Lecture Notes on Harmony

15-816: Linear Logic
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Lecture 3
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In the last lecture we started developing the propositional connectives of linear logic. Specifically, we introduced simultaneous conjunction $A \otimes B$ and linear implication $A \multimap B$. Each connective is determined by giving left and right rules in the sequent calculus that specify how to use a resource or prove a goal with that connective. We also explicated two tests we apply to determine if the logical connectives are meaningful: identity expansion and cut reduction. If they both hold we say that the left and right rules for the connectives are *in harmony*. We continue to develop additional connectives, and also explore some consequences of a failure of harmony between left and right rules.

1 Multiplicative Unit

Whenever we have a binary operator, $A \otimes B$ in this case, we should consider if it has a unit. In this case, the unit is written $\mathbf{1}$, and has the property that $A \otimes \mathbf{1}$ and $\mathbf{1} \otimes A$ are equivalent to A. We can try to derive the rules for $\mathbf{1}$ systematically, by thinking of it as a multiplicative conjunction with zero components. Below, we show the binary rule first, and the nullary rule second.

$$\frac{\Delta \vdash A \quad \Delta' \vdash B}{\Delta, \Delta' \vdash A \otimes B} \otimes R \qquad \frac{}{\cdot \vdash \mathbf{1}} \mathbf{1}R$$

$A \otimes B$ has two components, so $\otimes R$ has two premises. Correspondingly, $\mathbf{1}$ has no components and no premises. This also means that we require that there are no resources, indicated above by writing '$\cdot$'. In summary, $\mathbf{1}$ holds as a goal if we have no resources.

Looking at the left rules:

$$\frac{\Delta, A, B \vdash C}{\Delta, A \otimes B \vdash C}\ \otimes L \qquad \frac{\Delta \vdash C}{\Delta, \mathbf{1} \vdash C}\ \mathbf{1}L$$

$A \otimes B$ has two components, so they both appear as new resources in the premise. $\mathbf{1}$ has no components, so no new resources appear in the premise.

The left and right rules for $\mathbf{1}$ are in balance. First, the local expansion:

$$\frac{}{\mathbf{1} \vdash \mathbf{1}}\ \mathsf{id}_{\mathbf{1}} \quad \longrightarrow_E \quad \frac{\dfrac{}{\cdot \vdash \mathbf{1}}\ \mathbf{1}R}{\mathbf{1} \vdash \mathbf{1}}\ \mathbf{1}L$$

and then the reduction

$$\frac{\dfrac{}{\cdot \vdash \mathbf{1}}\ \mathbf{1}R \quad \dfrac{\Delta' \vdash C}{\Delta', \mathbf{1} \vdash C}\ \mathbf{1}L}{\Delta' \vdash C}\ \mathsf{cut}_{\mathbf{1}} \quad \longrightarrow_R \quad \Delta' \vdash C$$

We should verify that $A \otimes \mathbf{1}$ is equivalent to A. But what does this mean? We take it to mean that both $A \otimes \mathbf{1} \vdash A$ and $A \vdash A \otimes \mathbf{1}$. We just give a short example proof, the other direction and other versions are similarly easy to prove.

$$\frac{\dfrac{\dfrac{}{A \vdash A}\ \mathsf{id}_A}{A, \mathbf{1} \vdash A}\ \mathbf{1}L}{A \otimes \mathbf{1} \vdash A}\ \otimes L$$

The multiplicative unit is useful in specifications to eliminate resources, using the idiom $A \multimap \mathbf{1}$.

