev}\ \mathbf{E_2}\ \mathbf{V_2} \\ & \quad \leftarrow\ \text{ev}\ (\mathbf{E_1'}\ \mathbf{V_2})\ \mathbf{V} \\ \text{ev\_fix} & : \text{ev}\ (\text{fix}\ \mathbf{E})\ \mathbf{V} \\ & \quad \leftarrow\ \text{ev}\ (\mathbf{E}\ (\text{fix}\ \mathbf{E}))\ \mathbf{V} \end{array}$$

In this example we reversed the function arrows, writing $A_2 \leftarrow A_1$ instead of $A_1 \to A_2$ following logic programming notation. Since $\to$ is right associative, $\leftarrow$ is left associative. The free variables in each declaration, written with bold uppercase names, are implicitly $\Pi$-quantified with an appropriate type which can be deduced from the context in which the variable appears. For example, the fully explicit form of the first declaration would be ev_lam : $\Pi E : \text{exp} \to \text{exp}.\, \text{ev}\ (\text{lam}\ E)\ (\text{lam}\ E)$. These *index variables* play an important role, as they must be treated differently from other variables during proof search, as will be discussed in Section 4.

Mini-ML types are denoted by $\tau$, where $\tau_1 \Rightarrow \tau_2$ stand for function types. They are represented by canonical LF objects of type tp over the (trivial) signature below.

$$\ulcorner \tau_1 \Rightarrow \tau_2 \urcorner = \text{arr}\ \ulcorner \tau_1 \urcorner\ \ulcorner \tau_2 \urcorner \qquad \begin{array}{l} \text{tp} \ : \text{type} \\ \text{arr} : \text{tp} \to \text{tp} \to \text{tp} \end{array}$$

We use $\Delta$ to denote Mini-ML typing assumptions for variables in the typing judgment $\Delta \triangleright e : \tau$. This is an example of a so-called *hypothetical judgment*, which means that higher-order representation techniques can be applied to great advantage.

$$\frac{\Delta, x : \tau_1 \triangleright e : \tau_2}{\Delta \triangleright \mathbf{lam}\ x.\, e : \tau_1 \Rightarrow \tau_2}\ \mathsf{ofLam} \qquad \frac{\Delta \triangleright e_1 : \tau_2 \Rightarrow \tau_1 \qquad \Delta \triangleright e_2 : \tau_2}{\Delta \triangleright e_1 @ e_2 : \tau_1}\ \mathsf{ofApp}$$

$$\frac{\Delta, x : \tau \triangleright e : \tau}{\Delta \triangleright \mathbf{fix}\ x.\, e : \tau}\ \mathsf{ofFix}$$

A derivation of a judgment $\Delta \triangleright e : \tau$ is represented by canonical LF objects of type of $\ulcorner e \urcorner\ \ulcorner \tau \urcorner$ in an LF context $\ulcorner \Delta \urcorner$ which represents the Mini-ML typing assumptions in $\Delta$ by corresponding LF typing assumption.

$$
\begin{array}{ll}
\mathsf{of} & : \mathrm{exp} \to \mathrm{tp} \to \mathrm{type} \\
\mathsf{of\_lam} & : \mathsf{of}\ (\mathsf{lam}\ \mathbf{E})\ (\mathsf{arr}\ \mathbf{T_1}\ \mathbf{T_2}) \\
& \quad \leftarrow\ (\Pi x : \mathrm{exp}.\ \mathsf{of}\ x\ \mathbf{T_1} \to \mathsf{of}\ (\mathbf{E}\ x)\ \mathbf{T_2}) \\
\mathsf{of\_app} & : \mathsf{of}\ (\mathsf{app}\ \mathbf{E_1}\ \mathbf{E_2})\ \mathbf{T_1} \\
& \quad \leftarrow\ \mathsf{of}\ \mathbf{E_1}\ (\mathsf{arr}\ \mathbf{T_2}\ \mathbf{T_1}) \\
& \quad \leftarrow\ \mathsf{of}\ \mathbf{E_2}\ \mathbf{T_2} \\
\mathsf{of\_fix} & : \mathsf{of}\ (\mathsf{fix}\ \mathbf{E})\ \mathbf{T} \\
& \quad \leftarrow\ (\Pi x : \mathrm{exp}.\ \mathsf{of}\ x\ \mathbf{T} \to \mathsf{of}\ (\mathbf{E}\ x)\ \mathbf{T})
\end{array}
$$

This representation is once again a compositional bijection between Mini-ML typing derivations and canonical LF objects. Compositionality here means that if we have a derivation $\mathcal{P}'$ of $\Delta \rhd e' : \tau'$ and a derivation $\mathcal{P}$ of $\Delta, x{:}\tau' \rhd e : \tau$, then $\mathcal{P}[\mathcal{P}'/X]$ is a derivation of $\Delta \rhd e[e'/x] : \tau$ and $\ulcorner \mathcal{P}[\mathcal{P}'/X] \urcorner = \ulcorner \mathcal{P} \urcorner [\ulcorner \mathcal{P}' \urcorner / X]$. Here, $X$ is a derivation variable labelling the assumption $x{:}\tau'$.

Thus, compositionality of the encoding gives us the the following substitution lemma "for free", since it can be represented simply by substitution in LF whose correctness has been proven once and for all for the logical framework.

