Lecture Notes on Sequent Calculus

15-816: Modal Logic
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1 Introduction

In this lecture we present the *sequent calculus* and its theory. The sequent calculus was originally developed by Gentzen [Gen35] as a means to establish properties of a system of natural deduction. In particular, this included *consistency*, which means that not every proposition is provable. But there are many other properties that naturally follow from the sequent calculus that are much more difficult to see on natural deduction.

For us, the sequent calculus provides a bridge between the truth and verification judgments. It will finally let us finish the internal global soundness and completeness theorems for the eliminations with respect to the introductions discussed in earlier lectures.

2 Searching for Verifications

In ordinary intuitionistic logic, when searching for a verification we are in a situation

$$\begin{array}{c} A_1\downarrow \;\ldots\; A_n\downarrow \\ \vdots \\ C\uparrow \end{array}$$

where we try to select introductions to deduce $C\uparrow$ and eliminations to apply to $A_i\downarrow$ until we meet in the middle. The sequent calculus codifies these three kinds of steps as inference rules that are all read from the conclusion to the premises.

A *sequent* is a particular form of hypothetical judgment

$$A_1\ \mathit{left}, \ldots, A_n\ \mathit{left} \vdash C\ \mathit{right}$$

where $A\ \mathit{left}$ corresponds to a proposition that can be used ($A\!\downarrow$) and $C\ \mathit{right}$ corresponds to a proposition we have to verify ($C\!\uparrow$). The *right rules* decompose C in analogy with the introduction rules, while the *left rules* decompose one of the hypotheses, in analogy with the elimination rules, but "upside-down". For the moment, we ignore the fine structure of proofs and do not label the hypotheses. We return to this in Exercise 2.

We now go through each of the rules for verifications, constructing analogous sequent rules. We still write Γ for a collection of hypotheses, where in the sequent calculus they all have the form $A\ \mathit{left}$.

Judgmental rule. The rule

$$\frac{P\!\downarrow}{P\!\uparrow}\ {\downarrow\uparrow}$$

means that if we have P on the left we can conclude P on the right.

$$\frac{}{\Gamma, P\ \mathit{left} \vdash P\ \mathit{right}}\ \mathsf{init}$$

Conjunction. The introduction rule translates straightforwardly to the right rule.

$$\frac{\Gamma \vdash A\ \mathit{right} \quad \Gamma \vdash B\ \mathit{right}}{\Gamma \vdash A \wedge B\ \mathit{right}}\ {\wedge}R$$

The elimination rule

$$\frac{\Gamma \vdash A \wedge B\!\downarrow}{\Gamma \vdash A\!\downarrow}\ {\wedge}E_L$$

means that if we have license to use $A \wedge B$, then we are justified in using A. During proof search, we are still licensed in using $A \wedge B$ again, since we do have a justification for using $A \wedge B$. So we obtain the following left rule:

$$\frac{\Gamma, A \wedge B\ \mathit{left}, A\ \mathit{left} \vdash C\ \mathit{right}}{\Gamma, A \wedge B\ \mathit{left} \vdash C\ \mathit{right}}\ {\wedge}L_1$$

Here, $A \wedge B$ is called the *principal formula* of the inference. Even though we always write this as if it were on the right end of the hypotheses, the rule can be applied to any hypothesis since we consider their order to be irrelevant. If we ignore the additional assumption $A \wedge B$ *left*, this is just the elimination rules upside-down, on the left of the hypothetical judgment rather than to the right. We also obtain a second left rule, from the second elimination rule.

$$\frac{\Gamma, A \wedge B\ \mathit{left}, B\ \mathit{left} \vdash C\ \mathit{right}}{\Gamma, A \wedge B\ \mathit{left} \vdash C\ \mathit{right}}\ \wedge L_2$$

Implication. As is typical, the introduction rule translates straightforwardly to a right rule.

$$\frac{\Gamma, A{\downarrow} \vdash B{\uparrow}}{\Gamma \vdash A \supset B{\uparrow}}\ {\supset}I \qquad \frac{\Gamma, A\ \mathit{left} \vdash B\ \mathit{right}}{\Gamma \vdash A \supset B\ \mathit{right}}\ {\supset}R$$

