Chapter 6

Non-Commutative Linear Logic

[*warning: this chapter is even more tentative some most of the other material in these lecture notes.*]

The goal of this chapter is to develop a system of pure natural deduction which encompasses the (ordinary) intuitionistic simply-typed λ-calculus, the intuitionistic linear λ-calculus, and new constructs for a non-commutative linear λ-calculus. It is important that this calculus be conservative over the intuitionistic and linear fragments, so that we do not lose any expressive power and the new features can be introduced gently into the intended application domains.

The system has applications in functional languages, logic programming languages, and logical frameworks. In functional languages, the non-commutative type system allows us to capture strong stackability properties, thereby, for example, giving a logical and general foundation for observations made about terms in continuation-passing style and monadic style [DP95, **?**]. In logic programming languages, it allows us to remove some uses of cut which arise from don't-care non-determinism in languages based on linear logic such as Lolli [HM94, CHP97]. In logical frameworks, non-commutative connectives allow us drastically simplify the representations of problems involving stacks or languages such as the ones above.

We start with the simplest fragment which fixes the basic concepts and auxiliary definitions and then add other connectives and modalities incrementally.

Various formulations of non-commutative linear logic have been considered, both in their classical [**?**, Roo92, Abr95] and intuitionistic [BG91, Abr90a, Abr90b] variants, including various modal operators, analyzed in particular depth in [**?**]. Except for a brief mention in [Abr90a], we are not aware of any systematic study of natural deduction, the Curry-Howard isomorphism, and the computational consequences of non-commutativity in the λ-calculus. The material in this chapter may grow to eventually fill this gap in the literature and sketch some applications of non-commutativity in the area of logic programming, logical frameworks, and functional programming, complementing Ruet's investigation of concurrent constraint programming from the point of view of mixed classical non-commutative linear logic [**?**].

6.1 The Implicational Fragment

In this first section we present the pure implicational fragment, containing only the intuitionistic implication ($\rightarrow$), the linear implication ($\multimap$), ordered right implication ($\twoheadrightarrow$) and ordered left implication ($\rightarrowtail$).

We use a formulation of the main judgment using multiple zones: one for *intuitionistic assumptions*, one for *linear assumptions*, and one for *ordered assumptions*. While this may not be the best formulation for all purposes, it is the one we found most easy to understand. We will also freely go back and forth between propositions and types, using the well-known Curry-Howard correspondence.

$$
\begin{array}{llcll}
\textit{Types} & A & ::= & P & \text{atomic types} \\
 & & | & A_1 \rightarrow A_2 & \text{intuitionistic implication} \\
 & & | & A_1 \multimap A_2 & \text{linear implication} \\
 & & | & A_1 \twoheadrightarrow A_2 & \text{ordered right implication} \\
 & & | & A_1 \rightarrowtail A_2 & \text{ordered left implication}
\end{array}
$$

Objects of the λ-calculus (or proof terms for the underlying logic) are defined in a straightforward fashion. We do not formally distinguish different kinds of variables, although we later use the convention that x stands for intuitionistic assumptions, y for linear assumptions, and z for ordered assumptions.

$$
\begin{array}{llcll}
\textit{Objects} & M & ::= & x & \text{variables} \\
 & & | & \lambda x{:}A.\ M \mid M_1\, M_2 & \text{intuitionistic functions} \\
 & & | & \hat{\lambda} x{:}A.\ M \mid M_1 \hat{\ } M_2 & \text{linear functions} \\
 & & | & \overset{>}{\lambda} x{:}A.\ M \mid M_1 \overset{>}{\ } M_2 & \text{right ordered functions} \\
 & & | & \overset{<}{\lambda} x{:}A.\ M \mid M_1 \overset{<}{\ } M_2 & \text{left ordered functions}
\end{array}
$$

Contexts are simply lists of assumptions $x{:}A$ with distinct variables x. In a triple of contexts $\Gamma; \Delta; \Omega$ needed for the typing judgment, we also assume that no variable occurs more than once.

$$
\textit{Contexts} \quad \Gamma \quad ::= \quad \cdot \mid \Gamma, x{:}A
$$

We use the convention that Γ stands for an intuitionistic context, Δ for a linear context, and Ω for an ordered context. We abbreviate $\cdot, x{:}A$ as $x{:}A$.

