3.4 Cut Elimination

This section is devoted to proving that the rule of cut is redundant in the sequent calculus. First we prove that cut is *admissible*: whenever the premises of the cut rule are derivable in the sequent calculus *without cut*, then the conclusion is. It is a simple observation that adding an admissible rule to a deductive system does not change the derivable judgments. Formally, this second step is an induction over the structure of a derivation that may contain cuts, proving that if $\Gamma \stackrel{+}{\Longrightarrow} C$ then $\Gamma \Longrightarrow C$.

There is a stronger property we might hope to prove for cut: it could be a *derived* rule of inference. Derived rules have a direct deduction of the conclusion from the premises within the given system. For example,

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash B \qquad \Gamma \vdash C}{\Gamma \vdash A \wedge (B \wedge C)}$$

is a derived rule, as evidenced by the following deduction:

$$\dfrac{\Gamma \vdash A \qquad \dfrac{\Gamma \vdash B \qquad \Gamma \vdash C}{\Gamma \vdash B \wedge C} \wedge\mathrm{I}}{\Gamma \vdash A \wedge (B \wedge C)} \wedge\mathrm{I}.$$

Derived rules have the property that they remain valid under all extensions of a given system. Admissible rules, on the other hand, have to be reconsidered when new connectives or inference rules are added to a system, since these rules may invalidate the proof of admissibility.

It turns out that cut is only admissible, but not derivable in the sequent calculus. Therefore, we will prove the following theorem:

If $\Gamma \Longrightarrow A$ *and* $\Gamma, A \Longrightarrow C$ *then* $\Gamma \Longrightarrow C$.

We call A the *cut formula*. Also, each left or right rule in the sequent calculus focuses on an occurrence of a proposition in the conclusion, called the *principal formula* of the inference.

The proof combines two ideas: induction over the structure of the cut formula with induction over the structures of the two given derivations. They are combined into one nested induction: an outer induction over the structure of the cut formula and an inner induction over the structure of the derivations of the premises. The outer induction over the structure of the cut formula is related to local reductions in natural deduction (see Exercise 3.7).

Theorem 3.11 (Admissibility of Cut)
If $\Gamma \Longrightarrow A$ *and* $\Gamma, A \Longrightarrow C$ *then* $\Gamma \Longrightarrow C$.

Proof: By nested inductions on the structure of A, the derivation $\mathcal{D}$ of $\Gamma \Longrightarrow A$ and $\mathcal{E}$ of $\Gamma, A \Longrightarrow C$. More precisely, we appeal to the induction hypothesis either with a strictly smaller cut formula, or with an identical cut formula and

two derivations, one of which is strictly smaller while the other stays the same. The proof is constructive, which means we show how to transform

$$\begin{array}{c}\mathcal{D}\\ \Gamma \Longrightarrow A\end{array} \quad\text{and}\quad \begin{array}{c}\mathcal{E}\\ \Gamma, A \Longrightarrow C\end{array} \quad\text{to}\quad \begin{array}{c}\mathcal{F}\\ \Gamma \Longrightarrow C.\end{array}$$

The proof is divided into several classes of cases. More than one case may be applicable, which means that the algorithm for constructing the derivation of $\Gamma \Longrightarrow C$ from the two given derivations is naturally non-deterministic.

Case: $\mathcal{D}$ is an initial sequent.

$$\mathcal{D} = \frac{}{\Gamma', A \Longrightarrow A}\,\text{init}$$

$\Gamma = \Gamma', A$	This case
$\Gamma', A, A \Longrightarrow C$	Derivation $\mathcal{E}$
$\Gamma', A \Longrightarrow C$	By contraction (Lemma 3.7)
$\Gamma \Longrightarrow C$	By equality

Case: $\mathcal{E}$ is an initial sequent using the cut formula.

$$\mathcal{E} = \frac{}{\Gamma, A \Longrightarrow A}\,\text{init}$$

$C = A$	This case
$\Gamma \Longrightarrow A$	Derivation $\mathcal{D}$

Case: $\mathcal{E}$ is an initial sequent not using the cut formula.

$$\mathcal{E} = \frac{}{\Gamma', C, A \Longrightarrow C}\,\text{init}$$

$\Gamma = \Gamma', C$	This case
$\Gamma', C \Longrightarrow C$	By rule init
$\Gamma \Longrightarrow C$	By equality

Case: A is the principal formula of the final inference in both $\mathcal{D}$ and $\mathcal{E}$. There are a number of subcases to consider, based on the last inference in $\mathcal{D}$ and $\mathcal{E}$. We show some of them.