To transcribe the elimination rule into a left rule, we just have to follow carefully the idea of bidirectional proof construction with introductions and eliminations.

$$\frac{\Gamma \vdash A \supset B{\downarrow} \quad \Gamma \vdash A{\uparrow}}{\Gamma \vdash B{\downarrow}}\ {\supset}E$$

In order to use $A \supset B$ we have to verify A, which then licenses us to use B.

$$\frac{\Gamma, A \supset B\ \mathit{left} \vdash A\ \mathit{right} \quad \Gamma, A \supset B\ \mathit{left}, B\ \mathit{left} \vdash C\ \mathit{right}}{\Gamma, A \supset B\ \mathit{left} \vdash C\ \mathit{right}}\ {\supset}L$$

There is some redundancy in this rule. The hypothesis $A \supset B$ *left* in the second premise is not needed, because the assumption B *left* is strictly stronger. We retain it in this calculus for two reasons. For one, it makes the connections to verifications stronger, because in the construction of a verification one could re-use the assumption even if that is not strictly necessary. Secondly, this means that all rules (in the non-modal case) preserve *monotonicity of hypotheses*: an assumption, once made, will be available in the remainder of the bottom-up proof construction process.

Disjunction. Again, the introduction rules correspond directly to the following right rules.

$$\frac{\Gamma \vdash A\ \mathit{right}}{\Gamma \vdash A \vee B\ \mathit{right}}\ \vee R_1 \qquad \frac{\Gamma \vdash B\ \mathit{right}}{\Gamma \vdash A \vee B\ \mathit{right}}\ \vee R_2$$

Perhaps surprisingly, the somewhat awkward elimination rule

$$\frac{\Gamma \vdash A \vee B \downarrow \quad \Gamma, A \downarrow \vdash C \uparrow \quad \Gamma, B \downarrow \vdash C \uparrow}{\Gamma \vdash C \uparrow} \vee E$$

becomes much more similar to the other left rules.

$$\frac{\Gamma, A \vee B \; \mathit{left}, A \; \mathit{left} \vdash C \; \mathit{right} \quad \Gamma, A \vee B \; \mathit{left}, B \; \mathit{left} \vdash C \; \mathit{right}}{\Gamma, A \vee B \; \mathit{left} \vdash C \; \mathit{right}} \vee L$$

Again, we accept the redundancy for the sake of uniformity.

Truth. There is no elimination rule, so we only have a right rule corresponding to the introduction rule.

$$\frac{}{\Gamma \vdash A \; \mathit{right}} \top R$$

Falsehood. There is no introduction rule, so we only have a left rule corresponding the the elimination rule.

$$\frac{\Gamma \vdash \bot \downarrow}{\Gamma \vdash C \uparrow} \bot E \qquad \frac{}{\Gamma, \bot \; \mathit{left} \vdash C \; \mathit{right}} \bot L$$

This concludes the rule for the purely (non-modal) intuitionistic sequent calculus. Instead of

$$A_1 \; \mathit{left}, \ldots, A_n \; \mathit{left} \vdash C \; \mathit{right}$$

we write

$$A_1, \ldots, A_n \Longrightarrow C$$

because with the new notation for sequents, the judgments *left* and *right* are determined by the position of the formula in the sequent. The rules are summarized in Figure 1.

We take weakening and contraction properties for the sequent assumptions for granted; their proof is entirely straightforward and just follows from the general principles behind hypothetical judgments.

$$\frac{}{\Gamma, P \Longrightarrow P}\ \mathsf{init}$$

$$\frac{\Gamma \Longrightarrow A \quad \Gamma \Longrightarrow B}{\Gamma \Longrightarrow A \wedge B}\ {\wedge}R \qquad \frac{\Gamma, A \wedge B, A \Longrightarrow C}{\Gamma, A \wedge B \Longrightarrow C}\ {\wedge}L_1 \qquad \frac{\Gamma, A \wedge B, B \Longrightarrow C}{\Gamma, A \wedge B \Longrightarrow C}\ {\wedge}L_2$$

$$\frac{\Gamma, A \Longrightarrow B}{\Gamma \Longrightarrow A \supset B}\ {\supset}R \qquad \frac{\Gamma, A \supset B \Longrightarrow A \quad \Gamma, A \supset B, B \Longrightarrow C}{\Gamma, A \supset B \Longrightarrow C}\ {\supset}L$$

