5.5 Forward Subsumption

For the propositional case, we can obtain a decision procedure from the inverse method. We stop with success if we have reached the goal sequent (or a strengthened form of it) and with failure if any possible application of an inference rule leads to a sequent that is already present. This means we should devise a data structure or algorithm which allows us to check easily if the conclusion of an inference rule application is already present in the database of derived sequents. This check for equality should allow for permutations of hypotheses.

We can improve this further by not just checking equality modulo permutations, but taking weakening into account. For example, if we have derived $L_1^-, L_2^- \longrightarrow L_4^+$ then the sequent $L_1^-, L_2^-, L_3^- \longrightarrow L_4^+$ is redundant and could simply be obtained from the previous sequent by weakening. Similarly, $L_1^- \longrightarrow \cdot$ has more information than $L_1^- \longrightarrow L_2^+$, so the latter clause does not need to be kept if we have the former clause. Note that we already need this form of weakening to determine success if the goal sequent has assumptions. We say the a sequent S subsumes a sequent S' (written as $S \leq S'$) if S' can be obtains from S by weakening on the right and left.

In the propositional case, there is a relatively simple way to implement subsumption. We introduce a total ordering among all atomic propositions and also the new literals introduced during the naming process. Then we keep the antecedents of each sequent as an ordered list of atoms and literals. The union operation required in the implementation of inference rules with two premises, and the subset test required for subsumption can now both be implemented efficiently.

The reverse, called *backward subsumption* discards a previously derived sequent S if the new sequent S' subsumes S. Generally, backward subsumption is considered less fundamentally important. For example, it is not necessary to obtain a decision procedure for the propositional case. Implementations generally appear to be optimized for efficient forward subsumption.

[*the remainder of this section is speculative*]

However, it seems possible to exploit backward subsumption in a stronger way. Instead of simply deleting the subsumed sequent, we could strengthen its consequences, essentially by replaying the rules applied to it on the stronger sequent.

5.6 Proof Terms for the Inverse Method

The simplicity of the proof for the completeness theorem (Theorem 5.4) indicates that a proof term assignment should be relatively straightforward. The implicit contraction necessary when taking the union of two sets of antecedents presents the only complication. A straightforward solution seems to be to label each antecedent not with just a single variable, but with a set of variables. When taking the union of two sets of antecedents, we also need to take the union of

the corresponding label sets. But this would require globally different variables for labeling antecedents in order to avoid interference between the premises of two-premise rules. Another possibility would be to assign a unique label to each negative subformula of the goal sequent and simply use this label in the proof term. This strategy will have to be reexamined in the first-order case, since a given literal may appear with different arguments.

Note that proof term assignment in the forward sequent calculus can be done *on-line* or *off-line*. In the on-line method we construct an appropriate proof term for each sequent at each inference step in a partial derivation. In the off-line method we keep track of the minimal information so we can recover the actual sequence of inference steps to arrive a the final conclusion. From this we reconstruct a proof term only once a complete sequent derivation has been found.

The on-line method would be preferable if we could use the proof term information to guide further inferences or subsumption; otherwise the off-line method is preferable since the overhead is reduced to a a validation phase once a proof has been found.

5.7 Inverse Focusing

In the system presented so far the non-determinism in forward reasoning is still unacceptable, despite the use of subsumption. We can now analyze the rules in a way that is analogous to Chapter 4, taking advantage of inversion and focusing properties. This eliminates many derivations, significantly improving overall efficiency at a high level of abstraction. Similar optimizations have been proposed by Tammet [Tam96] and Mints [Min94], although the exact relationship between these system and the one presented below have yet to be investigated.

In focused derivations, we reason from the goal sequent upward, first applying invertible rules. This means that in the inverse method, invertible rules will be applied last, since we reason from the initial to the goal sequent. Conversely, in focused derivations we finish proofs in the focusing phase, which is therefore the first phase to be applied in the forward direction.

We first show the rules for focus sequents. These rules roughly correspond to the Tammet's *reduction strategy.*

$\Delta; \cdot > A; \cdot$	Right focus on A
$\Delta; A > \cdot; R$	Left focus on A

We apply the following restrictions.

Left passive	L	$::=$	$P \mid A \wedge B \mid \top \mid A \supset B \mid \forall x.\, A$
Passive context	Δ	$::=$	$\cdot \mid \Delta, L$
Right passive	R	$::=$	$P \mid A \vee B \mid \bot \mid \exists x.\, A$

These are slightly different from the previous definitions since conjunction and truth is treated in a different manner.

