5.9 Inverse Focusing

In the system presented so far the non-determinism in forward reasoning is still unacceptable, despite the use of subsumption. We can now analyze the rules in a way that is analogous to Chapter 4, taking advantage of inversion and focusing properties. This eliminates many derivations, significantly improving overall efficiency at a high level of abstraction. Similar optimizations have been proposed by Tammet [Tam96] and Mints [Min94], although the exact relationship between these system and the one presented below have yet to be investigated. The work here exploits Andreoli's observation [**?**] that focused derivations correspond to interpreting propositions as derived rules of inference. Much of this section, however, is speculative in the sense that no formal properties have been proven.

Our overall strategy is to restrict the inverse method further so that it can find only focused proofs.

Given a sequent $\Gamma \Longrightarrow A$, use ordinary backward reasoning to decompose $\cdot; \Gamma \stackrel{a}{\Longrightarrow} A; \cdot$ into a collection of subgoals, all of which have the form $\Delta; \cdot \stackrel{a}{\Longrightarrow} \cdot; R$. Note that this set is uniquely determined (modulo names of parameters that may be introduced). We prove each of the subgoals completely independently.

We call sequents of the form $\Delta; \cdot \stackrel{a}{\Longrightarrow} \cdot; R$ *stable sequents*. Recall that Δ consists only of left synchronous propositions (P, $A \supset B$, $\forall x.\ A$) and R only of right synchronous propositions (P, $A \vee B$, $\bot$, $\exists x.\ A$). In our version of the inverse method, we only generate stable sequents, so we write them as $\Delta \longrightarrow R$. Furthermore, instead of naming all subformulas, we only name the subformulas that could occur in stable sequents in the search for a proof of the given proposition.

As a first example, consider proving $(A \supset (B \supset C)) \supset ((A \wedge B) \supset C)$. Here, A, B, and C are considered atomic formulas. We begin by decomposing all top-level asynchronous connectives.

$$\begin{array}{l}
B, A, A \supset (B \supset C); \cdot \stackrel{a}{\Longrightarrow} \cdot; C \\
B, A; A \supset (B \supset C) \stackrel{a}{\Longrightarrow} \cdot; C \\
B; A \supset (B \supset C), A \stackrel{a}{\Longrightarrow} \cdot; C \\
\cdot; A \supset (B \supset C), A, B \stackrel{a}{\Longrightarrow} \cdot; C \\
\cdot; A \supset (B \supset C), A \wedge B \stackrel{a}{\Longrightarrow} \cdot; C \\
\cdot; A \supset (B \supset C), A \wedge B \stackrel{a}{\Longrightarrow} C; \cdot \\
\cdot; A \supset (B \supset C) \stackrel{a}{\Longrightarrow} (A \wedge B) \supset C; \cdot \\
\cdot; \cdot \stackrel{a}{\Longrightarrow} (A \supset (B \supset C)) \supset ((A \wedge B) \supset C); \cdot
\end{array}$$

Next we consider how a bottom-up proof search could proceed: it could focus on any of the three propositions in the premise, but not on the conclusion (since C is atomic).

Focusing on B or A succeeds only if the conclusion is B and A, respectively. This leads to the initial sequents

$$\begin{array}{l}
A \longrightarrow A \\
B \longrightarrow B
\end{array}$$

If we focus on $A \supset (B \supset C)$, in a situation where have some unknown Δ and R, the proof fragment would have to look as follows:

$$
\begin{array}{lll}
(1) & \Delta; \cdot \stackrel{a}{\Longrightarrow} \cdot; A & \\
(2) & \Delta; \cdot \stackrel{s}{\Longrightarrow} A; \cdot & \mathrm{blur}R\ 1 \\
(3) & \Delta; \cdot \stackrel{a}{\Longrightarrow} \cdot; B & \\
(4) & \Delta; \cdot \stackrel{s}{\Longrightarrow} B; \cdot & \mathrm{blur}R\ 3 \\
(5) & \Delta; C \stackrel{s}{\Longrightarrow} \cdot; R & \\
(6) & \Delta; B \supset C \stackrel{s}{\Longrightarrow} \cdot; R & \supset\mathrm{L}\ 5\ 4 \\
(7) & \Delta; A \supset (B \supset C) \stackrel{s}{\Longrightarrow} \cdot; R & \supset\mathrm{L}\ 6\ 2
\end{array}
$$