**Lemma 1 (Substitution).** *For all contexts $\Delta$,*
*if $\Delta \rhd e' : \tau'$ and $\Delta, x : \tau' \rhd e : \tau$ then $\Delta \rhd e[e'/x] : \tau$.*

A substitution lemma of this or a similar form is an important ingredient in many theorems in logic (e.g., cut-elimination, normalization, or the Church-Rosser theorem) or the theory of programming languages (e.g., subject reduction or type preservation).

As a point of reference, we now give an informal proof of type preservation for Mini-ML, which is also *exactly* the proof found automatically rendered into informal notation. We write $\mathcal{D} :: J$ if $\mathcal{D}$ is a derivation of a judgment $J$ to avoid two-dimensional notation. Besides structural induction, we also use *inversion*, when the shape of the conclusion determines the inference rule which must have been applied last.

**Theorem 1 (Type preservation).** *For all expressions $e, v$, types $\tau$, and derivations $\mathcal{D} :: e \hookrightarrow v$ and $\mathcal{P} :: \cdot \rhd e : \tau$, there exists a derivation $\mathcal{Q} :: \cdot \rhd v : \tau$.*

*Proof.* By structural induction over $\mathcal{D}$.

**Case:** $\mathcal{D} = \dfrac{}{\mathbf{lam}\ x.\, e \hookrightarrow \mathbf{lam}\ x.\, e}\ \mathsf{evLam}$.

This case is immediate because we can use $\mathcal{P}$ for $\mathcal{Q}$.

**Case:** $\mathcal{D} = \dfrac{e_1 \hookrightarrow \mathbf{lam}\ x.\, e_1' \quad (\mathcal{D}_1) \qquad e_2 \hookrightarrow v_2 \quad (\mathcal{D}_2) \qquad e_1'[v_2/x] \hookrightarrow v \quad (\mathcal{D}_3)}{e_1 @ e_2 \hookrightarrow v}\ \mathsf{evApp}$.

We have $\mathcal{P} :: \cdot \rhd e_1 @ e_2 : \tau$. By inversion, we obtain $\mathcal{P}_1 :: \cdot \rhd e_1 : \tau_2 \Rightarrow \tau$ and $\mathcal{P}_2 :: \cdot \rhd e_2 : \tau_2$ for some $\tau_2$. By induction hypothesis on $\mathcal{D}_1$ and $\mathcal{P}_1$ we obtain $\mathcal{Q}_1 :: \cdot \rhd \mathbf{lam}\ x.\, e_1' : \tau_2 \Rightarrow \tau$. Another inversion step gives us that $\mathcal{Q}_1' ::$

$x : \tau_2 \triangleright e_1' : \tau$. A second appeal to the induction hypothesis on $\mathcal{D}_2$ and $\mathcal{P}_2$ yields that $\mathcal{Q}_2 :: \cdot \triangleright v_2 : \tau_2$. The application of the substitution lemma gives us a $\mathcal{Q}_2' :: \cdot \triangleright e_1'[v_2/x] : \tau$. Finally, applying the induction hypothesis to $\mathcal{D}_3$ and $\mathcal{Q}_2'$ results in the required derivation $\mathcal{Q}_3 :: \cdot \triangleright v : \tau$.

**Case:** $\mathcal{D} = \dfrac{\begin{array}{c}\mathcal{D}_1 \\ e[\mathbf{fix}\ x.\, e/x] \hookrightarrow v\end{array}}{\mathbf{fix}\ x.\, e \hookrightarrow v}\ \mathsf{evFix}$.

We have $\mathcal{P} :: \cdot \triangleright \mathbf{fix}\ x.\, e : \tau$. By inversion we immediately obtain a $\mathcal{P}_1 :: x : \tau \triangleright e : \tau$. Applying the substitution lemma yields $\mathcal{P}_1' :: \cdot \triangleright e[\mathbf{fix}\ x.\, e/x] : \tau$. By applying the induction hypothesis to $\mathcal{D}_1$ and $\mathcal{P}_1'$ we obtain the desired derivation $\mathcal{Q}_1 :: \cdot \triangleright v : \tau$.

How can we render this informal proof in a formal system? The proof is clearly constructive and contains a method for constructing a derivation $\mathcal{Q} :: \cdot \triangleright v : \tau$ from $\mathcal{D} :: e \hookrightarrow v$ and $\mathcal{P} :: \cdot \triangleright e : \tau$. By an extension of the Curry-Howard correspondence one might hope to represent this as an LF function

$$\text{tps} : \text{ev}\ \mathbf{E}\ \mathbf{V} \rightarrow \text{of}\ \mathbf{E}\ \mathbf{T} \rightarrow \text{of}\ \mathbf{V}\ \mathbf{T}.$$

In fact, if we could exhibit a total function of this type, we would know that type preservation holds. Unfortunately, such a function does not exist in LF, since it would have to be defined by primitive recursion over its first argument, and primitive recursion is not available in LF. Moreover, straightforward attempts to add primitive recursion render the higher-order representations inadequate. This is discussed in detail in [DPS97].

Instead we define a meta-logic for LF in which it is possible to express and prove:

> For all closed LF objects $E : \text{exp}$, $V : \text{exp}$, $T : \text{tp}$, $D : \text{ev}\ E\ V$, and $P : \text{of}\ E\ T$ there exists a closed LF object $Q : \text{of}\ V\ T$.