$$\frac{\Gamma \Longrightarrow A}{\Gamma \Longrightarrow A \vee B}\ {\vee}R_1 \qquad \frac{\Gamma \Longrightarrow B}{\Gamma \Longrightarrow A \vee B}\ {\vee}R_2 \qquad \frac{\Gamma, A \vee B, A \Longrightarrow C \quad \Gamma, A \vee B, B \Longrightarrow C}{\Gamma, A \vee B \Longrightarrow C}\ {\vee}L$$

$$\frac{}{\Gamma \Longrightarrow \top}\ {\top}R \qquad \text{no } {\top}L \text{ rule}$$

$$\text{no } {\bot}R \text{ rule} \qquad \frac{}{\Gamma, \bot \Longrightarrow C}\ {\bot}L$$

Figure 1: Intuitionistic Sequent Calculus

3 Verifications and Sequents

The meaning of propositions is defined by their verifications. To show that the sequent calculus is *sound* with respect to this definition we want to show that if A *right* then $A\uparrow$ and conversely for completeness.

To translate verifications to sequent deductions we have to demonstrate that if $A_1\downarrow, \ldots, A_n\downarrow \vdash C\uparrow$ then A_1 *left*, $\ldots$, A_n *left* $\vdash C$ *right*. The difficulty is how to translate proofs of the second judgment, namely $A_1\downarrow, \ldots, A_n\downarrow \vdash A\downarrow$. The correct property is not at all obvious, and the reader is invited to attempt to generalize the induction hypothesis appropriately before reading on.

In the statement of the soundness theorem and its proof, we have to translate hypotheses $\Gamma = (A_1 \downarrow, \ldots, A_n \downarrow)$ to $\hat{\Gamma} = (A_1\, \mathit{left}, \ldots, A_n\, \mathit{left})$.

Theorem 1 (From Verifications to Sequent Calculus)

(i) If $\Gamma \vdash C \uparrow$ *then* $\hat{\Gamma} \vdash C\, \mathit{right}$.

(ii) If $\Gamma \vdash A \downarrow$ *and* $\hat{\Gamma}, A\, \mathit{left} \vdash C\, \mathit{right}$ *then* $\hat{\Gamma} \vdash C\, \mathit{right}$.

Proof: By mutual induction on the deduction of $\Gamma \vdash C \uparrow$ and $\Gamma \vdash A \downarrow$. We show some representative cases.

Case:

$$\frac{\Gamma, C_1 \downarrow \vdash C_2 \uparrow}{\Gamma \vdash C_1 \supset C_2 \uparrow} \supset I$$

$\hat{\Gamma}, C_1\, \mathit{left} \vdash C_2\, \mathit{right}$	By i.h.(i)
$\hat{\Gamma} \vdash C_1 \supset C_2\, \mathit{right}$	By $\supset R$

Case:

$$\frac{\Gamma \vdash P \downarrow}{\Gamma \vdash P \uparrow} \downarrow\uparrow$$

$\hat{\Gamma}, P\, \mathit{left} \vdash P\, \mathit{right}$	By rule init
$\hat{\Gamma} \vdash P\, \mathit{right}$	By i.h.(ii)

Case:

$$\frac{\Gamma \vdash A_1 \supset A_2 \downarrow \quad \Gamma \vdash A_1 \uparrow}{\Gamma \vdash A_2 \downarrow} \supset E$$

$\hat{\Gamma}, A_2\, \mathit{left} \vdash C\, \mathit{right}$	Assumption
$\hat{\Gamma}, A_1 \supset A_2\, \mathit{left}, A_2\, \mathit{left} \vdash C\, \mathit{right}$	By weakening
$\hat{\Gamma} \vdash A_1\, \mathit{right}$	By i.h.(i)
$\hat{\Gamma}, A_1 \supset A_2\, \mathit{left} \vdash A_1\, \mathit{right}$	By weakening
$\hat{\Gamma}, A_1 \supset A_2\, \mathit{left} \vdash C\, \mathit{right}$	By rule $\supset L$
$\hat{\Gamma} \vdash C\, \mathit{right}$	By i.h.(ii)

Case:

$$\frac{}{\Gamma', A \downarrow \vdash A \downarrow} \mathsf{hyp}$$

$\hat{\Gamma}', A\ \mathit{left}, A\ \mathit{left} \vdash C\ \mathit{right}$ — Assumption and $\Gamma = (\Gamma', A\downarrow)$
$\hat{\Gamma}', A\ \mathit{left} \vdash C\ \mathit{right}$ — By contraction

□

For the completeness theorem, we use the substitution property for $A\downarrow$. This is just an example of the general substitution principle since the conclusion $A\downarrow$ matches the hypothesis of the second judgment.