Here (1) and (3) are stable sequents, but what about (5)? The only rule that applies is init, for $C = R$. This means any use of focusing with the hypothesis $A \supset (B \supset C)$ will reduce the goal of proving C to the goal of proving A and B. Writing is as a derived rule (and changing it to the forward direction)

$$\frac{\Delta_1 \longrightarrow A \qquad \Delta_2 \longrightarrow B}{\Delta_1, \Delta_2, A \supset (B \supset C) \longrightarrow C}$$

Next we observe that B, A, and $A \supset (B \supset C)$ would occur in any (bottom-up) focused sequent that is part of the proof search tree. We call such assumptions *global* and do not explicitly write them in our sequents. As a result we have exactly two axioms and one rule of inference.

$$\cdot \longrightarrow A$$

$$\cdot \longrightarrow B$$

$$\frac{\Delta_1 \longrightarrow A \qquad \Delta_2 \longrightarrow B}{\Delta_1, \Delta_2 \longrightarrow C}$$

The overall goal is to prove $\cdot \longrightarrow C$, which follows in one inference. We also observe that Δ_1 and Δ_2 in the sole inference rule will always be empty, since all initial sequents have an empty right-hand side, and all inference rules (only one, here) preserve this property. We claim, without substantiating it, that this is true for any Horn theory.

As a second example, consider $((A \vee C) \wedge (B \supset C)) \supset ((A \supset B) \supset C)$. First, we decompose the asynchronous connectives, which yields two independent theorems to prove.

$$
\begin{array}{c}
A \supset B, B \supset C, A; \cdot \stackrel{a}{\Longrightarrow} \cdot; C \\
A \supset B, B \supset C, C; \cdot \stackrel{a}{\Longrightarrow} \cdot; C
\end{array}
$$

The second one is trivial. For the first one, focusing on the three possible assumptions, and deleting global assumptions, yields one starting sequent and two rules of inference.

$$\cdot \Longrightarrow A$$

$$\frac{\Delta \Longrightarrow B}{\Delta \Longrightarrow C} \qquad \frac{\Delta \Longrightarrow A}{\Delta \Longrightarrow B}$$

Now C follows in two (forced) steps.

As an example of an unprovable sequent, consider the reverse implication $((A \supset B) \supset C) \supset ((A \vee C) \wedge (B \supset C))$. Asynchronous decomposition yields two independent stable sequents to be proven.

$$\begin{array}{c}(A \supset B) \supset C; \cdot \stackrel{a}{\Longrightarrow} \cdot; A \vee C \\ B, (A \supset B) \supset C; \cdot \stackrel{a}{\Longrightarrow} \cdot; C\end{array}$$

Analysing the first one, we apply synchronous decomposition on the right and on the left. On the right we have two possible derivation fragments.

$$\dfrac{\dfrac{\dfrac{\Delta; \cdot \stackrel{a}{\Longrightarrow} \cdot; A}{\Delta; \cdot \stackrel{s}{\Longrightarrow} A; \cdot}}{\Delta; \cdot \stackrel{s}{\Longrightarrow} A \vee C; \cdot}}{\Delta; \cdot \stackrel{a}{\Longrightarrow} \cdot; A \vee C} \qquad \dfrac{\dfrac{\dfrac{\Delta; \cdot \stackrel{a}{\Longrightarrow} \cdot; C}{\Delta; \cdot \stackrel{s}{\Longrightarrow} C; \cdot}}{\Delta; \cdot \stackrel{s}{\Longrightarrow} A \vee C; \cdot}}{\Delta; \cdot \stackrel{a}{\Longrightarrow} \cdot; A \vee C}$$

This yields two derived rules of inference in the forward direction.

$$\dfrac{\Delta \longrightarrow A}{\Delta \longrightarrow A \vee C} \qquad \dfrac{\Delta \longrightarrow C}{\Delta \longrightarrow A \vee C}$$

Focusing on the left-hand side instead, we obtain:

$$\dfrac{\Delta; C \stackrel{s}{\Longrightarrow} \cdot; R \qquad \dfrac{\dfrac{\Delta, A; \cdot \stackrel{a}{\Longrightarrow} \cdot; B}{\Delta; \cdot \stackrel{a}{\Longrightarrow} A \supset B; \cdot}\text{3 steps}}{\Delta; \cdot \stackrel{s}{\Longrightarrow} A \supset B; \cdot}}{\Delta; (A \supset B) \supset C \stackrel{s}{\Longrightarrow} \cdot; R}$$

The leftmost open premise forces $R = C$, so we obtain the rule

$$\dfrac{\Delta, A \longrightarrow B}{\Delta \longrightarrow C}$$

We have to be careful when applying this rule, because A may not acutally be present on the left-hand side (or, if we consider empty succedents, B on the right-hand side). We should mark A and B as optional (although at least one of them must be there, otherwise the rule makes no progress and the conclusion is subsumed by the premise). We indicate this with square brackets.