Subcase:

$$\mathcal{D} = \frac{\begin{array}{c}\mathcal{D}_1\\ \Gamma \Longrightarrow A_1\end{array} \qquad \begin{array}{c}\mathcal{D}_2\\ \Gamma \Longrightarrow A_2\end{array}}{\Gamma \Longrightarrow A_1 \wedge A_2}\,\wedge\text{R}$$

$$\text{and} \qquad \mathcal{E} = \frac{\begin{array}{c}\mathcal{E}_1\\ \Gamma, A_1 \wedge A_2, A_1 \Longrightarrow C\end{array}}{\Gamma, A_1 \wedge A_2 \Longrightarrow C}\,\wedge\text{L}_1$$

$\Gamma, A_1 \Longrightarrow C$ By i.h. on $A_1 \wedge A_2$, $\mathcal{D}$ and $\mathcal{E}_1$
$\Gamma \Longrightarrow C$ By i.h. on A_1 from above and $\mathcal{D}_1$

Actually we have ignored a detail: in the first appeal to the induction hypothesis, $\mathcal{E}_1$ has an additionaly hypothesis (A_1 *left*) and therefore does not match the statement of the theorem precisely. However, we can always weaken $\mathcal{D}$ to include this additional hypothesis without changing the structure of $\mathcal{D}$ (see the proof of Lemma 3.7) and then appeal to the induction hypothesis. We will not be explicit about these trivial weakening steps in the remaining cases.

Subcase:

$$\mathcal{D} = \dfrac{\begin{array}{c}\mathcal{D}_2\\ \Gamma, A_1 \Longrightarrow A_2\end{array}}{\Gamma \Longrightarrow A_1 \supset A_2} \supset\!\mathrm{R}$$

$$\text{and} \quad \mathcal{E} = \dfrac{\begin{array}{c}\mathcal{E}_1\\ \Gamma, A_1 \supset A_2 \Longrightarrow A_1\end{array} \qquad \begin{array}{c}\mathcal{E}_2\\ \Gamma, A_1 \supset A_2, A_2 \Longrightarrow C\end{array}}{\Gamma, A_1 \supset A_2 \Longrightarrow C} \supset\!\mathrm{L}$$

$\Gamma \Longrightarrow A_1$ By i.h. on $A_1 \supset A_2$, $\mathcal{D}$ and $\mathcal{E}_1$
$\Gamma \Longrightarrow A_2$ By i.h. on A_1 from above and $\mathcal{D}_2$
$\Gamma, A_2 \Longrightarrow C$ By i.h. on $A_1 \supset A_2$, $\mathcal{D}$ and $\mathcal{E}_2$
$\Gamma \Longrightarrow C$ By i.h. on A_2 from above

Subcase:

$$\mathcal{D} = \dfrac{\begin{array}{c}\mathcal{D}_1\\ \Gamma, A_1 \Longrightarrow p\end{array}}{\Gamma \Longrightarrow \neg A_1} \neg\mathrm{R}^p$$

$$\text{and} \quad \mathcal{E} = \dfrac{\begin{array}{c}\mathcal{E}_1\\ \Gamma, \neg A_1 \Longrightarrow A_1\end{array}}{\Gamma, \neg A_1 \Longrightarrow C} \neg\mathrm{L}$$

$\Gamma \Longrightarrow A_1$ By i.h. on $\mathcal{D}$ and $\mathcal{E}_1$
$\Gamma, A_1 \Longrightarrow C$ By substitution for parameter C in $\mathcal{D}_1$
$\Gamma \Longrightarrow C$ By i.h. on A_1 from above

Note that the condition that p be a new parameter in $\mathcal{D}_1$ is necessary to guarantee that in the substitution step above we have $[C/p]A_1 = A_1$ and $[C/p]\Gamma = \Gamma$.