Left Focus Rules. As mentioned at the end of Section 5.4, we can restrict initial sequents to be atomic or literals which name compound formulas which appear both positively and negatively. As given below, however, the literals would have to stand for R-formulas for this rule to be applicable.

$$\frac{}{\Delta; P > \cdot; P}\text{init}$$

$$\frac{\Delta; A > \cdot; R}{\Delta; A \wedge B > \cdot; R}\wedge\text{L}_1 \qquad \frac{\Delta; B > \cdot; R}{\Delta; A \wedge B > \cdot; R}\wedge\text{L}_2$$

$$\frac{\Delta_1; B > \cdot; R \qquad \Delta_2; \cdot \longrightarrow A; \cdot}{\Delta_1 \cup \Delta_2; A \supset B > \cdot; R}\supset\text{L} \qquad \text{no rule } \top\text{L}$$

$$\frac{\Delta; [t/x]A > \cdot; R}{\Delta; \forall x.\, A > \cdot; R}\forall\text{L}$$

Right Focus Rules.

$$\frac{\Delta; \cdot > A; \cdot}{\Delta; \cdot > A \vee B; \cdot}\vee\text{R}_1 \qquad \frac{\Delta; \cdot > B; \cdot}{\Delta; \cdot > A \vee B; \cdot}\vee\text{R}_2$$

$$\frac{\Delta; \cdot > [t/x]A; \cdot}{\Delta; \cdot > \exists x.\, A; \cdot}\exists\text{R} \qquad \text{no rule } \bot\text{R}$$

Transition Rules.

$$\frac{\Delta; L > \cdot; R}{(\Delta, L); \cdot \longrightarrow \cdot; R} \qquad \frac{\Delta; \cdot > R_+; \cdot}{\Delta; \cdot \longrightarrow \cdot; R_+}$$

Here, R_+ is an non-atomic R-formula.

In the backwards directed focusing calculus we can choose between the active right rules in a don't-care non-deterministic manner. This means the calculus admits many derivations; choosing any one of them is complete. For the forward directions we should eliminate this so that exactly one of the derivations will be generated. While the forward active rules below cut down some on the non-determinism, they do not yet have this stronger property. For this will need to add some ordering properties[2] (see Section **??**). This roughly corresponds to Tammet's inversion strategy.

Our partial solution is to force the right active rules to be applied first (in the downward direction) and the left active rules second. Moreover, we restrict ourselves to at most one active proposition on the left and right. Therefore we have the following three judgments (the first being an auxiliary one).

[2] [*I speculate*]

$$\begin{array}{ll} \Delta; \cdot \longrightarrow \cdot; R & \text{Neutral sequent} \\ \Delta; \cdot \longrightarrow A; \cdot & \text{Right active } A \\ \Delta; A \longrightarrow B; \cdot & \text{Left active } A \text{ and right active } B \end{array}$$

Right Active Rules.

$$\frac{\Delta; \cdot \longrightarrow \cdot; R}{\Delta; \cdot \longrightarrow R; \cdot}$$

$$\frac{\Delta_1; \cdot \longrightarrow A; \cdot \qquad \Delta_2; \cdot \longrightarrow B; \cdot}{\Delta_1 \cup \Delta_2; \cdot \longrightarrow A \wedge B; \cdot} \wedge\mathrm{R} \qquad \frac{}{\cdot; \cdot \longrightarrow \top; \cdot} \top\mathrm{R}$$

$$\frac{\Delta; \cdot \longrightarrow B; \cdot}{\Delta; \cdot \longrightarrow A \supset B; \cdot} \supset\mathrm{R}_2$$

$$\frac{\Delta; \cdot \longrightarrow [a/x]A; \cdot}{\Delta; \cdot \longrightarrow \forall x.\ A; \cdot} \forall\mathrm{R}^a$$

Left Active Rules.

$$\frac{(\Delta, L); \cdot \longrightarrow C; \cdot}{\Delta; L \longrightarrow C; \cdot}$$

$$\frac{\Delta; A \longrightarrow B; \cdot}{\Delta; \cdot \longrightarrow A \supset B; \cdot} \supset\mathrm{R}_1$$

$$\frac{\Delta_1; A \longrightarrow C; \cdot \qquad \Delta_2; B \longrightarrow C; \cdot}{\Delta_1 \cup \Delta_2; A \vee B \longrightarrow C; \cdot} \vee\mathrm{L} \qquad \frac{}{\cdot; \bot \longrightarrow C; \cdot} \bot\mathrm{L}$$

$$\frac{\Delta; [a/x]A \longrightarrow C; \cdot}{\Delta; \exists x.\ A \longrightarrow C; \cdot} \exists\mathrm{L}^a$$

Note that the left rule for $\bot$ is not satisfactory because of the unconstrained proposition C on the right. A satisfactory treatment can be given using empty right-hand sides (see Section 5.8) or by replacing C by a propositional parameter p. However, this latter solution has further consequences (see Exercise **??**).