In order to describe the inference rules, we need some auxiliary operations on contexts, *context join* Ω, Ω' and *context merge* $\Delta \times \Delta'$. Context join preserves the order of the assumption, while the non-deterministic merge allows any interleaving of assumption.

$$
\begin{array}{llcl}
\textit{Context Join} & \Omega, \cdot & = & \Omega \\
 & \Omega, (\Omega', x{:}A) & = & (\Omega, \Omega'), x{:}A \\
\textit{Context Merge} & \cdot \times \cdot & = & \cdot \\
 & (\Delta, x{:}A) \times \Delta' & = & (\Delta \times \Delta'), x{:}A \\
 & \Delta \times (\Delta', x{:}A) & = & (\Delta \times \Delta'), x{:}A
\end{array}
$$

The typing rules below are perhaps most easily understood when reading them from the conclusion to the premises, as rules for the construction of a typing derivation for a term. We have designed the language of objects so that the rules are completely syntax directed, and that every well-typed object has a unique type (but not necessarily a unique typing derivation).

When viewing a derivation bottom-up, we think of context join Ω_1, Ω_2 as *ordered context split* and context merge $\Delta_1 \times \Delta_2$ as *context split*. Both of these are non-deterministic when read in this way, that is, there may be many way to split a context $\Omega = \Omega_1, \Omega_2$ or $\Delta = \Delta_1 \times \Delta_2$.

The typing judgment has the form

$$\Gamma; \Delta; \Omega \vdash M : A$$

where Γ is the context of intuitistic assumptions, Δ is the context of linear assumptions, and Ω is the context of ordered assumptions.

Intuitionistic Functions $A {\rightarrow} B$**.**

$$\frac{}{(\Gamma_1, x{:}A, \Gamma_2); \cdot; \cdot \vdash x : A}\ \mathbf{ivar}$$

$$\frac{(\Gamma, x{:}A); \Delta; \Omega \vdash M : B}{\Gamma; \Delta; \Omega \vdash \lambda x{:}A.\ M : A \rightarrow B}\ {\rightarrow}I$$

$$\frac{\Gamma; \Delta; \Omega \vdash M : A \rightarrow B \qquad \Gamma; \cdot; \cdot \vdash N : A}{\Gamma; \Delta; \Omega \vdash M\, N : B}\ {\rightarrow}E$$

Linear Functions $A \multimap B$**.**

$$\frac{}{\Gamma; y{:}A; \cdot \vdash y : A}\ \mathbf{lvar}$$

$$\frac{\Gamma; (\Delta, y{:}A); \Omega \vdash M : B}{\Gamma; \Delta; \Omega \vdash \hat{\lambda} y{:}A.\ M : A \multimap B}\ {\multimap}I$$

$$\frac{\Gamma; \Delta_1; \Omega \vdash M : A \multimap B \qquad \Gamma; \Delta_2; \cdot \vdash N : A}{\Gamma; (\Delta_1 \times \Delta_2); \Omega \vdash M\,\hat{}\,N : B}\ {\multimap}E$$

Ordered Variables.

$$\frac{}{\Gamma; \cdot; z{:}A \vdash z : A}\,\mathbf{ovar}$$

Right Ordered Functions $A \twoheadrightarrow B$**.**

$$\frac{\Gamma; \Delta; (\Omega, z{:}A) \vdash M : B}{\Gamma; \Delta; \Omega \vdash \overset{>}{\lambda} z{:}A.\ M : A \twoheadrightarrow B}\,{\twoheadrightarrow}I$$

$$\frac{\Gamma; \Delta_1; \Omega_1 \vdash M : A \twoheadrightarrow B \qquad \Gamma; \Delta_2; \Omega_2 \vdash N : A}{\Gamma; (\Delta_1 \times \Delta_2); (\Omega_1, \Omega_2) \vdash M \overset{>}{\ } N : B}\,{\twoheadrightarrow}E$$

Left Ordered Functions $A \rightarrowtail B$**.**

$$\frac{\Gamma; \Delta; (z{:}A, \Omega) \vdash M : B}{\Gamma; \Delta; \Omega \vdash \overset{<}{\lambda} z{:}A.\ M : A \rightarrowtail B}\,{\rightarrowtail}I$$

$$\frac{\Gamma; \Delta_2; \Omega_2 \vdash M : A \rightarrowtail B \qquad \Gamma; \Delta_1; \Omega_1 \vdash N : A}{\Gamma; (\Delta_1 \times \Delta_2); (\Omega_1, \Omega_2) \vdash M \overset{<}{\ } N : B}\,{\rightarrowtail}E$$

These rules enforce linearity and ordering constraints on assumptions through the restrictions placed upon contexts.