Theorem 2 (Substitution for Uses) *Assume* $\Gamma \vdash A\downarrow$. *Then*

(i) if $\Gamma, A\downarrow \vdash B\downarrow$ *then* $\Gamma \vdash B\downarrow$, *and*

(ii) if $\Gamma, A\downarrow \vdash C\uparrow$ *then* $\Gamma \vdash C\uparrow$.

Proof: By mutual induction on the structure of the deduction of $\Gamma, A\downarrow \vdash B\downarrow$ and $\Gamma, A\downarrow \vdash C\uparrow$. □

Now we can prove the completeness of the sequent calculus. For a context $\Gamma = (A_1\ \mathit{left}, \ldots, A_n\ \mathit{left})$ we write $\check{\Gamma} = (A_1\downarrow, \ldots, A_n\downarrow)$

Theorem 3 (From Sequent Calculus to Verifications)
If $\Gamma \vdash C\ \mathit{right}$ *then* $\check{\Gamma} \vdash C\uparrow$.

Proof: By induction on the structure of the given derivation. We give some representative cases

Case:

$$\frac{\Gamma, C_1 \mathit{left} \vdash C_2\ \mathit{right}}{\Gamma \vdash C_1 \supset C_2\ \mathit{right}} \supset\! R$$

$\check{\Gamma}, C_1\downarrow \vdash C_2\uparrow$ — By i.h.
$\check{\Gamma} \vdash C_1 \supset C_2, \uparrow$ — By rule $\supset\! R$

Case:

$$\frac{\Gamma, A \supset B\ \mathit{left} \vdash A\ \mathit{right} \quad \Gamma, A \supset B\ \mathit{left}, B\ \mathit{left} \vdash C\ \mathit{right}}{\Gamma, A \supset B\ \mathit{left} \vdash C\ \mathit{right}} \supset\! L$$

$\check{\Gamma}, A \supset B\downarrow \vdash A \supset B\downarrow$ — By hypothesis rule
$\check{\Gamma}, A \supset B\downarrow \vdash A\uparrow$ — By i.h.
$\check{\Gamma}, A \supset B\downarrow \vdash B\downarrow$ — By rule $\supset\! E$
$\check{\Gamma}, A \supset B\downarrow, B\downarrow \vdash C\uparrow$ — By i.h.
$\check{\Gamma}, A \supset B\downarrow \vdash C\uparrow$ — By substitution (Theorem 2)

□

4 Cut Elimination

Our sequent calculus so far lacks the rule of *cut*:

$$\frac{\Gamma \vdash A\ right \quad \Gamma, A\ left \vdash C\ right}{\Gamma \vdash C\ right}\ \mathsf{cut}$$

It is the absence of this rule (a variant of which was present in Gentzen's original formulation) which allows us to easily relate sequent deductions to verifications. From the proof search perspective, the rule above corresponds to the introduction of a lemma A into a proof. In the one premise we prove this lemma, which justifies its use in the other. While this makes sense from the perspective of truth, it could not be part of a verification. This is because a verification should only consult the subformulas of a given proposition, while A in the cut rule is arbitrary.

Rather than taking cut as a rule, we show that whenever the premises have proofs in the (cut-free) sequent calculus, then the conclusion also has a proof. This reminiscent of the substitution principle, and yet it is different. As we can see from the rule above, the conclusion $A\ right$ is a different judgment than the hypothesis $A\ left$. Its proof is consequently more involved, because it does not follow merely by substitution. We will see later that it is the key to the global (internal) soundness and completeness theorems for intuitionistic natural deduction as presented in earlier lectures.