Subcase:

$$\mathcal{D} = \dfrac{\begin{array}{c}\mathcal{D}_1 \\ \Gamma \Longrightarrow [t/x]A_1\end{array}}{\Gamma \Longrightarrow \exists x.\ A_1}\ \exists\mathrm{R}$$

$$\text{and} \quad \mathcal{E} = \dfrac{\begin{array}{c}\mathcal{E}_1 \\ \Gamma, \exists x.\ A_1, [a/x]A_1 \Longrightarrow C\end{array}}{\Gamma, \exists x.\ A_1 \Longrightarrow C}\ \exists\mathrm{L}^a$$

$\Gamma, [t/x]A_1 \Longrightarrow C$	By substitution for parameter a in $\mathcal{E}_1$
$\Gamma, [t/x]A_1 \Longrightarrow C$	By i.h. on $\exists x.\ A_1$, $\mathcal{D}$ and $[t/a]\mathcal{E}_1$
$\Gamma \Longrightarrow C$	By i.h. on $[t/x]A_1$ from $\mathcal{D}_1$ and above

Note that this case requires that $[t/x]A_1$ is considered smaller than $\exists x.\ A_1$. Formally, this can be justified by counting the number of quantifiers and connectives in a proposition and noting that the term t does not contain any. A similar remark applies to check that $[t/a]\mathcal{E}_1$ is smaller than $\mathcal{E}$. Also note how the side condition that a must be a new parameter in the $\exists$L rule is required in the substitution step to conclude that $[t/a]\Gamma = \Gamma$, $[t/a][a/x]A_1 = [t/x]A_1$, and $[t/a]C$.

Case: A is not the principal formula of the last inference in $\mathcal{D}$. In that case $\mathcal{D}$ must end in a left rule and we can appeal to the induction hypothesis on one of its premises. We show some of the subcases.

Subcase:

$$\mathcal{D} = \dfrac{\begin{array}{c}\mathcal{D}_1 \\ \Gamma', B_1 \wedge B_2, B_1 \Longrightarrow A\end{array}}{\Gamma', B_1 \wedge B_2 \Longrightarrow A}\ \wedge\mathrm{L}_1$$

$\Gamma = \Gamma', B_1 \wedge B_2$	This case
$\Gamma', B_1 \wedge B_2, B_1 \Longrightarrow C$	By i.h. on A, $\mathcal{D}_1$ and $\mathcal{E}$
$\Gamma', B_1 \wedge B_2 \Longrightarrow C$	By rule $\wedge\mathrm{L}_1$
$\Gamma \Longrightarrow C$	By equality

Subcase:

$$\mathcal{D} = \dfrac{\begin{array}{c}\mathcal{D}_1 \\ \Gamma', B_1 \supset B_2 \Longrightarrow B_1\end{array} \qquad \begin{array}{c}\mathcal{D}_2 \\ \Gamma', B_1 \supset B_2, B_2 \Longrightarrow A\end{array}}{\Gamma', B_1 \supset B_2 \Longrightarrow A}\ \supset\mathrm{L}$$

$\Gamma = \Gamma', B_1 \supset B_2$	This case
$\Gamma', B_1 \supset B_2, B_2 \Longrightarrow C$	By i.h. on A, $\mathcal{D}_2$ and $\mathcal{E}$
$\Gamma', B_2 \supset B_2 \Longrightarrow C$	By rule $\supset$L on $\mathcal{D}_1$ and above
$\Gamma \Longrightarrow C$	By equality

Case: A is not the principal formula of the last inference in $\mathcal{E}$. This overlaps with the previous case, since A may not be principal on either side. In this case, we appeal to the induction hypothesis on the subderivations of $\mathcal{E}$ and directly infer the conclusion from the results. We show some of the subcases.