$$\dfrac{\Delta, [A] \longrightarrow [B]}{\Delta \longrightarrow C}$$

Because focusing on $(A \supset B) \supset C$ adds A as a new assumption, and B as a new conclusion we need to iterate the process of deriving rules. B in the conclusion yields no rule (we cannot focus on an atomic succedent), but A in the premise does because we also have a positive occurrence of A in a prior rule, which yields the initial sequent

$$A \longrightarrow A$$

Summarizing the situation, we have

$$\frac{\Delta \longrightarrow A}{\Delta \longrightarrow A \vee C} \qquad \frac{\Delta \longrightarrow C}{\Delta \longrightarrow A \vee C}$$

$$\frac{\Delta, [A] \longrightarrow [B]}{\Delta \longrightarrow C}$$

$$A \longrightarrow A$$

Forward reasoning saturates after one step, without proving C, which means that this sequent we started with is unprovable. This means the original formula is (intuitionistically) unprovable. However, we can still consider the second subgoal

$$B, (A \supset B) \supset C; \cdot \stackrel{a}{\Longrightarrow} \cdot; C$$

Applying a similiar analysis to before, we obtain:

$$\frac{\Delta, [A] \longrightarrow [B]}{\Delta \longrightarrow C}$$

$$B \longrightarrow B$$

After one step we obtain $\cdot \longrightarrow C$ (applying the only rule without matching A), which is what we needed to prove. So this subgoal is indeed provable.

Next we consider a first-order example, $(\forall x.\ A(x) \supset C) \supset ((\exists x.\ A(x)) \supset C)$, where x not free in C. First, we decompose the asynchronous connectives.

$$\begin{array}{l}
A(b), \forall x.\ A(x) \supset C; \cdot \stackrel{a}{\Longrightarrow} \cdot; C \\
A(b); \forall x.\ A(x) \supset C \stackrel{a}{\Longrightarrow} \cdot; C \\
\cdot; \forall x.\ A(x) \supset C, A(b) \stackrel{a}{\Longrightarrow} \cdot; C \\
\cdot; \forall x.\ A(x) \supset C, \exists x.\ A(x) \stackrel{a}{\Longrightarrow} \cdot; C \\
\cdot; \forall x.\ A(x) \supset C, \exists x.\ A(x) \stackrel{a}{\Longrightarrow} C; \cdot \\
\cdot; \forall x.\ A(x) \supset C \stackrel{a}{\Longrightarrow} (\exists x.\ A(x)) \supset C; \cdot \\
\cdot; \cdot \stackrel{a}{\Longrightarrow} (\forall x.\ A(x) \supset C) \supset ((\exists x.\ A(x)) \supset C); \cdot
\end{array}$$

In the resulting stable sequent we have a new parameter b. Since it will be available in any sequent of its proof, we can consider it as a parameter with global scope (that is, a constant). Focusing on $A(b)$ (and erasing the global assumption $A(b)$) yields the initial sequent

$$\cdot \longrightarrow A(b)$$

Focusing on the implication yields

$$\dfrac{\Delta; C \overset{s}{\Longrightarrow} \cdot; R \qquad \dfrac{\Delta; \cdot \overset{a}{\Longrightarrow} \cdot; A(t)}{\Delta; \cdot \overset{s}{\Longrightarrow} A(t); \cdot}}{\dfrac{\Delta; A(t) \supset C \overset{s}{\Longrightarrow} \cdot; R}{\Delta; \forall x.\, A(x) \supset C \overset{s}{\Longrightarrow} \cdot; R}}$$

The left-most open goal forces $R = C$, and we obtain the rule

$$\frac{\Delta \longrightarrow A(t)}{\Delta \longrightarrow C}$$

Note that this rule is schematic in t. Now we obtain our overall goal C in one step, using b for t.

It does not show up very prominently in our examples, but for completeness of this method it is critical that we continue the construction of derived rules with the new subformulas that arise when focusing on any proposition in a stable sequent ends in a collection of new stable sequent.

So, given a stable sequent to start with, we pick a synchronous proposition on the left or right. We iterate synchronous decomposition, obtaining asynchronous subgoals. Those asynchronous subgoals are now decomposed in turn, until we have again all stable sequents. The new propositions in these stable sequents must be named, and then recursively analyzed in the same way.

We must also take care to allow some formula occurrences in the premises of the derived rule to be absent from the sequents they are matched against. We only sketched this here in one of the examples. Finally, we conjecture that it is sufficient to consider contraction (factoring) on stable sequents.