The proof we follow here proceeds by nested structural induction, as proposed in [Pfe00]. This proof is amenable to formalization in Twelf [PS99], which was its origin. It can be presented more formally with proof terms, but we forego this extra step here.

Theorem 4 (Admissibility of Cut) *If* $\Gamma \vdash A\ right$ *and* $\Gamma, A\ left \vdash C\ right$ *then* $\Gamma \vdash C\ right$.

Proof: By nested induction, first on the structure of the cut formula A, and second on the structure of the two given deductions (either one may be decreased while the other remains the same). We also allow weakening on either of the two given deductions and exploit that weakening does not change the size or structure of a deduction at all, so we can appeal to the induction hypothesis on a weakened subdeduction. We do this silently, in order to avoid cluttering the proof. In one case, we also explicitly appeal to *contraction*.

So we can refer to proofs, we use the abbreviated form of sequents and write

$$\text{If} \quad \Gamma \overset{\mathcal{D}}{\Longrightarrow} A \quad \text{and} \quad \Gamma, A \overset{\mathcal{E}}{\Longrightarrow} C \quad \text{then} \quad \Gamma \overset{\mathcal{F}}{\Longrightarrow} C$$

We also write $\mathcal{D} :: (\Gamma \Longrightarrow A)$ instead of $\Gamma \overset{\mathcal{D}}{\Longrightarrow} A$ to indicate that $\mathcal{D}$ is a deduction of the sequent $\Gamma \Longrightarrow A$.

We distinguish a variety of cases. First, two cases where either $\mathcal{D}$ or $\mathcal{E}$ is an initial sequent.

Case:

$$\mathcal{D} = \frac{}{\Gamma', P \Longrightarrow P}\ \mathsf{init}$$

and $\Gamma', P, P \overset{\mathcal{E}}{\Longrightarrow} C$ is arbitrary. Note that $\Gamma = (\Gamma', P)$ in this case.

$\Gamma', P, P \Longrightarrow C$	Deduction $\mathcal{E}$
$\Gamma', P \Longrightarrow C$	By contraction

Case: $\Gamma \overset{\mathcal{D}}{\Longrightarrow} P$ is arbitrary and

$$\mathcal{E} = \frac{}{\Gamma, P \Longrightarrow P}\ \mathsf{init}$$

$\Gamma \Longrightarrow P$	Deduction $\mathcal{D}$

Next an example of a case where the principal formula of the cut was just introduced by the most recent inference on both side. This is a prototype of similar cases for other connectives; we call these *principal cases*.

Case:

$$\mathcal{D} = \frac{\overset{\mathcal{D}_2}{\Gamma, A_1 \Longrightarrow A_2}}{\Gamma \Longrightarrow A_1 \supset A_2}\ {\supset}R$$

and

$$\mathcal{E} = \frac{\overset{\mathcal{E}_1}{\Gamma, A_1 \supset A_2 \Longrightarrow A_1} \quad \overset{\mathcal{E}_2}{\Gamma, A_1 \supset A_2, A_2 \Longrightarrow C}}{\Gamma, A_1 \supset A_2 \Longrightarrow C}\ {\supset}L$$

This case proceeds in four major steps. First, we apply the induction hypothesis to remove the "extra" copies of $A_1 \supset A_2$ from the two subdeductions $\mathcal{E}_1$ and $\mathcal{E}_2$. Then we cut A_1 and A_2 from the results. The latter two are on subformulas of the original cut formula.

$\Gamma \Longrightarrow A_1 \supset A_2$	Given ($\mathcal{D}$)
$\Gamma, A_1 \supset A_2 \Longrightarrow A_1$	Subdeduction $\mathcal{E}_1$
$\mathcal{E}_1' :: (\Gamma \Longrightarrow A_1)$	By i.h. on $A_1 \supset A_2$, $\mathcal{D}$, and $\mathcal{E}_1$
$\Gamma, A_1 \Longrightarrow A_2$	Subdeduction $\mathcal{D}_2$
$\mathcal{D}_2' :: (\Gamma \Longrightarrow A_2)$	By i.h. on A_1, $\mathcal{E}_1'$, and $\mathcal{D}_2$
$\Gamma, A_1 \supset A_2, A_2 \Longrightarrow C$	Subdeduction $\mathcal{E}_2$
$\mathcal{E}_2' :: (\Gamma, A_2 \Longrightarrow C)$	By i.h. on $A_1 \supset A_2$, $\mathcal{D}$, and $\mathcal{E}_2$
$\Gamma \Longrightarrow C$	By i.h. on A_2, $\mathcal{D}_2'$, and $\mathcal{E}_2'$

Finally, we have cases where the cut formula is *not* the principal formula of the last inference, either in $\mathcal{D}$ or in $\mathcal{E}$. In that case the principal formula can be found in one or more of the premises and we just cut it from these premises and reapply the rule. We "push" the cut up the derivation, into subderivations.