2 Failure of Harmony

We investigate the consequences of a failure of harmony in one case. Let's assume had replaced the left rule for simultaneous conjunction with the following two rules:

$$\frac{\Delta, A \vdash C}{\Delta, A \otimes B \vdash C}\ \otimes L_1?? \qquad \frac{\Delta, B \vdash C}{\Delta, A \otimes B \vdash C}\ \otimes L_2??$$

First we note that identity expansion indeed fails. We show one attempt; others fail similarly.

$$\frac{}{A \otimes B \vdash A \otimes B}\ \mathsf{id}_{A\otimes B} \qquad \frac{\dfrac{\dfrac{}{A \vdash A}\ \mathsf{id}_A \quad \dfrac{??}{\cdot \vdash B}}{A \vdash A \otimes B}\ {\otimes}R}{A \otimes B \vdash A \otimes B}\ {\otimes}L_1??$$

It turns out that with these the two incorrect left rules, *weakening*

$$\frac{\Delta \vdash C}{\Delta, A \vdash C}\ \mathsf{weaken}$$

becomes a derived rule of inference. This, of course, violates the basic principle that ephemeral resources must be used exactly once, because A is indeed not used. Before reading on, we suggest you try to prove this yourself.

$$\dfrac{\dfrac{}{A \vdash A}\,\mathsf{id}_A \quad \dfrac{}{\cdot \vdash \mathbf{1}}\,\mathbf{1}R}{A \vdash A \otimes \mathbf{1}}\,\otimes R \quad \dfrac{\dfrac{\Delta \vdash C}{\Delta, \mathbf{1} \vdash C}\,\mathbf{1}L}{\Delta, A \otimes \mathbf{1} \vdash C}\,\otimes L_2??$$
$$\dfrac{A \vdash A \otimes \mathbf{1} \quad \Delta, A \otimes \mathbf{1} \vdash C}{\Delta, A \vdash C}\,\mathsf{cut}_{A\otimes\mathbf{1}}$$

3 Alternative Conjunction

Using our rules for coin exchange, we can prove $\mathsf{q} \vdash \mathsf{d}\otimes\mathsf{d}\otimes\mathsf{n}$ and also $\mathsf{q} \vdash \mathsf{q}$. But clearly we can *not* prove $\mathsf{q} \vdash (\mathsf{d} \otimes \mathsf{d} \otimes \mathsf{n}) \otimes \mathsf{q}$, because we would have to use q twice in the proof. Still, with the resource q we can actually prove both! To express this situation inside the logic as a connective we write $A \mathbin{\&} B$, pronounced "*A with B*". It is variously called *alternative conjunction* or *additive conjunction*. With this connective we can prove $\mathsf{q} \vdash (\mathsf{d}\otimes\mathsf{d}\otimes\mathsf{n}) \mathbin{\&} \mathsf{q}$. We can achieve this as a goal from resources Δ exactly if we can achieve A from Δ, and also B from Δ.

$$\dfrac{\Delta \vdash A \quad \Delta \vdash B}{\Delta \vdash A \mathbin{\&} B}\,\& R$$

This appears to be a violation of linearity: the resources in Δ seem to be duplicated. But we are saved, because when we have a resource $A \mathbin{\&} B$ we have to choose between A and B. We cannot get both. This means that Δ is not really duplicated, since only one of the two premises will ever be used, as we will check shortly.

$$\dfrac{\Delta, A \vdash C}{\Delta, A \mathbin{\&} B \vdash C}\,\& L_1 \qquad \dfrac{\Delta, B \vdash C}{\Delta, A \mathbin{\&} B \vdash C}\,\& L_2$$

The informal reason that the "duplication" of Δ does not destroy linearity is reflected in the identity expansion and cut reduction. It is most clear in the reduction, so we show this first.

$$\dfrac{\dfrac{\Delta \vdash A \quad \Delta \vdash B}{\Delta \vdash A \mathbin{\&} B}\,\& R \quad \dfrac{\Delta', A \vdash C}{\Delta', A \mathbin{\&} B \vdash C}\,\& L_1}{\Delta, \Delta' \vdash C}\,\mathsf{cut}_{A\&B} \quad \longrightarrow_R \quad \dfrac{\Delta \vdash A \quad \Delta', A \vdash C}{\Delta, \Delta' \vdash C}\,\mathsf{cut}_A$$