Subcase:

$$\mathcal{E} = \frac{\begin{array}{c}\mathcal{E}_1 \\ \Gamma, A \Longrightarrow C_1\end{array} \qquad \begin{array}{c}\mathcal{E}_2 \\ \Gamma, A \Longrightarrow C_2\end{array}}{\Gamma, A \Longrightarrow C_1 \wedge C_2} \wedge\mathrm{R}$$

$C = C_1 \wedge C_2$	This case
$\Gamma \Longrightarrow C_1$	By i.h. on A, $\mathcal{D}$ and $\mathcal{E}_1$
$\Gamma \Longrightarrow C_2$	By i.h. on A, $\mathcal{D}$ and $\mathcal{E}_2$
$\Gamma \Longrightarrow C_1 \wedge C_2$	By rule $\wedge$R on above

Subcase:

$$\mathcal{E} = \frac{\begin{array}{c}\mathcal{E}_1 \\ \Gamma', B_1 \wedge B_2, B_1, A \Longrightarrow C\end{array}}{\Gamma', B_1 \wedge B_1, A \Longrightarrow C} \wedge\mathrm{L}_1$$

$\Gamma = \Gamma', B_1 \wedge B_2$	This case
$\Gamma', B_1 \wedge B_2, B_1 \Longrightarrow C$	By i.h. on A, $\mathcal{D}$ and $\mathcal{E}_1$
$\Gamma', B_1 \wedge B_2 \Longrightarrow C$	By rule $\wedge\mathrm{L}_1$ from above

□

As mentioned above, it is a general property of deductive system that adding an admissible rule does not change the derivable judgments. We show the argument in this special case.

Theorem 3.12 (Cut Elimination)
If $\Gamma \stackrel{+}{\Longrightarrow} C$ *then* $\Gamma \Longrightarrow C$.

Proof: In each case except cut we simply appeal to the induction hypotheses and reapply the same rule on the resulting cut-free derivations. So we write out only the case of cut.

Case:

$$\mathcal{D}^+ = \frac{\begin{array}{c}\mathcal{D}_1^+ \\ \Gamma \stackrel{+}{\Longrightarrow} A\end{array} \qquad \begin{array}{c}\mathcal{D}_2^+ \\ \Gamma, A \stackrel{+}{\Longrightarrow} C\end{array}}{\Gamma \stackrel{+}{\Longrightarrow} C} \mathrm{cut}$$

$\Gamma \Longrightarrow A$	By i.h. on $\mathcal{D}_1^+$
$\Gamma, A \Longrightarrow C$	By i.h. on $\mathcal{D}_2^+$
$\Gamma \Longrightarrow C$	By admissibility of cut (Theorem 3.11)

□

The cut elimination theorem is the final piece needed to complete our study of natural deduction and normal natural deduction and at the same time the springboard to the development of efficient theorem proving procedures. Our proof above is constructive and therefore contains an algorithm for cut elimination. Because the cases are not mutually exclusive, the algorithm is non-deterministic. However, the resulting derivation should always be the same. This is called the *confluence* property for intuitionistic cut elimination and it is not implicit in our proof, but has to be established separately.[2] On the other hand, our proof shows that any possible execution of the cut-elimination algorithm terminates. This is called the *strong normalization* property for the sequent calculus.

By putting the major results of this section together we can now prove the normalization theorem for natural deduction.

Theorem 3.13 (Normalization for Natural Deduction)
If $\Gamma \vdash A$ *then* $\Gamma^{\downarrow} \vdash A \Uparrow$.

Proof: Direct from previous theorems.

