Lecture Notes on
Identity and Inversion

15-816: Linear Logic
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Lecture 8
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In the last lecture we saw cut elimination as the global version of cut reduction. In this lecture we begin with identity, which is the global version of identity expansion. Together, they provide the basis for understanding the left and right rules in the sequent calculus as *meaning explanations* of the logical connectives, a program with a long history [Dum91, ML83].

The cut-free sequent calculus is a good basis for proof search, but it still has too much nondeterminism. One way to reduce this nondeterminism is *inversion*, which we discuss in this lecture. Another is *chaining*, which will be subject of the next lecture. Together, these two strategies make up *focusing* [And92], which has found many applications in logic and programming languages.

1 Admissibility of Identity

The admissibility of cut in the cut-free sequent calculus

$$\frac{\Gamma \,;\, \Delta \Rightarrow A \quad \Gamma \,;\, \Delta', A \Rightarrow C}{\Gamma \,;\, \Delta, \Delta' \Rightarrow C}\ (\mathsf{cut}_A)$$

means that we do not have to consider this rule when searching for a proof. This is extremely important because it means we do not have to pluck arbitrary formulas A out of thin air to prove and then use as a lemma. All the remaining rules just decompose the connectives when the rules are viewed from the conclusion to the premises.

Cut also establishes that if we can prove A, we can use A. The opposite, namely that if we can use A then we can prove A is the contents of the *admissibility of identity*:

$$\frac{}{\Gamma \,;\, A \Rightarrow A}\ (\mathsf{id}_A)$$

Of course, so far this has been a rule in our cut-free sequent calculus, so for this theorem to make sense, we first have to remove it. Intuitively, the reason we might think this is true because identity expansions reduce identity at compound formulas to identity used only at subformulas. We can carry this only so far, however: atomic formulas P have no constituents. We therefore need to retain identity as a rule, but for atomic formulas only.

$$\frac{}{\Gamma \,;\, P \Rightarrow P}\ \mathsf{id}_P$$

where P must be an atomic formula. Fortunately, the proof of the admissibility of cut from the previous lecture is not affected by this change.

Theorem 1 (Admissibility of Identity) $\Gamma \,;\, A \Rightarrow A$ *for any* Γ *and* A*.*

Proof: By induction on the structure of A. In each case we take the idea embodied in the identity expansion. For example:

Case: $A = A_1 \multimap A_2$. We construct:

$$\cfrac{\cfrac{\begin{array}{c}\text{i.h.}(A_1)\\ \Gamma \,;\, A_1 \Rightarrow A_1\end{array} \quad \begin{array}{c}\text{i.h.}(A_2)\\ \Gamma \,;\, A_2 \Rightarrow A_2\end{array}}{\Gamma \,;\, A_1 \multimap A_2, A_1 \Rightarrow A_2}\ {\multimap}L}{\Gamma \,;\, A_1 \multimap A_2 \Rightarrow A_1 \multimap A_2}\ {\multimap}R$$

Case: $A = {!}A'$. We construct:

$$\cfrac{\cfrac{\cfrac{\begin{array}{c}\text{i.h.}(A')\\ \Gamma, A' \,;\, A' \Rightarrow A'\end{array}}{\Gamma, A' \,;\, \cdot \Rightarrow A'}\ \mathsf{copy}_{A'}}{\Gamma, A' \,;\, \cdot \Rightarrow {!}A'}\ {!}R}{\Gamma \,;\, {!}A' \Rightarrow {!}A'}\ {!}L$$

□

We can now restate the previous cut elimination theorem as a cut and identity elimination theorem.

Theorem 2 (Cut and Identity Elimination) *If* $\Gamma \,;\, \Delta \vdash A$ *then* $\Gamma \,;\, \Delta \Rightarrow A$*.*

Proof: By induction on the structure of the given derivation. In all cases except cut and identity we replay the given rule in the cut-free calculus. In these two cases we appeal to the admissibility of cut and identity. □

2 Inversion

We say an inference rule is *invertible* if whenever the conclusion holds, then so do the premises. We will refine this notion slightly for our purposes, but let's examine this straightforward definition first. For example, consider

$$\frac{\Delta, A \Rightarrow B}{\Delta \Rightarrow A \multimap B}\ {\multimap}R$$

To show that this rule is invertible, we would have to show:

$$\frac{\Delta \Rightarrow A \multimap B}{\Delta, A \Rightarrow B}\ ({\multimap}R^{-1})$$

Our, by now hopefully highly tuned instinct would tell us to prove this by induction on the structure of the proof of $\Delta \Rightarrow A \multimap B$. If the last rule is ${\multimap}R$, we are done, because the premise matches what we have to show. In other cases we appeal to the induction hypothesis and then re-apply the last rule.

This is a reasonable approach and meets with success. But there is a more succinct way of proving this, exploiting the admissibility of identity and cut.

Theorem 3 (Invertibility of ${\multimap}R$) *The rule*

$$\frac{\Delta \Rightarrow A \multimap B}{\Delta, A \Rightarrow B}\ ({\multimap}R^{-1})$$

is admissible.

Proof: Using admissibility of cut and identity, we construct

$$\dfrac{\Delta \Rightarrow A \multimap B \qquad \dfrac{\dfrac{}{A \Rightarrow A}\,(\mathsf{id}_A) \qquad \dfrac{}{B \Rightarrow B}\,(\mathsf{id}_B)}{A, A \multimap B \Rightarrow B}\,{\multimap}L}{\Delta, A \Rightarrow B}\,(\mathsf{cut}_{A \multimap B})$$

□

One nice property of this proof is that we do not need to reconsider it when we extend the logic, as long as we make sure cut and identity remain admissible (which should *always* be the case). Our previous inductive proof would have to be reconsidered, because additional cases arise.