By the adequacy of the encodings, the existence of such an LF object $Q$ means the existence of a typing derivation $\mathcal{Q}$ of $\cdot \triangleright v : \tau$, where $\ulcorner v \urcorner = V$ and $\ulcorner \tau \urcorner = T$, guaranteeing the type preservation property for Mini-ML.

## 3 Meta-Logic $\mathcal{M}_2$

The purpose of the meta-logic $\mathcal{M}_2$ is formal reasoning about properties of LF signatures, with the goal of automating the proof of such properties. Since LF signatures implement object languages and their semantics, this provides for automatic proofs of properties of logics and programming languages.

$\mathcal{M}_2$ is a restricted constructive first-order logic where quantifiers range over closed LF objects constructed over a given signature $\Sigma$. Its formal definition is a sequent calculus endowed with realizing proof terms.

The formal system of $\mathcal{M}_2$ in its full generality is rather complex. We therefore present here only a restriction of $\mathcal{M}_2$, where pattern matching subjects must be of atomic type. For a complete presentation of the meta logic we refer the interested reader to the detailed forthcoming technical report [Sch98]. We introduce $\mathcal{M}_2$ in four steps: in Section 3.1 we describe a constructive logic over LF which we augment

$$\frac{\Gamma \vdash \sigma : \Gamma_1 \qquad \Gamma; \Delta_1, \forall \Gamma_1.\, F_1, \Delta_2, F_1[\sigma] \vdash F_2}{\Gamma; \Delta_1, \forall \Gamma_1.\, F_1, \Delta_2 \vdash F_2}\,\forall \mathsf{L} \qquad \frac{\Gamma, \Gamma_1; \Delta \vdash F}{\Gamma; \Delta \vdash \forall \Gamma_1.\, F}\,\forall \mathsf{R}^*$$

$$\frac{\Gamma, \Gamma_1; \Delta_1, \exists \Gamma_1.\, \top, \Delta_2 \vdash F}{\Gamma; \Delta_1, \exists \Gamma_1.\, \top, \Delta_2 \vdash F}\,\exists \mathsf{L}^* \qquad \frac{\Gamma \vdash \sigma : \Gamma_1}{\Gamma; \Delta \vdash \exists \Gamma_1.\, \top}\,\exists \mathsf{R}$$

* Eigenvariable condition: $\vdash \Gamma, \Gamma_1$ ctx

**Fig. 1.** Constructive Logic Over LF

with proof terms in Section 3.2 and by well-founded recursion in Section 3.3. In Section 3.4 we introduce definition by cases and in Section 3.5 we establish the necessary meta-theoretic properties of $\mathcal{M}_2$ which make it an appropriate meta-logic.

### 3.1 A Constructive Sequent Calculus Over LF

Formulas in $\mathcal{M}_2$ have the form $\forall x_1 : A_1. \ldots \forall x_n : A_n.\, \exists y_1 : B_1. \ldots \exists y_m : B_m.\, \top$ (which we write as $\forall \Gamma_1.\, \exists \Gamma_2.\, \top$, where $\Gamma_1 = x_1 : A_1, \ldots, x_n : A_n$ and $\Gamma_2 = y_1 : B_1, \ldots, y_m : B_m$). Here all $A_i$ and $B_j$ are LF types, and for a formula to be well-formed the combined context $\Gamma_1, \Gamma_2$ must be a valid LF context.

While this may not seem very expressive, it is sufficient for many theorems in the realm of logic and the theory of programming languages we have examined, since other connectives (such as disjunction) and even more complex quantifier alternations can be incorporated at the level of LF. The main limitation is that the quantifiers range only over closed LF objects of the given types; a generalization is the subject of current research.

$$\begin{array}{ll} \textit{Formulas} & F ::= \forall \Gamma_1.\, \exists \Gamma_2.\, \top \\ \textit{Assumptions} & \Delta ::= \cdot \mid \Delta, F \end{array}$$

The judgment for provability for this sequent calculus is $\Gamma; \Delta \vdash F$, where the LF context $\Gamma$ makes all Eigenvariables explicit with their type. The judgment is also indexed by an LF signature $\Sigma$ which we suppress for the sake of brevity.

The rules for the judgment are in the form of a sequent calculus and defined in Figure 1. Because of the way our search engine actually works and the restriction on quantifier alternations, it is convenient to instantiate all quantified variables of the same kind simultaneously by means of a substitution $\sigma$ explained below. This applies to the $\forall \mathsf{L}$ and $\exists \mathsf{R}$ rules, where the latter also incorporates an axiom rule for $\top$.

$$\textit{Substitutions}\ \sigma ::= \cdot \mid \sigma, M/x$$

Valid substitutions map variables in a context $\Gamma'$ to valid objects in a context $\Gamma$. This judgment is written as $\Gamma \vdash \sigma : \Gamma'$ and defined by the following inference rules, which guarantees that dependencies are properly respected.