In fact, there is another symmetric reduction, when the second premise of the cut is inferred with the $\&L_2$ rule. The identity expansion is

$$\dfrac{}{A \mathbin{\&} B \vdash A \mathbin{\&} B}\ \mathsf{id}_{A\&B} \quad \longrightarrow_E \quad \dfrac{\dfrac{\dfrac{}{A \vdash A}\ \mathsf{id}_A}{A \mathbin{\&} B \vdash A}\ \&L_1 \quad \dfrac{\dfrac{}{B \vdash B}\ \mathsf{id}_B}{A \mathbin{\&} B \vdash B}\ \&L_2}{A \mathbin{\&} B \vdash A \mathbin{\&} B}\ \&R$$

4 Additive Unit

As for multiplicative conjunction with unit $\mathbf{1}$, there is a nullary version of additive conjunction. The unit is $\top$ (pronounced "*top*"). Again, its rules can be derived systematically from the rules for additive conjunction. The right rule has zero premises, propagating the resources Δ to all of them. $A \mathbin{\&} B$ has two conjuncts and therefore two left rules; $\top$ has zero conjuncts and therefore zero left rules.

$$\dfrac{}{\Delta \vdash \top}\ \top R \qquad \text{no } \top L \text{ rule}$$

Of course, there is now no cut reduction since there is no left rule for $\top$. But there is still an identity expansion.

$$\dfrac{}{\top \vdash \top}\ \mathsf{id}_\top \quad \longrightarrow_E \quad \dfrac{}{\top \vdash \top}\ \top R$$

5 Disjunction

We have seen that alternative conjunction stands for a kind of choice. If we have a resource $A \mathbin{\&} B$ we have a choice wether to use A or B in our proof. If we have a goal $A \mathbin{\&} B$ we have to account for both uses.

Disjunction is symmetric to this. If we have a resource $A \oplus B$ the environment must provide either A or B, so we have to account for both possibilities. Conversely, if we have a goal $A \oplus B$ we can satisfy it by providing A or by providing B.

$$\dfrac{\Delta \vdash A}{\Delta \vdash A \oplus B}\ \oplus R_1 \qquad \dfrac{\Delta \vdash B}{\Delta \vdash A \oplus B}\ \oplus R_2 \qquad \dfrac{\Delta, A \vdash C \quad \Delta, B \vdash C}{\Delta, A \oplus B \vdash C}\ \oplus L$$

We leave the identity expansion and cut reduction to Exercise 5.

6 Disjunctive Unit

There is also a unit of disjunction $\mathbf{0}$, which is a form of falsehood. We can obtain its rules as the nullary versions of disjunction.

$$\text{no } \mathbf{0}R \text{ rule} \qquad \frac{}{\Delta, \mathbf{0} \vdash C}\ \mathbf{0}L$$

We leave its properties to Exercise 6.

7 Persistent Truth

The most difficult connective to add is one to internalize persistent truth as a proposition. It turns out for many purposes it is convenient to retain the judgment $A\ \mathit{pers}$, although this is not the only possible choice. This necessitates a judgmental rule, connection the judgments of ephemeral and persistent truth. But first, we generalize the sequents themselves by segregating ephemeral and persistent propositions. We write

$$\underbrace{B_1\ \mathit{pers}, \ldots, B_k\ \mathit{pers}}_{\Gamma}\ ;\ \underbrace{A_1\ \mathit{eph}, \ldots, A_n\ \mathit{eph}}_{\Delta} \vdash C\ \mathit{eph}$$

Now our rules will have to respect the ephemeral nature of the The persistent and ephemeral resources are separated by a semicolon ";" so that we can tell the status of a resource from its position in the sequent.