Case:

$$\mathcal{D} = \dfrac{\overset{\mathcal{D}_1}{\Gamma', B_1 \supset B_2 \Longrightarrow B_1} \quad \overset{\mathcal{D}_2}{\Gamma', B_1 \supset B_2, B_2 \Longrightarrow A}}{\Gamma', B_1 \supset B_2 \Longrightarrow A} \supset L$$

and $\overset{\mathcal{E}}{\Gamma', B_1 \supset B_2, A \Longrightarrow C}$ is arbitrary.

$\mathcal{D}_2' :: (\Gamma', B_1 \supset B_2, B_2 \Longrightarrow C)$	By i.h. on A, $\mathcal{D}_2$, and $\mathcal{E}$
$\Gamma', B_1 \supset B_2 \Longrightarrow C$	By rule $\supset L$ on $\mathcal{D}_1$ and $\mathcal{D}_2'$

Case: $\overset{\mathcal{D}}{\Gamma \Longrightarrow A}$ is arbitrary and

$$\mathcal{E} = \dfrac{\overset{\mathcal{E}_1}{\Gamma', B_1 \supset B_2, A \Longrightarrow B_1} \quad \overset{\mathcal{E}_2}{\Gamma', B_1 \supset B_2, B_2, A \Longrightarrow C}}{\Gamma', B_1 \supset B_2, A \Longrightarrow C} \supset L$$

$\mathcal{E}_1' :: (\Gamma', B_1 \supset B_2 \Longrightarrow B_1)$	By i.h. on A, $\mathcal{D}$, and $\mathcal{E}_1$
$\mathcal{E}_2' :: (\Gamma', B_1 \supset B_2, B_2 \Longrightarrow C)$	By i.h. on A, $\mathcal{D}$, and $\mathcal{E}_2$
$\Gamma', B_1 \supset B_2 \Longrightarrow C$	By rule $\supset L$ on $\mathcal{E}_1'$ and $\mathcal{E}_2'$

Case: $\Gamma \overset{\mathcal{D}}{\Longrightarrow} A$ is arbitrary and

$$\mathcal{E} = \dfrac{\begin{array}{c}\mathcal{E}_2 \\ \Gamma, A, C_1 \Longrightarrow C_2\end{array}}{\Gamma, A \Longrightarrow C_1 \supset C_2} \supset\! R$$

$\mathcal{E}_2' :: (\Gamma, C_1 \Longrightarrow C_2)$ — By i.h. on A, $\mathcal{D}$, and $\mathcal{E}_2$
$\Gamma \Longrightarrow C_1 \supset C_2$ — By rule $\supset\! R$ on $\mathcal{E}_2'$

□

5 Identity

Cut corresponds to a global soundness property: if we have proved *A right*, we are justified in assuming *A left* Identity is the corresponding completeness property: if we have an assumption *A left* we can prove *A right*. Together they show that we can move back and forth between *A left* and *A right* as long as we respect the side they can appear on. The identity theorem below shows that the rule

$$\dfrac{}{\Gamma, A\, left \vdash A\, right}\ \mathsf{id}$$

is admissible and can be used in proofs without changing the provable sequents.

Theorem 5 (Identity) *$\Gamma, A\, left \vdash A\, right$ for arbitrary propositions A and contexts Γ.*

Proof: By induction on the structure of A. We again use the abbreviated form of sequents, showing two representative cases.

Case: $A = P$, an atomic proposition.

$\Gamma, P \Longrightarrow P$ — By rule init

Case: $A = A_1 \supset A_2$.