In the three variable rules **ivar**, **lvar**, and **ovar**, the linear and ordered contexts must either be empty or contain only the subject variable, while the intuitionistic context is unrestricted. This forces linear and ordered assumptions made in the $\multimap I$ and $\twoheadrightarrow I$ rules to be appear at least once in a term.

In the $\multimap E$ and $\twoheadrightarrow E$ rules, the linear context is split into two disjoint parts (when reading from the bottom up), which means that each assumption can be used at most once. In the $\rightarrow E$ rules, all linear assumption propagate to the left premise. These observations together show that each linear variable is used at most once. Since it is also used at least once by the observation made about the variable rules, linear assumptions occur exactly once.

In the $\twoheadrightarrow E$ rules, the ordered context is split in an order-preserving way, with the leftmost assumptions Ω_1 going to the left premise and the rightmost assumptions Ω_2 going to the right premise. In the $\multimap E$ and $\rightarrow E$ rules the whole ordered context Ω goes to the left premise. These observations, together with the observation on the variable rules, show that ordered assumptions occur exactly once and in the order they were made.

As we will see, the emptiness restrictions on the linear and ordered contexts in the $\multimap E$ and $\rightarrow E$ rules are necessary to guarantee subject reduction. The reduction rules, of course, are simply β-reduction for all three kinds of functions. We will later also consider a form of η-expansion.

Reduction Rules.

$$\begin{array}{rcl}
(\lambda x.\ M)\, N & \Longrightarrow & [N/x]M \\
(\hat{\lambda} x.\ M)\,\hat{}\, N & \Longrightarrow & [N/x]M \\
(\overset{>}{\lambda} x.\ M)^{>}\, N & \Longrightarrow & [N/x]M \\
(\overset{<}{\lambda} x.\ M)^{<}\, N & \Longrightarrow & [N/x]M
\end{array}$$

In order to prove subject reduction we proceed to establish the expected structural properties for contexts and then verify the expected substitution lemmas.

Lemma 6.1 *The following structural properties hold for derivations in the implicational fragment of INCLL.*

1. *(Intuitionistic Exchange)*
   If $(\Gamma_1, x{:}A, x'{:}A', \Gamma_2); \Delta; \Omega \vdash M : B$ *then* $(\Gamma_1, x'{:}A', x{:}A, \Gamma_2); \Delta; \Omega \vdash M : B$.

2. *(Intuitionistic Weakening)*
   If $(\Gamma_1, \Gamma_2); \Delta; \Omega \vdash M : B$ *then* $(\Gamma_1, x{:}A, \Gamma_2); \Delta; \Omega \vdash M : B$.

3. *(Intuitionistic Contraction)*
   If $(\Gamma_1, x{:}A, \Gamma_2, x'{:}A, \Gamma_3); \Delta; \Omega \vdash M : B$ *then* $(\Gamma_1, x{:}A, \Gamma_2, \Gamma_3); \Delta; \Omega \vdash [x/x']M : B$.

4. *(Linear Exchange)*
   If $\Gamma; (\Delta_1, y{:}A, y'{:}A', \Delta_2); \Omega M : B$ *then* $\Gamma; (\Delta_1, y'{:}A', y{:}A, \Delta_2); \Omega \vdash M : B$.

Proof: By straightforward induction on the structure of the given derivations.
□

It is easy to construct counterexamples to the missing properties such as "linear contraction" or "ordered exchange". With these properties we can now establish the critical substitution lemmas.

Lemma 6.2 *The following substitution properties hold for the implicational fragment of INCLL.*

1. *(Intuitionistic Substitution)*
   If $(\Gamma_1, x{:}A, \Gamma_2); \Delta; \Omega \vdash M : B$ *and* $\Gamma_1; \cdot; \cdot \vdash N : A$ *then* $(\Gamma_1, \Gamma_2); \Delta; \Omega \vdash [N/x]M : B$.