First, we have a judgmental rule that allows us to make an ephemeral copy of a persistent resource.

$$\frac{\Gamma, A\ \mathit{pers}\ ;\ \Delta, A\ \mathit{eph} \vdash C\ \ \mathit{eph}}{\Gamma, A\ \mathit{pers}\ ;\ \Delta \vdash C\ \ \mathit{eph}}\ \mathsf{copy}$$

This rule applies to arbitrary A. One may also wonder if there is a correspoding structural rule for a goal $C\ \mathit{pers}$. If there were such a rule, it would be:

$$\left[\ \frac{\Gamma\ ;\ \cdot \vdash A\ \mathit{eph}}{\Gamma\ ;\ \cdot \vdash A\ \mathit{pers}}\ \mathsf{valid}\ \right]$$

It expresses that we judge something to be persistently true if its proof does not depend on any ephemeral resources. Since all resources in Γ are persistent, this means we can produce as many copies of A as we like, thereby satisfying the demain of the copy rule. However, as the brackets indicate, we do not have an explicit rule of this form, but immediately reduce $A\ pers$ when it appears on the right-hand side of a sequent to $A\ eph$. This is justified when we view $A\ pers$ as being defined by the property of being true without the use of any ephemeral resources: the premise and conclusion of this are equal by definition, and we therefore always replace the conclusion by the premise.

Just as before, we should have rules that connect the role of persistent truth as a resource and goal. The new identity rule

$$\left[\ \frac{}{\Gamma, A\ pers\ ;\ \cdot \vdash A\ eph}\ \mathsf{id}^!_A\ \right]$$

is immediately derivable from the copy rule and identity at A, so we do not need this as an explicit rule. The converse, namely that proving that A is persistently true allows us to make persistent use of A is needed, however.

$$\frac{\Gamma\ ;\ \cdot \vdash A\ eph \quad \Gamma, A\ pers\ ;\ \Delta \vdash C\ eph}{\Gamma\ ;\ \Delta \vdash C\ eph}\ \mathsf{cut}^!_A$$

Note that in both $\mathsf{id}^!_A$ (which we omitted) and $\mathsf{cut}^!_A$, we have exploited the judgmental definition of persistent truth in the conclusion, replacing $A\ pers$ by $A\ eph$ with an empty collection of ephemeral resources.

At this point all the rules of linear logic so far have to be generalize to allow persistent resources. Just as in the $\mathsf{cut}^!$, persistent resources in the conclusion and all premises are all the same in the rules we have discussed so far. This is correct since persistent resources may be duplicated and need not be consumed. We will also tacitly adjoin an unused persistent resource to a sequent whenever this turns out to be necessary.

8 Of Course!

Having laid down the extension of sequents to encompass persistence, we can now integrate it into propositions. We write $!A$ (read *"of course A"* or *"bang A"*) for the proposition which expresses that A is persistently true. We also refer to $!A$ as the *exponential modality*. The left rule takes an ephemeral resource of $!A$ and marks it as persistent.

$$\frac{\Gamma, A\ pers\ ;\ \Delta \vdash C\ eph}{\Gamma\ ;\ \Delta, !A\ eph \vdash C\ eph}\ !L$$

The right rule requires that A is persistently true, which is the same as A being ephemerally true, without ephemeral resources.

$$\frac{\Gamma \,;\, \cdot \vdash A \; eph}{\Gamma \,;\, \cdot \vdash\, !A \; eph} \; !R$$

Are these in harmony? Let's check. We omit the judgments eph and $pers$ now, because we can always tell from the position within the sequent. First identity expansion:

$$\dfrac{}{\cdot \,;\, !A \vdash\, !A}\;\mathsf{id}_{!A} \quad \longrightarrow_E \quad \dfrac{\dfrac{\dfrac{\dfrac{}{A \,;\, A \vdash A}\;\mathsf{id}_A}{A \,;\, \cdot \vdash A}\;\mathsf{copy}}{A \,;\, \cdot \vdash\, !A}\;!R}{\cdot \,;\, !A \vdash\, !A}\;!L$$

Second, cut reduction, which reduces a cut at $!A$ by a $\mathsf{cut}^!$ at A, reducing the complexity of the proposition.