$$\frac{\Gamma \vdash \sigma : \Gamma_1 \qquad \Gamma; \Delta_1, \mathbf{x} \in \forall \Gamma_1.\, F_1, \Delta_2, \mathbf{y} \in F_1[\sigma] \vdash P \in F_2}{\Gamma; \Delta_1, \mathbf{x} \in \forall \Gamma_1.\, F_1, \Delta_2 \vdash \text{let } \mathbf{y} = \mathbf{x}\ \sigma \text{ in } P \in F_2}\ \forall\mathsf{L} \qquad \frac{\Gamma, \Gamma_1; \Delta \vdash P \in F}{\Gamma; \Delta \vdash \Lambda\Gamma_1.\, P \in \forall \Gamma_1.\, F}\ \forall\mathsf{R}^*$$

$$\frac{\Gamma, \Gamma_1; \Delta_1, \mathbf{x} \in \exists \Gamma_1.\, \top, \Delta_2 \vdash P \in F}{\Gamma; \Delta_1, \mathbf{x} \in \exists \Gamma_1.\, \top, \Delta_2 \vdash \text{split } \mathbf{x} \text{ as } \langle \Gamma_1 \rangle \text{ in } P \in F}\ \exists\mathsf{L}^* \qquad \frac{\Gamma \vdash \sigma : \Gamma_1}{\Gamma; \Delta \vdash \langle \sigma \rangle \in \exists \Gamma_1.\, \top}\ \exists\mathsf{R}$$

* Eigenvariable condition: $\vdash \Gamma, \Gamma_1$ ctx

**Fig. 2.** $\mathcal{M}_2$ without recursion or pattern matching

$$\frac{}{\Gamma \vdash \cdot : \cdot}\ \mathsf{subId} \qquad \frac{\Gamma \vdash \sigma : \Gamma' \qquad \Gamma \vdash M : A[\sigma]}{\Gamma \vdash (\sigma, M/x) : (\Gamma', x : A)}\ \mathsf{subDot}$$

When $\Gamma \vdash \sigma : \Gamma'$ and $\Gamma' \vdash M : A$ then we write $M[\sigma]$ for the result of applying the substitution $\sigma$ to $M$, and similarly for types, contexts, etc. The result will satisfy $\Gamma \vdash M[\sigma] : A[\sigma]$. This is also reflected in our implementation of the system, which employs dependently typed explicit substitutions.

We write $\mathrm{id}_\Gamma$ for the identity substitution on $\Gamma$ satisfying $\Gamma \vdash \mathrm{id}_\Gamma : \Gamma$. The formulation of the calculus incorporates the structural rules: weakening is implicit in $\exists\mathsf{R}$, contraction and exchange are implicit in the left rules $\forall\mathsf{L}$ and $\exists\mathsf{L}$.

The type preservation theorem (Theorem 1) can be expressed in $\mathcal{M}_2$ as

$$\forall E : \text{exp}, V : \text{exp}, T : \text{tp}, D : \text{ev } E\ V, P : \text{of } E\ T.\, \exists Q : \text{of } V\ T.\, \top$$

Using our implicit index variable convention, this may be abbreviated to $\forall D : \text{ev } \mathbf{E}\ \mathbf{V}, P : \text{of } \mathbf{E}\ \mathbf{T}.\, \exists Q : \text{of } \mathbf{V}\ \mathbf{T}.\, \top$.

### 3.2 Adding Proof Terms

$\mathcal{M}_2$ extends the calculus in Figure 1 by proof terms denoted by $P$. Assumptions now have the form $\mathbf{x} \in F$, where $\mathbf{x}$ is a new kind of variable ranging over proof terms (which must not be confused with LF variables). The provability judgment then has the form $\Gamma; \Delta \vdash P \in F$ where $P$ is a proof term for $F$ and the assumptions in $\Delta$ are now labelled.

The result of endowing all inference rules from Figure 1 with proof terms is given in Figure 2. Every inference rule introduces a new proof term constructor. The proof term for the $\exists\mathsf{R}$-rule is simply the witnessing substitution $\langle \sigma \rangle$, and the proof term for the $\forall\mathsf{L}$ rule is 'let $\mathbf{y} = \mathbf{x}\ \sigma$ in $P$', where $\mathbf{x}\ \sigma$ stands for the "application" of $\mathbf{x}$ to the LF-objects in $\sigma$.

The system presented in Figure 2 is the core of the meta logic $\mathcal{M}_2$ for LF. In the next two sections we strengthen $\mathcal{M}_2$ by introducing well-founded recursion and definition by cases for closed LF objects. This will allow us to represent many proofs by structural induction, case distinction, and inversion in $\mathcal{M}_2$. A further extension of $\mathcal{M}_2$ is the introduction of conjunction as formula and corresponding

pairs as proof terms. These constructs are required for the representation of mutual inductive proofs, but omitted here for the sake of brevity (see [Sch98]).

### 3.3 Adding Recursion

The recursion operator $\mu \mathbf{x} \in F.\, P$ is the standard fixed point operator at the level of proof terms with the following introduction rule.

$$\frac{\Gamma; \Delta, \mathbf{x} \in F \vdash P \in F}{\Gamma; \Delta \vdash \mu \mathbf{x} \in F.\, P \in F}\ \mathsf{fix} \quad (\text{where } \mu \mathbf{x} \in F.\, P \text{ terminates in } \mathbf{x})$$

It is obvious that a proof term represents a total function only if it terminates independently of the arguments it is applied to. Thus the side condition on the rule. For termination we use arbitrary lexicographic extensions of the subterm ordering on LF objects described in [RP96], all of which are well-founded orderings and easy to check due to the restricted nature of our meta-logic.

### 3.4 Adding Case Analysis

The context of Eigenvariables $\Gamma$ in the judgment $\Gamma; \Delta \vdash P \in F$ represents all LF variables which might occur free in the proof term $P$. Because of the assumption that proof terms are only applied to closed LF objects, all variables in $\Gamma$ stand for closed LF objects. It is therefore possible to determine all possible cases for the top-level constructor of such objects.