2. *(Linear Substitution)*
   If $\Gamma; (\Delta_1, y{:}A, \Delta_2); \Omega \vdash M : B$ *and* $\Gamma; \Delta'; \cdot \vdash N : A$ *then* $\Gamma; (\Delta_1, \Delta', \Delta_2); \Omega \vdash [N/x]M : B$.

3. *(Ordered Substitution)*
   If $\Gamma; \Delta; (\Omega_1, x{:}A, \Omega_2) \vdash M : B$ *and* $\Gamma; \Delta'; \Omega' \vdash N : A$ *then* $\Gamma; (\Delta \times \Delta'); (\Omega_1, \Omega', \Omega_2) \vdash [N/x]M : B$.

Proof: By induction over the structure of the given typing derivation for M in each case, using Lemma 6.1. □

Subject reduction now follows immediately.

Theorem 6.3 (Subject Reduction) *If $M \Longrightarrow M'$ and $\Gamma; \Delta; \Omega \vdash M : A$ then $\Gamma; \Delta; \Omega \vdash M' : A$.*

Proof: For each reduction, we apply inversion to the giving typing derivation and then use the substitution lemma 6.2 to obtain the typing derivation for the conclusion. □

We also have three forms of η-expansion.

Theorem 6.4 (Subject Expansion) *The following η-expansion properties hold for the implication fragment of INCLL.*

1. *(Intuitionistic Expansion) If $\Gamma; \Delta; \Omega \vdash M : A \to B$ then $\Gamma; \Delta; \Omega \vdash \lambda x{:}A.\ M\ x : A \to B$.*
2. *(Linear Expansion) If $\Gamma; \Delta; \Omega \vdash M : A \multimap B$ then $\Gamma; \Delta; \Omega \vdash \hat{\lambda} y{:}A.\ M\,\hat{}\,y : A \multimap B$.*
3. *(Right Ordered Expansion) If $\Gamma; \Delta; \Omega \vdash M : A \twoheadrightarrow B$ then $\Gamma; \Delta; \Omega \vdash \overset{>}{\lambda} z{:}A.\ M \,^{>}\, z : A \twoheadrightarrow B$.*
4. *(Left Ordered Expansion) If $\Gamma; \Delta; \Omega \vdash M : A \rightarrowtail B$ then $\Gamma; \Delta; \Omega \vdash \overset{<}{\lambda} z{:}A.\ M \,^{<}\, z : A \rightarrowtail B$.*

Proof: By a straightforward derivation in each case, using intuitionistic weakening for intuitionistic expansion. □

We also believe that our calculus satisfies the normalization and Church-Rosser properties, and that canonical form (that is, long $\beta\eta$-normal forms) exist for well-typed objects. We have not checked all details for the above formulation, but it appears that these properties can be established by straightforward logical relations arguments.

6.2 Other Logical Connectives

Putting off the ordered left implication for the moment, there are other multiplicative and additive connectives. However, there is no explosion in the number of connectives, since the link between linear and ordered hypotheses rules out certain possibilities. The coupling between the linear and ordered hypotheses arises from a desire to look at linear hypotheses as a form of intuitionistic hypotheses whose use is restricted, and at ordered hypotheses as a form of linear hypotheses whose use is even further restricted. Extension of our core so far should therefore preserve the following property of *demotion*.

Lemma 6.5 *The following structural properties hold for derivations in the right implicational fragment of INCLL.*

1. *(Linear Demotion)*
   If $(\Gamma_1, \Gamma_2); (\Delta_1, y{:}A, \Delta_2); \Omega \vdash M : B$ *then* $(\Gamma_1, x{:}A, \Gamma_2); (\Delta_1, \Delta_2); \Omega \vdash [x/y]M : B$.

2. *(Ordered Demotion)*
   If $\Gamma; (\Delta_1, \Delta_2); (\Omega_1, z{:}A, \Omega_2) \vdash M : B$ *then* $\Gamma; (\Delta_1, y{:}A, \Delta_2); (\Omega_1, \Omega_2) \vdash [y/z]M : B$.