$$\dfrac{\dfrac{\Gamma \,;\, \cdot \vdash A}{\Gamma \,;\, \cdot \vdash\, !A}\;!R \quad \dfrac{\Gamma, A \,;\, \Delta \vdash C}{\Gamma \,;\, \Delta, !A \vdash C}\;!L}{\Gamma \,;\, \Delta \vdash C}\;\mathsf{cut}_{!A}$$

$$\longrightarrow_R$$

$$\dfrac{\Gamma \,;\, \cdot \vdash A \quad \Gamma, A \,;\, \Delta \vdash C}{\Gamma \,;\, \Delta \vdash C}\;\mathsf{cut}^!_A$$

We will talk more about this and the other reductions in future lectures. The approach to modeling persistent truth in this manner goes back to Andreoli's *dyadic system* [And92] in classical linear logic and Hodas and Miller [HM94] in intuitionistic linear logic, which is a fragment of ours. A judgmental justification for this system can be found in Chang et al. [CCP03] which drew its inspiration from Pfenning and Davies [PD01].

9 Example: Beggars Revisited

Previously, we could not express the proverb "*If wishes were horses, beggars would ride.*" Unfortunately, we do not yet have quantifiers, so let me just say that $\forall x.\, A(x)$ is true if $A(y)$ is true for an arbitrary parameter y. Similarly, $\exists x.\, A(x)$ is true, if $A(t)$ is true for some object t. We can then say:

$$!(\forall x.\, \mathsf{wish}(x) \multimap !\mathsf{horse}(x)) \multimap !\forall y.\, !\mathsf{beggar}(y) \multimap \exists z.\, !\mathsf{horse}(z) \otimes \mathsf{rides}(y, z)$$

The whole proposition itself should be considered persistent, which we could express by wrapping another "!" around it on the outside.

A few words on operator precedence: "!" binds mostly tightly, "$\otimes$" is next followed by "$\multimap$". The quantifiers are a weak prefix, so their scope extends as far to the right as possible while remaining consistent with the parentheses that are present. This means that $\forall x.\, !A(x) \otimes B \multimap !C \otimes D$ is interpreted as $\forall x.\, (((!A(x)) \otimes B) \multimap ((!C) \otimes D))$.

10 Example: Two Nickels

We transcribe:

> *If I had a nickel for every time I had a nickel, I would have two nickels.*
> — Anonymous

as the persistent

$$!(\mathsf{n} \multimap \mathsf{n}) \multimap \mathsf{n} \otimes \mathsf{n}$$

It should be clear that this statement is not particularly true.

11 Example: Menu

We go through the exercise of representing a fixed price menu, as opposed ordering a la carte. Here is the menu:

Onion Soup or Mixed Salad

T-Bone Steak or Maine Lobster
(or both, for $20 more)

Peas and Carrots, or Fennel (according to season)

Chocolate Cake (or $5 off price)

Coffee (unlimited refills) with cream (optional)

$50

In linear logic:

$$
\begin{array}{rcl}
\$50 & \multimap & (\mathsf{soup} \mathbin{\&} \mathsf{salad}) \\
 & & \otimes (\mathsf{steak} \mathbin{\&} \mathsf{lobster} \mathbin{\&} (\$20 \multimap \mathsf{steak} \otimes \mathsf{lobster})) \\
 & & \otimes ((\mathsf{peas} \otimes \mathsf{carrot}) \oplus \mathsf{fennel}) \\
 & & \otimes (\mathsf{cake} \mathbin{\&} \$5) \\
 & & \otimes\, !\mathsf{coffee} \otimes (\mathsf{cream} \mathbin{\&} \mathbf{1})
\end{array}
$$

We encourage you to study this carefully to understand the use of each connective.

The menu represents a contract between two parties. As such it is not something that we prove directly. If we are the restaurant we expect to get $\$50$ (or a promise of such) and then engage in an interaction with the diner during which we have to provide more specific services, such as a soup. As a restaurant we satisfy these goals with additional resources (such as the soup ingredients), which are not part of this particular example.