Assume we would like to distinguish all possible cases for a a given LF variable $x$ of type $A$ declared in $\Gamma$. For simplicity, we assume that $A$ is a base type, even though in the full system [Sch98] function types are also permitted which is needed, for example, in the proof of the deduction theorem. The top-level structure of a closed canonical term of base type is always $c\, x_1 \ldots x_n$, where $x_i$ are new variables[3]. If $c$ has type $\Pi x_1 : A_1. \ldots \Pi x_n : A_n.\, B$, then this is a possible candidate for the shape of $x : A$ if $B$ unifies with $A$.

This idea is very similar to the realization of partial inductive definitions and definitional reflection [SH93], except that dependent types can eliminate more cases statically. Also, because of the higher-order nature of the term language, we need to deal with the undecidability of the full higher-order unification problem. Our solution is to restrict the analysis of possible cases to Miller's higher-order patterns, generalized to the setting of dependent types [Pfe91]. However, we do not restrict our system to patterns statically, since this would preclude, for example, the direct appeal to substitution or substitution lemmas at the level of LF. Instead, we simply rule out definition by cases where determining the possible cases would require unification beyond the pattern fragment.

Formally, we extend the language of proof terms by a case construct.

$$\begin{array}{ll} \textit{Patterns} & R ::= \Gamma'; \Gamma'' \triangleright M \\ \textit{Cases} & \Omega ::= \cdot \mid \Omega, R \mapsto P \\ \textit{Proof Terms} & P ::= \ldots \mid \text{case } x \text{ of } \Omega \end{array}$$

[3] technically, they may need to be $\eta$-expanded

$$\frac{}{\Gamma_1; x : B_x; \Gamma_2; \Delta \vdash_{\cdot} \cdot \in F}\ \textsf{sigempty}$$

$$\frac{\Gamma_1; x : B_x; \Gamma_2; \Delta \vdash_{\Sigma} \Omega \in F}{\Gamma_1; x : B_x; \Gamma_2; \Delta \vdash_{\Sigma, c:\Pi\Gamma_c.\, B_c} \Omega \in F}\ \textsf{signonuni}\quad (B_x, B_c \text{ do not unify})$$

$$\frac{\Gamma', \Gamma_2[\sigma]; \Delta[\sigma'] \vdash P \in F[\sigma'] \qquad \Gamma_1; x : B_x; \Gamma_2; \Delta \vdash_{\Sigma} \Omega \in F}{\Gamma_1; x : B_x; \Gamma_2; \Delta \vdash_{\Sigma, c:\Pi\Gamma_c.\, B_c} \Omega, (\Gamma'; \Gamma_2[\sigma] \triangleright (c\ \Gamma_c)[\sigma] \mapsto P) \in F}\ \textsf{siguni}$$

$$\Gamma' \vdash \sigma = \text{mgu}\ (B_x \doteq B_c, x \doteq c\ \Gamma_c) : (\Gamma_1, x : B_x, \Gamma_c)$$
$$\Gamma', \Gamma_2[\sigma] \vdash \sigma' = (\sigma, \text{id}_{\Gamma_2}) : (\Gamma_1, x : B_x, \Gamma_c, \Gamma_2)$$

**Fig. 3.** Selection rules for $\Gamma_1; x : B_x; \Gamma_2; \Delta \vdash_{\Sigma} \Omega \in F$

The objects $M$ in patterns are strongly restricted by the rules which check valid patterns; usually it will be a constant applied to variable arguments, but because of dependencies, it might be more complex than that. Contexts $\Gamma'$ and $\Gamma''$ are separated for technical reasons, where $\Gamma'$ contains the variables which will be instantiated when the case subject is matched against the object $M$, while $\Gamma''$ contains those variables which will not be instantiated (although their types could still be instantiated). We always have that $\Gamma', \Gamma'' \vdash M : A'$ for some type $A'$ which is equal to or more specific than the type $A$ of the case subject $x$.

The judgment for checking the validity of a case construct has the form $\Gamma_1; x : B_x; \Gamma_2; \Delta \vdash_{\Sigma} \Omega \in F$, where we maintain the invariant that $B_x$ depends on all variables in $\Gamma_1$, which therefore collects the variables which will be instantiated by pattern matching. By using the limited permutation properties of LF [HHP93] this can always be established. The following rule then completes the definition of derivability in $\mathcal{M}_2$.

$$\frac{\Gamma(x) = B_x \qquad \Gamma_1; x : B_x; \Gamma_2; \Delta \vdash_{\Sigma} \Omega \in F}{\Gamma; \Delta \vdash \text{case } x \text{ of } \Omega \in F}\ \textsf{case}$$

where $\Gamma_1, x : B_x, \Gamma_2$ is a valid permutation of $\Gamma$, and $B_x$ depends on all variables in $\Gamma_1$. The judgment $\Gamma_1; x : B_x; \Gamma_2; \Delta \vdash_{\Sigma} \Omega \in F$ selects all constants from $\Sigma$ which are possible shapes of a closed object of type $B_x$. The rules for the judgment are given in Figure 3. This judgment iterates through the signature $\Sigma$, trying each constant $c$ in turn. If the target type $B_c$ unifies with the type $B_x$ of the case subject, a new case is added to $\Omega$. Otherwise, $c$ cannot be a top-level constructor for a closed term $M$ of type $B_x$ and no case is added.