$\Gamma, A_1 \supset A_2, A_1 \Longrightarrow A_1$ — By i.h. on A_2
$\Gamma, A_1 \supset A_2, A_1, A_2 \Longrightarrow A_2$ — By i.h. on A_2
$\Gamma, A_1 \supset A_2, A_1 \Longrightarrow A_2$ — By rule $\supset\! L$
$\Gamma, A_1 \supset A_2 \Longrightarrow A_1 \supset A_2$ — By rule $\supset\! R$

□

6 Truth and Verifications, Revisited

We can use the sequent calculus as a bridge to show that every true proposition has a verification. We show this by first showing that every true proposition has a sequent calculus proof. Since we already know every proposition with a sequent proof has a verification, the desired result is a simple consequence.

The key to mapping arbitrary natural deductions to sequent derivations is the admissibility of cut. We write $\hat{\Gamma}$ for the translation of hypotheses $A\ true$ to $A\ left$.

Theorem 6 (From Natural Deductions to Sequent Calculus) *If* $\Gamma \vdash A\ true$ *then* $\hat{\Gamma} \vdash A\ right$*.*

Proof: By induction on the structure of the given proof $\Gamma \vdash \overset{\mathcal{D}}{A\ true}$. We show some representative cases.

Case:

$$\mathcal{D} = \frac{}{\Gamma', A\ true \vdash A\ true}\ \mathsf{hyp}$$

$\hat{\Gamma}', A \Longrightarrow A$ By identity (Theorem 5)

Case:

$$\mathcal{D} = \frac{\begin{array}{c}\mathcal{D}_2 \\ \Gamma, A_1\ true \vdash A_2\ true\end{array}}{\Gamma \vdash A_1 \supset A_2\ true} \supset I$$

$\hat{\Gamma}, A_1 \Longrightarrow A_2$ By i.h. on $\mathcal{D}_2$

$\hat{\Gamma} \Longrightarrow A_1 \supset A_2$ By rule $\supset R$

Case:

$$\mathcal{D} = \frac{\begin{array}{c}\mathcal{D}_2 \\ \Gamma \vdash A_1 \supset A_2\ true\end{array} \quad \begin{array}{c}\mathcal{D}_1 \\ \Gamma \vdash A_1\ true\end{array}}{\Gamma \vdash A_2\ true} \supset E$$

$\hat{\Gamma} \Longrightarrow A_1 \supset A_2$	By i.h. on $\mathcal{D}_2$
$\hat{\Gamma} \Longrightarrow A_1$	By i.h. on $\mathcal{D}_1$
$\hat{\Gamma}, A_1 \supset A_2, A_1 \Longrightarrow A_1$	By identity
$\hat{\Gamma}, A_1 \supset A_2, A_1, A_2 \Longrightarrow A_2$	By identity
$\hat{\Gamma}, A_1 \supset A_2, A_1 \Longrightarrow A_2$	By rule $\supset L$
$\hat{\Gamma}, A_1 \Longrightarrow A_2$	By cut (Theorem 4)
$\hat{\Gamma} \Longrightarrow A_2$	By cut (Theorem 4)

□

Recall that verifications are sound for truth. This follows by induction on verifications and uses, since each introduction and elimination rule applies equally to truth if we conflate verifications and uses. The rule $\downarrow\uparrow$ is the only exception, but after replacing $P\uparrow$ and $P\downarrow$ both by P *true* the premise and conclusion are identical.

Corollary 7 (Truth and Verifications) *A true iff $A\uparrow$.*

Proof: "⇒": Assume A *true*.

$\cdot \vdash A$ *right*	By Theorem 6
$\cdot \vdash A\uparrow$	By Theorem 3

"⇐": See above remark.

□

Of course, there will in general be many more proofs of A *true* than $A\uparrow$, because verifications are purposefully restricted to be constructed in a certain analytic manner. The sequent calculus from this lecture can be seen as a rudimentary basis for a search procedure for verifications, which is therefore sound and complete for truth.

Computationally, the proof of the above corollary is not particularly satisfactory. A natural deduction of A *true* is converted to a sequent proof using a plethory of cuts, employing cut elimination to go back to a verification. The computational interpretation of cut elimination is not nearly as elegant as proof term reduction directly on natural deductions. One can also directly transform a proof into a verification using a technique called *hereditary substitution*, which we will probably not cover in this course.