Both sides make choices during this sequence of interactions. For example, once we get to the last stage the contract has been reduced to $\mathsf{cream} \mathbin{\&} \mathbf{1}$. From the guest's perspective this is an available resource, that is, part of the left-hand side of the sequent. To use this part, the guest must choose either $\& L_1$ or $\& L_2$, which means he or she continues either with cream or $\mathbf{1}$. In one case the guest obtains cream, in the other the empty resource, which subsequently disappears by using the $\mathbf{1}L$ rules. In this manner, $A \mathbin{\&} \mathbf{1}$ represents a resource whose use is optional.

Exercises

Exercise 1 *Does cut reduction fail for the incorrect* $\otimes L_1$?? *and* $\otimes L_2$?? *rules? Either show the reduction or explain informally why it fails.*

Exercise 2 *Explain the consequence of replacing the two left rules for alternative conjunction with*

$$\frac{\Delta, A, B \vdash C}{\Delta, A \mathbin{\&} B \vdash C} \ \&L??$$

Do identity expansion and cut reduction still hold? Exhibit a derived rule of inference that illustrates the dire consequence of this incorrect rule.

Exercise 3 We have explained logical equivalence between A and B as $A \vdash B$ and $B \vdash A$. Internalize logical equivalence as a connective $A \multimapboth B$ in linear logic and show identity expansion as well as local reductions. Briefly discuss alternative formulations (if there any) and explain why you chose yours.

Exercise 4 Show that $A \mathbin{\&} \top$ is equivalent to A.

Exercise 5 Show the identity expansion and cut reductions for disjunction $A \oplus B$.

Exercise 6 Show the identity expansion and cut reductions for the nullary disjunction $\mathbf{0}$.

Exercise 7 Show that $A \oplus \mathbf{0}$ is equivalent to A.

Exercise 8 With which additional resources can we prove the saying: "*If I had a nickel for every time I had a nickel, I would have two nickels.*" from Section 10?

Exercise 9 We can generally examine the interaction of the different connectives with each other to develop some logical equivalences. When they exist, they generally take the form of associativity or distributivity, where (iii) below is a somewhat special case, swapping a connective while re-associating. For example, here are some laws connecting $\otimes$ and $\multimap$ with themselves and each other.

(i) $(A \otimes B) \otimes C \dashv\vdash A \otimes (B \otimes C)$

(ii) $(A \multimap B) \multimap C$ do not interact.

(iii) $A \multimap (B \multimap C) \dashv\vdash (A \otimes B) \multimap C$

(iv) $A \multimap (B \otimes C)$ do not interact.

Prove (i) and (iii). We do not yet have the tools to prove (ii) and (iv).

Exercise 10 Examine if interaction equivalences like the ones in Exercise 9 hold in the following cases. Write out and prove those that hold, thereby implicitly conjecturing that their are no interaction for the remaining connectives.

(i) $(A \mathbin{\&} B) \multimap C$ and $A \multimap (B \mathbin{\&} C)$

(ii) $(A \mathbin{\&} B) \otimes C$

(iii) $(A \mathbin{\&} B) \oplus C$

(iv) $(A \otimes B) \mathbin{\&} C$

(v) $(A \otimes B) \oplus C$

(vi) $(A \oplus B) \otimes C$

(vii) $(A \oplus B) \mathbin{\&} C$

(viii) $(A \oplus B) \multimap C$ and $A \multimap (B \oplus C)$

Exercise 11 The multiplicative and additive units satisfy the following interaction properties.

(i) $A \otimes \mathbf{1} \dashv\vdash A$

(ii) $A \mathbin{\&} \top \dashv\vdash A$

(iii) $A \oplus \mathbf{0} \dashv\vdash A$

Examine interactions of the units with other connectives of linear logic and write down those equivalences that hold.

Exercise 12 Investigate which equivalence laws hold for the interaction of ! with all connectives and units, including itself. You do not need to prove if there is no interaction, but give proofs of the equivalence where there is one.

References

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