Proof: In both cases by induction on the structure of the given derivation. □

Preserving this property means that we cannot have a connective which, for example, behaves multiplicatively on the linear context and additively on the ordered context. Some other connectives which we do not show below are definable through use of the modal operators in a way which even preserves the structure of proofs.

Tensor $A \otimes B$**.** This is adjoint to the right ordered implication, that is, $(A \otimes B) \twoheadrightarrow C$ iff $A \twoheadrightarrow (B \twoheadrightarrow C)$. Its rules introduce commutative conversions into the proof term calculus, and canonical forms no longer exist.

$$\frac{\Gamma; \Delta_1; \Omega_1 \vdash M{:}A \qquad \Gamma; \Delta_2; \Omega_2 \vdash N{:}B}{\Gamma; (\Delta_1 \times \Delta_2); (\Omega_1, \Omega_2) \vdash M \otimes N : A \otimes B} \otimes\mathrm{I}$$

$$\frac{\Gamma; \Delta_2; \Omega_2 \vdash M : A \otimes B \qquad \Gamma; \Delta_1; (\Omega_1, z{:}A, z'{:}B, \Omega_3) \vdash N : C}{\Gamma; (\Delta_1 \times \Delta_2); (\Omega_1, \Omega_2, \Omega_3) \vdash \mathbf{let}\, z \otimes z' = M \,\mathbf{in}\, N : C} \otimes\mathrm{E}$$

Besides destroying the existence of canonical forms, this connective also complicates the simple functional interpretation of the ordered context Ω as describing a stack. The problem is foreshadowed in the substitution lemma, where we also have to allow ordered variables to the left and right of the variable to be substituted. The new reduction rule is rather straightforward.[1]

$$\mathbf{let}\, z \otimes z' = M \otimes M' \,\mathbf{in}\, N \Longrightarrow [M/z, M'/z']N$$

Multiplicative Unit 1. The is the right and left unit element for the tensor connective. We have $\mathbf{1} \twoheadrightarrow C$ iff C off $\mathbf{1} \rightarrowtail C$, and $A \otimes \mathbf{1}$ iff A iff $\mathbf{1} \otimes A$. The

[1] It seems plausible that the restriction of this rule to $\Omega_3 = \cdot$ is also sound and complete, and that the general form is admissible in the system with the restricted rule. This would form a much better basis for functional language applications of this calculus, since the stack-like nature of accesses to the ordered context is preserved. Similar remarks may hold for the other elimination rules of a similar shape.

introduction rule shows why there is only one multiplicative unit.

$$\frac{}{\Gamma; \cdot; \cdot \vdash \star : \mathbf{1}}\ \mathbf{1}\mathrm{I}$$

$$\frac{\Gamma; \Delta_2; \Omega_2 \vdash M : \mathbf{1} \qquad \Gamma; \Delta_1; (\Omega_1, \Omega_3) \vdash N : C}{\Gamma; (\Delta_1 \times \Delta_2); (\Omega_1, \Omega_2, \Omega_3) \vdash \mathbf{let}\, \star = M \,\mathbf{in}\, N : C}\ \mathbf{1}\mathrm{E}$$

The reduction rule is straightforward.

$$\mathbf{let}\, \star = \star \,\mathbf{in}\, N \Longrightarrow N$$

Additive Conjunction $A \& B$. This is additive on both the linear an ordered contexts, in order to preserve demotion.

$$\frac{\Gamma; \Delta; \Omega \vdash M : A \qquad \Gamma; \Delta; \Omega \vdash N : B}{\Gamma; \Delta; \Omega \vdash \langle M, N \rangle : A \& B}\ \&\mathrm{I}$$

$$\frac{\Gamma; \Delta; \Omega \vdash M : A \& B}{\Gamma; \Delta; \Omega \vdash \mathbf{fst}\, M : A}\ \&\mathrm{E}_1 \qquad \frac{\Gamma; \Delta; \Omega \vdash M : A \& B}{\Gamma; \Delta; \Omega \vdash \mathbf{snd}\, M : B}\ \&\mathrm{E}_2$$

$$\begin{array}{rcl} \mathbf{fst}\, \langle M, N \rangle & \Longrightarrow & M \\ \mathbf{snd}\, \langle M, N \rangle & \Longrightarrow & N \end{array}$$

Additive Unit $\top$. Because it is additive, the left and right units for $\&$ coincide.

$$\frac{}{\Gamma; \Delta; \Omega \vdash \langle\,\rangle : \top}\ \top I$$

(no $\top E$ rule)

Since there is no elimination rule, there are no reduction for the additive unit.