We use $\Pi\Gamma_c.\, B_c$ to describe the combination of several abstractions and assume $B_c$ is atomic. We write $c\ \Gamma_c$ for the result of applying $c$ to the variables in $\Gamma_c$ in order, which gives us the most general form of a term in canonical form whose head is $c$.

### 3.5 Properties of $\mathcal{M}_2$

The principal property of $\mathcal{M}_2$ which justifies its use for reasoning about closed LF objects is the following.

**Theorem 2.** *If* $\cdot;\cdot \vdash P \in \forall\Gamma_1.\,\exists\Gamma_2.\,\top$ *is derivable for some* $P$*, then for every closed substitution* $\cdot \vdash \sigma_1 : \Gamma_1$ *there exists a substitution* $\cdot \vdash \sigma_2 : \Gamma_2[\sigma_1]$.

As indicated at the end of Section 2, this, together with the adequacy of the encodings guarantees the meta-theoretic properties of the object languages we can express in $\mathcal{M}_2$. Note that this is different from and in many ways simpler than a full cut-elimination result for $\mathcal{M}_2$.

The proof of this central property is non-trivial. What we show is that the realizing proof terms $P$ can be used to calculate $\sigma_2$ from $\sigma_1$. For this purpose, we define a small-step, call-by-value, continuation-passing operational semantics for proof terms $P$ with explicit environments and establish the following three properties.

**Type Soundness:** each step in the evaluation of $P$ preserves types and provability in $\mathcal{M}_2$ (the critical idea here is the use of explicit environments rather than substitution, since substitution may render some branches in a case distinction inapplicable, thereby invalidating it),

**Progress:** at each step we either have a final result, or a rule in the operational semantics applies (the critical step here shows that all possibilities are covered in a definition by cases);

**Termination:** all reduction sequences terminate (the critical step here uses well-foundedness restriction on recursion).

Unfortunately, space does not permit us to show the details of this proof or even the definition of the operational semantics. The interested reader is referred to [Sch98].

## 4 Twelf

Twelf is a theorem prover for LF which directly implements the meta-logic $\mathcal{M}_2$ (including mutual induction and distinction by cases over functions). It provides an interactive mode for experimentation and an automatic mode in which only the theorem and the termination ordering are specified. The deduction engine implements only a few elementary operations, which are used to formalize the four important basic proof principles: inversion (that is, determining all possible shapes of an LF object from its type), direct proofs (that is, direct construction of an LF object), and appeals to the induction hypothesis. The interactive mode also supports lemma application.

### 4.1 Elementary Operations

We discuss the elementary operations using the proof of the type preservation theorem as an example. The initial goal

$$\forall D : \text{ev}\ \mathbf{E}\ \mathbf{V}, P : \text{of}\ \mathbf{E}\ \mathbf{T}.\,\exists Q : \text{of}\ \mathbf{V}\ \mathbf{T}.\,\top$$

and the induction principle (induction over $D$) are specified by the user. Twelf uses only outermost induction, so there is an implicit application of the recursion rule before the real proof process is started. Then Twelf generates subgoals by applying its elementary operations until all subgoals are solved, using the strategy described in the next section.

The most basic step is directly constructing a substitution for the existentially quantified variables using the constants from the signature and the universally quantified variables. We call this step *filling*. It is basically a straightforward, iterative-deepening search over an LF signature and is derived from a related implementation of resolution for logic programming [Pfe94].

Such a substitution does not exist for the current state, so the system applies the *splitting* operation which performs a case analysis as part of the induction principle: it inspects the signature for possible "shapes" of $D$ and generates in our example a list of three subgoals, automatically updating the context of universal variables.

**Subgoal:** ev_lam:

$$\forall P : \text{of } (\text{lam } \mathbf{E})\ \mathbf{T}.\ \exists Q : \text{of } (\text{lam } \mathbf{E})\ \mathbf{T}.\ \top$$

**Subgoal:** ev_app:

$$\forall D_3 : \text{ev } (\mathbf{E_1'}\ \mathbf{V_2})\ \mathbf{V},\ D_2 : \text{ev } \mathbf{E_2}\ \mathbf{V_2}, D_1 : \text{ev } \mathbf{E_1}\ (\text{lam } \mathbf{E_1'}),$$
$$P : \text{of } (\text{app } \mathbf{E_1}\ \mathbf{E_2})\ \mathbf{T}.\ \exists Q : \text{of } \mathbf{V}\ \mathbf{T}.\ \top$$

**Subgoal:** ev_fix:

$$\forall D_1 : \text{ev } (\mathbf{E}\ (\text{fix } \mathbf{E}))\ \mathbf{V}, P : \text{of } (\text{fix } \mathbf{E})\ \mathbf{T}.\ \exists Q : \text{of } \mathbf{V}\ \mathbf{T}.\ \top$$

For the sake of brevity, we skip the discussion of the first two subgoals, and continue with the third. Inversion is now applied to $P$ in the informal proof. In Twelf inversion is realized by another splitting operation which generates only one subgoal in this example. The other two potential cases (of_lam, of_app) are rejected by Twelf, because their types are incompatible with the type of $P$. This leaves the subgoal

$$\forall D_1 : \text{ev } (\mathbf{E}\ (\text{fix } \mathbf{E}))\ \mathbf{V},\ P_1 : \Pi x : \text{exp.}\ \text{of } x\ \mathbf{T} \to \text{of } (\mathbf{E}\ x)\ \mathbf{T}.\ \exists Q : \text{of } \mathbf{V}\ \mathbf{T}.\ \top.$$

Note, that in this goal the variable $P_1$ is functional and represents a hypothetical derivation.