We can also observe that applying the algorithm implied by our proof of cut admissibility will essentially simulate our first inductive proof. We cannot push the cut upwards in the second premise, since the cut formula $A \multimap B$ is the principal formula. Instead, we push it up in the first premise until $A \multimap B$ is introduced by its right rule. There we apply a cut reduction. Since the premises are the identity the residual cuts at type A and B immediately disappear (see Exercise 1).

3 Negative and Positive Formulas

We can now systematically examine all the connectives of linear logic to see if their left or right rules are invertible. The idea is to apply this in proof search in the following way: if a formula appears on the right with an invertible right rule, we immediately decompose it by its right rule, and similarly when a formula appears on the left with an invertible left rule. We can do this for two reasons. First the rule is invertible so we do not lose provability of the sequent. We do not make any choices which may lead us down a dead-end path in the proof search. Second, each left or right rule eliminates a connective, so the sequents in the premises are all smaller in that they contain fewer connectives. This means we will not get into an infinite loop with this strategy alone.

However, there is a small caveat. Consider the rule

$$\dfrac{\Gamma\,;\cdot \Rightarrow A}{\Gamma\,;\cdot \Rightarrow\, !A}\,!R$$

This is invertible:

$$\dfrac{\Gamma\,;\,\cdot \Rightarrow\, !A \qquad \dfrac{\dfrac{\dfrac{}{\Gamma, A\,;\,A \Rightarrow A}\,(\mathsf{id}_A)}{\Gamma, A\,;\,\cdot \Rightarrow A}\,\mathsf{copy}}{\Gamma\,;\,!A \Rightarrow A}\,!L}{\Gamma\,;\,\cdot \Rightarrow A}\,(\mathsf{cut}_{!A})$$

But we cannot apply our strategy above: when we see $!A$ on the right of a sequent, we may not be able to apply $!R$ because there may be ephemeral resources we first have to consume.

When we refer to *inversion* of right or left rules for a connective we mean that the formula with that connective appears at the top level on the right or left of a sequent, it can immediately decomposed. Andreoli called such connectives *asynchronous*. Following standard terminology we call connectives *negative* if their right rule is invertible and *positive* if their left rule is invertible in this sense.

An easy way to conjecture the polarity of a connective is to consider the identity expansion. If it starts with a right rule, the connective is invertible on the right; if it starts with a left rule, the connective is invertible on the left. Of course, that's not a proof, but it appears that this always holds. Also, the identity expansion may provide an easy counterexample for invertibility of the connective on the other side of the sequent. We obtain the following table

Negative	$A \multimap B$, $A \mathbin{\&} B$, $\top$
Positive	$A \otimes B$, $\mathbf{1}$, $A \oplus B$, $\mathbf{0}$, $!A$

Atomic formulas don't have any particular polarity, because they do not have any left or right rules associated with them. We come back to this point in the next lecture.

Exercises

Exercise 1 In the proof of cut admissibility in Lecture 7 we noticed that a (cut_A) of a proof $\mathcal{D}$ with an instance identity id_A converts to $\mathcal{D}$, and similarly for a (cut_A) of the identity id_A with any $\mathcal{E}$.

Prove that this property continues to hold when one of the premises of the cut is an instance of the admissibility identity rule. That is,

$$\dfrac{\begin{array}{c}\mathcal{D}\\ \Gamma\,;\Delta \Rightarrow A\end{array} \quad \dfrac{}{\Gamma\,;A \Rightarrow A}\,(\mathsf{id}_A)}{\Gamma\,;\Delta \Rightarrow A}\,(\mathsf{cut}_A)$$

reduces to $\mathcal{D}$ by the construction in the proofs of cut admissibility and identity.

Exercise 2 In an earlier lecture we defined $A \dashv\vdash B$ as $A \vdash B$ and $B \vdash A$. Internalize $A \dashv\vdash B$ as a proposition $A \circ\!\!-\!\!\circ B$.

(i) Give its right and left rules. Make sure they are pure, that is, they do not refer to other connectives or constants, only the constituent propositions A and B and the judgments we have introduced.

(ii) Verify cut reduction.

(iii) Verify identity expansion.

(iv) Classify the connective as negative or positive, proving the appropriate inversion properties.

(v) Briefly discuss plausible alternative right and left rules for $A \circ\!\!-\!\!\circ B$ and justify your choice of rules.

Exercise 3 Proceed as in Exercise 2 for *unrestricted implication* $A \supset B$, which means that A can be used arbitrarily often in the proof of B. It should therefore be equivalent to $(!A) \multimap B$, so you are asked to explore if it would make sense as a primitive connective.

References

[And92] Jean-Marc Andreoli. Logic programming with focusing proofs in linear logic. *Journal of Logic and Computation*, 2(3):197–347, 1992.

[Dum91] Michael Dummett. *The Logical Basis of Metaphysics*. Harvard University Press, Cambridge, Massachusetts, 1991. The William James Lectures, 1976.

[ML83] Per Martin-Löf. On the meanings of the logical constants and the justifications of the logical laws. Notes for three lectures given in Siena, Italy. Published in *Nordic Journal of Philosophical Logic*, 1(1):11-60, 1996, April 1983.