Disjunction $\oplus$. The disjunction in intuitionistic linear logic and its non-commutative refinement is additive. Therefore the connective does not split into left and right disjunction.

$$\frac{\Gamma; \Delta; \Omega \vdash M : A}{\Gamma; \Delta; \Omega \vdash \mathbf{inl}^B\, M : A \oplus B}\ \oplus\mathrm{I}_1 \qquad \frac{\Gamma; \Delta; \Omega \vdash M : A}{\Gamma; \Delta; \Omega \vdash \mathbf{inr}^A\, M : A \oplus B}\ \oplus\mathrm{I}_2$$

$$\frac{\Gamma; \Delta_2; \Omega_2 \vdash M : A \oplus B \qquad \Gamma; \Delta_1; (\Omega_1, z{:}A, \Omega_3) \vdash N : C \qquad \Gamma; \Delta_1; (\Omega_1, z'{:}B, \Omega_3) \vdash N' : C}{\Gamma; (\Delta_1 \times \Delta_2); (\Omega_1, \Omega_2, \Omega_3) \vdash \mathbf{case}\, M \,\mathbf{of}\, \mathbf{inl}\, z \Rightarrow N \mid \mathbf{inr}\, z' \Rightarrow N' : C}\ \oplus\mathrm{E}$$

$$\begin{array}{lcl} \textbf{case}\,\textbf{inl}^B\,M\ \textbf{of}\ \textbf{inl}\,z \Rightarrow N \mid \textbf{inr}\,z' \Rightarrow N' & \Longrightarrow & [M/z]N \\ \textbf{case}\,\textbf{inr}^A\,M'\ \textbf{of}\ \textbf{inl}\,z \Rightarrow N \mid \textbf{inr}\,z' \Rightarrow N' & \Longrightarrow & [M'/z]N' \end{array}$$

Additive Falsehood 0. This is the unit for disjunction. Since it is additive, it does not split into left and right versions.

$$\textit{(no } \mathbf{0} \textit{ introduction rule)}$$

$$\frac{\Gamma; \Delta_2; \Omega_2 \vdash M : \mathbf{0}}{\Gamma; (\Delta_1 \times \Delta_2); (\Omega_1, \Omega_2, \Omega_3) \vdash \textbf{abort}^C\, M : C}\,\mathbf{0}\mathrm{E}$$

Since there is no introduction rule for **0**, there are no new reductions.

In analogy with linear logic, we have two modal operators: one allows an ordered assumption to become mobile (while it must remain linear), another one allows a linear assumption to become intuitionistic.

Mobility Modal ¡A.

$$\frac{\Gamma; \Delta; \cdot \vdash M : A}{\Gamma; \Delta; \cdot \vdash\ ¡M : ¡A}\,¡I$$

$$\frac{\Gamma; \Delta_2; \Omega_2 \vdash M : ¡A \qquad \Gamma; (\Delta_1, y{:}A); (\Omega_1, \Omega_3) \vdash N : C}{\Gamma; (\Delta_1 \times \Delta_2); (\Omega_1, \Omega_2, \Omega_3) \vdash \textbf{let}\ ¡y = M\ \textbf{in}\ N : C}\,¡E$$

$$\textbf{let}\ ¡y = ¡M\ \textbf{in}\ N \quad \Longrightarrow \quad [M/y]N$$

Linear Exponential $!A$.

$$\frac{\Gamma; \cdot; \cdot \vdash M : A}{\Gamma; \cdot; \cdot \vdash !M : !A}\,!I$$

$$\frac{\Gamma; \Delta_2; \Omega_2 \vdash M : !A \qquad (\Gamma, x{:}A); \Delta_1; (\Omega_1, \Omega_3) \vdash N : C}{\Gamma; (\Delta_1 \times \Delta_2); (\Omega_1, \Omega_2, \Omega_3) \vdash \textbf{let}\ !x = M\ \textbf{in}\ N : C}\,!E$$

$$\textbf{let}\ !x = !M\ \textbf{in}\ N \quad \Longrightarrow \quad [M/x]N$$