Transitions Rules.

$$\frac{\Delta; \overline{L}_+ \longrightarrow R; \cdot}{\Delta; \overline{L}_+ > \cdot; R} \qquad \frac{\Delta; \cdot \longrightarrow \overline{R}; \cdot}{\Delta; \cdot > \overline{R}; \cdot}$$

We restrict these rules to

$$\begin{array}{lcl} \overline{L}_+ & ::= & A \vee B \mid \bot \mid \exists x.\ A \\ \overline{R} & ::= & P \mid A \supset B \mid A \wedge B \mid \top \mid \forall x.\ A \end{array}$$

Soudness of inverse focused derivations is relatively easy to see, since the rules are just specializations of the usual forward sequent rules. Completeness is more difficult to show, especially since it does not hold for arbitrary active sequents $\Delta; \Gamma \Longrightarrow A; \cdot$ or $\Delta; \Gamma \Longrightarrow \cdot; A$. This means that we have to take advantage of the don't care non-determinism and choose (in the bottom-up direction) to always work on the left-hand side of the sequent first and then the right-hand side. One might proceed by first proving the completeness of this strategy with respect to the bottom-up focusing calculus, and then prove the completeness of the inverse focusing rules with respect to this intermediate calculus.

In practice, one might want to apply backwards rules in a don't-care non-deterministic fashion if a given goal sequent does not have the form $\Delta; \cdot \Longrightarrow A; \cdot$ and solve the remaining subgoals with the inverse method. Alternatively, one can "hide" left invertible connectives, for example, by discharging them and proving $\cdot; \cdot \longrightarrow A_1 \supset \ldots \supset A_n \supset A; \cdot$ for $\Delta = A_1, \ldots, A_n$.

[*Fill in appropriate soundness and completeness theorem here.*]

5.8 Inverse Focusing with Negation

As in Section 5.2, negation is handled by allowing empty succedents. We apply the following restrictions.

$$
\begin{array}{llll}
\text{Left passive} & L & ::= & P \mid A \wedge B \mid \top \mid A \supset B \mid \forall x.\, A \mid \neg A \\
\text{Passive context} & \Delta & ::= & \cdot \mid \Delta, L \\
\text{Right passive} & R & ::= & P \mid A \vee B \mid \bot \mid \exists x.\, A
\end{array}
$$

We have the following judgments, where ρ is either empty or a singleton R and γ is either empty or a singleton A.

$$
\begin{array}{ll}
\Delta; \cdot > A; \cdot & \text{Right focus on } A \\
\Delta; A > \cdot; \rho & \text{Left focus on } A \\
\Delta; \cdot \longrightarrow \cdot; \rho & \text{Neutral sequent} \\
\Delta; \cdot \longrightarrow A; \cdot & \text{Right active } A \\
\Delta; A \longrightarrow \gamma; \cdot & \text{Left active } A\text{, possible right active } \gamma
\end{array}
$$

Left Focus Rules.

$$
\frac{}{\Delta; P > \cdot; P}\,\text{init}
$$

$$
\frac{\Delta; A > \cdot; \rho}{\Delta; A \wedge B > \cdot; \rho}\,\wedge\text{L}_1
\qquad
\frac{\Delta; B > \cdot; \rho}{\Delta; A \wedge B > \cdot; \rho}\,\wedge\text{L}_2
$$

$$
\frac{\Delta_1; B > \cdot; \rho \qquad \Delta_2; \cdot \longrightarrow A; \cdot}{\Delta_1 \cup \Delta_2; A \supset B > \cdot; \rho}\,\supset\text{L}
\qquad \text{no rule } \top\text{L}
$$

$$
\frac{\Delta; \cdot \longrightarrow A; \cdot}{\Delta; \neg A > \cdot; \cdot}\,\neg\text{L}
\qquad
\frac{\Delta; [t/x]A > \cdot; \rho}{\Delta; \forall x.\, A > \cdot; \rho}\,\forall\text{L}
$$

Right Focus Rules.

$$\frac{\Delta; \cdot > A; \cdot}{\Delta; \cdot > A \vee B; \cdot}\vee\mathrm{R}_1 \qquad \frac{\Delta; \cdot > B; \cdot}{\Delta; \cdot > A \vee B; \cdot}\vee\mathrm{R}_2$$

$$\frac{\Delta; \cdot > [t/x]A; \cdot}{\Delta; \cdot > \exists x.\, A; \cdot}\exists\mathrm{R} \qquad \text{no rule } \bot\mathrm{R}$$

Transition Rules.

$$\frac{\Delta; L > \cdot; \rho}{(\Delta, L); \cdot \longrightarrow \cdot; \rho} \qquad \frac{\Delta; \cdot > R_+; \cdot}{\Delta; \cdot \longrightarrow \cdot; R_+}$$

Here, R_+ is a non-atomic R-formula.