$\Gamma \vdash A$	Assumption
$\Gamma^{\downarrow} \vdash^{+} A \Uparrow$	By completeness of annotated deductions (Theorem 3.3)
$\Gamma \overset{+}{\Longrightarrow} A$	By completeness of sequent calculus with cut (Theorem 3.10)
$\Gamma \Longrightarrow A$	By cut elimination (Theorem 3.12)
$\Gamma^{\downarrow} \vdash A \Uparrow$	By soundness of sequent calculus (Theorem 3.6)

□

3.5 Exercises

Exercise 3.1 Consider a system of normal deduction where the elimination rules for disjunction and existential are allowed to end in an extraction judgment.

$$\frac{\Gamma^{\downarrow} \vdash A \vee B \downarrow \qquad \Gamma^{\downarrow}, u{:}A \downarrow\ \vdash C \downarrow \qquad \Gamma^{\downarrow}, w{:}B \downarrow\ \vdash C \downarrow}{\Gamma^{\downarrow} \vdash C \downarrow} \vee\mathrm{E}^{u,w}$$

$$\frac{\Gamma^{\downarrow} \vdash \exists x.\ A \downarrow \qquad \Gamma^{\downarrow}, u{:}[a/x]A \downarrow\ \vdash C \downarrow}{\Gamma^{\downarrow} \vdash C \downarrow} \exists\mathrm{E}^{a,u}$$

Discuss the relative merits of allowing or disallowing these rules and show how they impact the subsequent development in this Chapter (in particular, bi-directional type-checking and the relationship to the sequent calculus).

[2] *[reference?]*

Exercise 3.2

1. Give an example of a natural deduction which is *not* normal (in the sense defined in Section 3.1), yet contains no subderivation which can be locally reduced.

2. Generalizing from the example, devise additional rules of reduction so that any natural deduction which is not normal can be reduced. You should introduce no more and no fewer rules than you need for this purpose.

3. Prove that your rules satisfy the specification in part (2).

Exercise 3.3 Write out the rules defining the judgments $\Gamma^{\downarrow} \vdash^{+} I : A \Uparrow$ and $\Gamma^{\downarrow} \vdash^{+} E : A \downarrow$ and prove Theorem 3.4. Make sure to carefully state the induction hypothesis (if it is different from the statement of the theorem) and consider all the cases.

Exercise 3.4 Fill in the missing subcases in the proof of the admissibility of cut (Theorem 3.11) where A is the principal formula in both $\mathcal{D}$ and $\mathcal{E}$.

Exercise 3.5 Consider an extension of intuitionistic logic by a universal quantifier over propositions, written as $\forall^2 p.\ A$, where p is variable ranging over propositions.

1. Show introduction and elimination rules for $\forall^2$.

2. Extend the calculus of normal and extraction derivations.

3. Show left and right rules of the sequent calulus for $\forall^2$.

4. Extend the proofs of soundness and completeness for the sequent calculus and sequent calculus with cut to accomodate the new rules.

5. Point out why the proof for admissibility of cut does not extend to this logic.

Exercise 3.6 Gentzen's original formulation of the sequent calculus for intuitionistic logic permitted the right-hand side to be empty. The introduction rule for negation then has the form

$$\frac{\Gamma, A \Longrightarrow}{\Gamma \Longrightarrow \neg A}\,\neg\text{R}.$$

Write down the corresponding left rule and detail the changes in the proof for admissibility of cut. Can you explain sequents with empty right-hand sides as judgments?

Exercise 3.7 Consider the fragment containing implication, conjunction, and truth. Analyze the relationship between local reductions in natural deduction and the proof for admissibility of cut. Once you have discovered a connection, make it precise in the form of a theorem and prove it.

Exercise 3.8

1. Prove that we can restrict initial sequents in the sequent calculus to have the form $\Gamma, P \Longrightarrow P$ where P is an atomic proposition without losing completeness.

2. Determine the corresponding restriction in normal and extraction derivations and prove that they preserve completeness.

3. If you see a relationship between these properties and local reductions or expansions, explain. If you can cast it in the form of a theorem, do so and prove it.

Exercise 3.9 For each of the following propositions, prove that they are derivable in classical logic using the law of excluded middle. Furthermore, prove that they are not true in intuitionistic logic for arbitrary A, B, and C.

1. $((A \supset B) \supset A) \supset A$.

2. Any entailment in Exercise 2.8 which is only classically, but not intuitionistically true.