It is now possible to appeal to the induction hypothesis in an operation we call *recursion*. The termination condition of the fix-rule requires that it is only applied to a term smaller than $D = \text{ev\_fix } D_1$. According to the termination ordering in [RP96] there is only one possibility, namely $D_1$.

We cannot appeal to the induction hypothesis without providing a typing derivation as second argument. Formally, the representation of this derivation must be of type 'of (**E** (fix **E**)) **T**'. Among others, Twelf constructs the term '$P_1$ (fix **E**) (of_fix $P_1$)' which represents the result of applying the substitution lemma (Lemma 1) as used in the proof of Theorem 1. Recursion then yields the following subgoal.

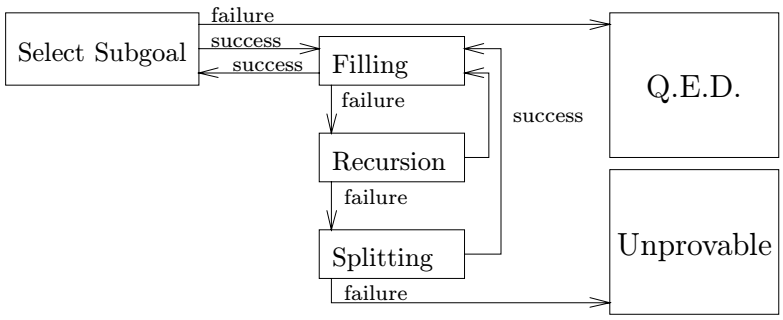


**Fig. 4.** Proof strategy

$$\forall D_1 : \text{ev}\ (\mathbf{E}\ (\text{fix}\ \mathbf{E}))\ \mathbf{V},\ P_1 : \Pi x : \text{exp. of}\ x\ \mathbf{T} \to \text{of}\ (\mathbf{E}\ x)\ \mathbf{T},\ Q_2 : \text{of}\ \mathbf{V}\ \mathbf{T}.$$
$$\exists Q_1 : \text{of}\ \mathbf{V}\ \mathbf{T}.\ \top$$

Twelf is now able to determine in a simple filling step that $Q_2$ is a possible instantiation for $Q_1$, thereby completing the ev_fix-branch of the proof. The other two branches can be solved similarly. Twelf then reports the proof term (currently shown in a more readable relational notation as an LF signature, rather than in the functional notation used to define $\mathcal{M}_2$).

### 4.2 Strategy

The proof strategy of Twelf is a simple combination of the three elementary operations. But each operation must be applied with care because they are inherently expensive in time and space. In particular, we completely avoid backtracking except during the filling step which is implemented as an iterative deepening depth-first search. Splitting and filling (and sometimes also recursion) use unification to analyze cases and to select constants. Recursion triggers the calculation of possible recursion arguments according to the termination ordering [RP96].

For a given theorem and induction principle, Twelf attempts to construct a derivation in $\mathcal{M}_2$ using the strategy sketched in Figure 4. There is a global store of yet to be proven subgoals, initialized with the formula representing the theorem. Once the automated proof process is started, the strategy *activates* a subgoal and tries to apply a filling operation.

**Filling:** The filling operation corresponds to an application of the $\exists$R-rule: it is applicable, if a substitution instantiating all existentially quantified variables can be constructed. Because index variables occur in the types of non-index variables, it is already enough to determine instantiations for all non-index variables (see Section 2). In general, infinitely many substitutions must be examined, but since our strategy is parametrized by a number to limit the depth of the search space,

the employed search algorithm is incomplete but will always terminate (even though failure is sometimes very slow).

If Twelf succeeds in constructing the substitution, the current subgoal is successfully completed and the next subgoal is selected if available, otherwise Twelf stops (Q.E.D.). If Twelf fails to construct the desired substitution the strategy tries to apply the recursion operation.

**Recursion:** The recursion operation corresponds to an application of the $\forall$L-rule, immediately followed by an application of the $\exists$L-rule: Twelf generates all possible recursive calls by constructing substitutions which correspond to the arguments of the recursive call. These substitutions must satisfy the side condition of the fix-rule. Because lower-ranked arguments in a lexicographic termination order actually may increase in size, there are potentially infinitely many different ways to appeal to the induction hypothesis. Limit the search space depth for such arguments guarantees that this step always terminates. If no new recursive calls can be generated, the strategy tries to apply the splitting operation.

**Splitting:** The splitting operation corresponds to an application of the case-rule. It may be applicable if there is at least one universally quantified variable. Among all those variables, a non-index variable is selected (see Section 2) whose type is then used to determine all possible cases (sigempty, siguni, and signonuni). For each case, a new subgoal is created and one of them is activated. Twelf then continues and tries then to apply the filling operation to the newly activated subgoal.

Among all universally quantified variables Twelf selects the one which generates the least number of subgoals first (which could be zero if a variable has a dependent type which does not unify with any constructor type—the subgoal succeeds immediately in that case). This heuristic works surprisingly well in all of our examples, we leave a refinement of this heuristic to future research. If no universally quantified variables are available, the strategy stops, reporting that the theorem is not provable with the current filling depth. It is also possible that splitting is not applicable because the types of all universally quantified variables fall outside Miller's pattern fragment. In this case Twelf reports that no proof can be found automatically.