Right Active Rules.

$$\frac{\Delta; \cdot \longrightarrow \cdot; R}{\Delta; \cdot \longrightarrow R; \cdot}$$

$$\frac{\Delta_1; \cdot \longrightarrow A; \cdot \qquad \Delta_2; \cdot \longrightarrow B; \cdot}{\Delta_1 \cup \Delta_2; \cdot \longrightarrow A \wedge B; \cdot}\wedge\mathrm{R} \qquad \frac{}{\cdot; \cdot \longrightarrow \top; \cdot}\top\mathrm{R}$$

$$\frac{\Delta; \cdot \longrightarrow B; \cdot}{\Delta; \cdot \longrightarrow A \supset B; \cdot}\supset\mathrm{R}_2$$

$$\frac{\Delta; \cdot \longrightarrow [a/x]A; \cdot}{\Delta; \cdot \longrightarrow \forall x.\, A; \cdot}\forall\mathrm{R}^a$$

Left Active Rules.

$$\frac{(\Delta, L); \cdot \longrightarrow \gamma; \cdot}{\Delta; L \longrightarrow \gamma; \cdot}$$

$$\frac{\Delta; A \longrightarrow B; \cdot}{\Delta; \cdot \longrightarrow A \supset B; \cdot}\supset\mathrm{R}_1 \qquad \frac{\Delta; A \longrightarrow \cdot; \cdot}{\Delta; \cdot \longrightarrow A \supset B; \cdot}\supset\mathrm{R}_3$$

$$\frac{\Delta; A \longrightarrow \cdot; \cdot}{\Delta; \cdot \longrightarrow \neg A; \cdot}\neg\mathrm{R}$$

$$\frac{\Delta_1; A \longrightarrow \gamma_1; \cdot \qquad \Delta_2; B \longrightarrow \gamma_2; \cdot}{\Delta_1 \cup \Delta_2; A \vee B \longrightarrow \gamma_1 \cup \gamma_2; \cdot}\vee\mathrm{L} \qquad \frac{}{\cdot; \bot \longrightarrow \cdot; \cdot}\bot\mathrm{L}$$

$$\frac{\Delta; [a/x]A \longrightarrow \gamma; \cdot}{\Delta; \exists x.\, A \longrightarrow C; \cdot}\exists\mathrm{L}^a$$

Note that $\vee$L requires $\gamma_1 \cup \gamma_2$ either to be a singleton C or empty.

Transitions Rules.

$$\frac{\Delta; \overline{L}_+ \longrightarrow R; \cdot}{\Delta; \overline{L}_+ > \cdot; R} \qquad \frac{\Delta; \cdot \longrightarrow \overline{R}; \cdot}{\Delta; \cdot > \overline{R}; \cdot}$$

We restrict these rules to

$$\begin{array}{lcl} \overline{L}_+ & ::= & A \vee B \mid \perp \mid \exists x.\, A \\ \overline{R} & ::= & P \mid A \supset B \mid A \wedge B \mid \top \mid \forall x.\, A \mid \neg A \end{array}$$

[*Fill in appropriate soundness and completeness theorems here. The same remarks as in the previous sections apply.*]

5.9 Exercises

Exercise 5.1 Show the forward sequent calculus on signed propositions and prove that if $\Gamma \longrightarrow A$ then $\Gamma^- \longrightarrow A^+$.

Exercise 5.2 In the exercise we explore add the connective $A \equiv B$ as a primitive to inverse method.

1. Following Exercise 2.6, introduce appropriate left and right rules to the backward sequent calculus.
2. Transform the rules to be appropriate for the forward sequent calculus.
3. Extend the notion of positive and negative subformula.
4. Extend the technique of subformula naming and inference rule specialization.
5. Show inverse derivations for each of the following.
   (a) Reflexivity: $\longrightarrow A \equiv A$.
   (b) Symmetry: $A \equiv B \longrightarrow B \equiv A$.
   (c) Transitivity: $A \equiv B, B \equiv C \longrightarrow A \equiv C$.
6. Compare your technique with thinking of $A \equiv B$ as a syntactic abbreviation for $(A \supset B) \wedge (B \supset A)$. Do you see significant advantages or disadvantages of your method?