### 4.3 Experimental Results

Twelf has been successfully employed to prove several non-trivial theorems automatically. In Figure 5 we give an overview over the experimental results from the areas of programming languages and logics. "Induction" states how many theorems have been simultaneously proven and "limit" the maximal search depth for LF objects. The left column of the "line" counter shows the length of the proof when implicit arguments are suppressed, the right column the natural length. All timings have been taken on a 300 Mhz Pentium-II machine, running Linux 2.30, New Jersey SML 109.26.1, and Twelf 5.0.

In the area of Mini-ML Twelf was used to prove value soundness, i.e. that if $e \hookrightarrow v$ then $v$ is a value, and type preservation (Theorem 1). Moreover, it was used

| Experiment | Induction | Limit | Recursive Calls | Lines (no imp) | Lines (imp) | Time |
|---|---|---|---|---|---|---|
| Value Soundness | 1 | 6 | 11 | 23 | 50 | 0.3 sec |
| Type Preservation | 1 | 6 | 15 | 35 | 106 | 1.9 sec |
| Completeness: Semantics | 1 | 20 | 16 | 100 | 316 | 8.3 sec |
| Proof equivalence: Completeness | 1 | 6 | 16 | 29 | 484 | 17.7 sec |
| Proof equivalence: Soundness | 1 | 6 | 16 | 29 | 506 | 17.1 sec |
| Deduction Theorem | 1 | 6 | 2 | 7 | 26 | 1.6 sec |
| Completeness: Logic Pogramming | 3 | 4 | 8 | 20 | 41 | 0.5 sec |
| Cartesian Closed Categories | 1 | 6 | 5 | 15 | 32 | 120.2 sec |

**Fig. 5.** Experimental results

to show the completeness of a continuation stack machine with respect to a natural semantics for Mini-ML. In the proof, Twelf constructed a mapping from Mini-ML evaluation derivations to computation traces of the abstract machine. But it could not verify the soundness direction, because the proof requires complete induction which is not supported by the current implementation. Nonetheless, Twelf could prove that the soundness proof (coded by hand) always resembles the completeness proof in structure. For this proof, it constructed a function mapping soundness proofs into completeness proofs and vice versa.

In the area of logic, Twelf was used to derive, among others, the proof of the deduction theorem for intuitionistic propositional logic using Hilbert's axiomatization which is used to translate pure functional programs into combinators. It also proved soundness and completeness of uniform derivations with respect to resolution for Horn-logic. From the area of category theory, it proved that Cartesian closed categories can be embedded into the simply-typed $\lambda$-calculus.

## 5 Related Work and Future Work

There have been many mechanized proofs of meta-theoretic properties of logics or programming languages in the literature (see the survey [Pfe96]). Most of these do not use techniques from logical frameworks, but represent the languages via standard inductive types and their semantics by inductively defined predicates. A popular choice for such encodings are de Bruijn indices, since they eliminate the problem of $\alpha$-conversion from consideration. However, various lemmas regarding substitution must still be shown and used, which severely limits the degree of automation which can be achieved. Most closely related to our own efforts in this area is the work on ALF [Mag95], since ALF also employ dependently typed pattern matching and termination orderings, although without the benefits of higher-order abstract syntax.

At the opposite extreme we have work on representing meta-theoretic proofs as relations in LF, which leaves the progress and termination properties above to an external check on relations (only type soundness is directly guaranteed by type-checking in LF) [PR92]. In this approach, we have no automation besides

type reconstruction. The expressive power of LF makes this feasible, but it remains tedious.

Most closely related to our approach is work by McDowell and Miller [MM97] who also define a meta-logic $FO\lambda^{\Delta IN}$ for a logical framework (hereditary Harrop formulas) and then reason in the meta-logic. Their approach is based entirely on simple types and does not incorporate proof terms, which makes it less suitable for automation. Moreover, in order to establish consistency for their meta-logic, they limit induction to natural numbers, which also complicates automation. In fact, their implementation based on the Pi proof editor [Eri94] is entirely interactive. On the other hand, McDowell has demonstrated the flexibility of his approach in his thesis [McD97] where he also treats a logical framework which incorporates linearity. Since the overall architecture is quite similar, this gives us confidence that our approach may be extended to a linear logical framework [CP96], which is planned in future work. We believe that the separation between logical framework and meta-logic, and the separation between definition by cases and well-founded recursion are all critical ingredients in making this idea successful for even richer logical frameworks than LF.

While the set of theorems we can prove at present is already surprisingly rich, they are limited by three factors: (1) we do not attempt to automatically use lemmas, (2) only lexicographic extensions of subterm orderings are permitted to show termination, and (3) $\mathcal{M}_2$ does not support reasoning about *open* LF objects. We believe that (1) and (2) can be addressed by incorporating standard techniques from inductive theorem proving, efficiency improvements such as indexing, and simply allowing more complex termination orderings. Nonetheless, we have currently no plans for developing Twelf into a general-purpose theorem prover, because we feel that its present success owes mostly to its design as a special-purpose prover for properties of programming languages and logics. We are currently investigating how to incorporate ideas from schema-checking [Roh94] and primitive recursion over higher-order abstract syntax [DPS97] into our meta-logical framework in order to make progress on item (3), that is, allow reasoning over terms which may have free variables from certain "regular" contexts which arises in many practical examples.

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