7 Modal Sequent Calculus

The system of natural deduction and verifications constructed in Lecture 3 and Lecture 4 to capture modal necessity and possibility has an elegant rendering as a sequent calculus, following exactly the same intuition as for (non-modal) intuitionistic logic. There are two forms of extended sequent judgments

$$\underbrace{B_1\, \mathit{lvalid}, \ldots, B_m\, \mathit{lvalid}}_{\Delta}\; ;\; \underbrace{A_1\, \mathit{left}, \ldots, A_n\, \mathit{left}}_{\Gamma} \vdash \gamma$$

where γ is either $C\, \mathit{right}$ or $C\, \mathit{rposs}$.

Validity. The judgmental rule

$$\frac{A\Downarrow\, \in \Delta}{\Delta; \Gamma \vdash A\downarrow}$$

is a transition from the assumption that A is valid which we may use ($A \Downarrow$) to the assumption that we may use A. It becomes

$$\frac{(\Delta, A\, \mathit{lvalid}); (\Gamma, A\, \mathit{left}) \vdash \gamma}{(\Delta, A\, \mathit{lvalid}); \Gamma \vdash \gamma}\ \mathsf{valid}$$

In addition, we have the introduction and elimination rules.

$$\frac{\Delta; \bullet \vdash A\uparrow}{\Delta; \Gamma \vdash \Box A\uparrow}\ \Box I \qquad \frac{\Delta; \Gamma \vdash \Box A\downarrow \quad (\Delta, A\Downarrow); \Gamma \vdash C\uparrow}{\Delta; \Gamma \vdash C\uparrow}\ \Box E$$

$$\frac{\Delta; \Gamma \vdash \Box A\downarrow \quad (\Delta, A\Downarrow); \Gamma \vdash C \cdot\!\!\uparrow\!\!\cdot}{\Delta; \Gamma \vdash C \cdot\!\!\uparrow\!\!\cdot}\ \Box E$$

They become (were γ is $C\, \mathit{right}$ or $C\, \mathit{rposs}$, as before):

$$\frac{\Delta; \bullet \vdash A\, \mathit{right}}{\Delta; \Gamma \vdash \Box A\, \mathit{right}}\ \Box R \qquad \frac{(\Delta, A\, \mathit{lvalid}); (\Gamma, \Box A\, \mathit{left}) \vdash \gamma}{\Delta; (\Gamma, \Box A\, \mathit{left}) \vdash \gamma}\ \Box L$$

As expected, the ordinary truth assumptions $A\, \mathit{left}$ are no longer monotonic, because they are erased when the $\Box R$ rule is applied. Validity assumptions in Δ, however, remain.

Possibility. The judgmental rule

$$\frac{\Delta; \Gamma \vdash A \uparrow}{\Delta; \Gamma \vdash A \,\cdot\!\!\uparrow\!\!\cdot}\ \mathsf{poss}$$

which constructs a verification of possibility from a verification of truth, translates easily into an analogous rule in the sequent calculus.

$$\frac{\Delta; \Gamma \vdash A\ \mathit{right}}{\Delta; \Gamma \vdash A\ \mathit{rposs}}\ \mathsf{poss}$$

The right and left rules for verifications of $\Diamond A$ do not pose any special challenges.

$$\frac{\Delta; \Gamma \vdash A\ \mathit{rposs}}{\Delta; \Gamma \vdash \Diamond A\ \mathit{right}}\ \Diamond R \qquad \frac{\Delta; (\bullet, A\ \mathit{left}) \vdash C\ \mathit{rposs}}{\Delta; (\Gamma, \Diamond A\ \mathit{left}) \vdash C\ \mathit{rposs}}\ \Diamond L$$

For the example, we use a shorthand notation, where the judgment is question is indicated by the position of the proposition in the sequent. The two forms of sequents are

$$\begin{array}{lll} \Delta; \Gamma \Longrightarrow C\ \mathit{right} & \text{written as} & \Delta; \Gamma \Longrightarrow C; \cdot \\ \Delta; \Gamma \Longrightarrow C\ \mathit{rposs} & \text{written as} & \Delta; \Gamma \Longrightarrow \cdot; C \end{array}$$

We summarize the modal rules in the abbreviated notation in Figure 2. For reference, we also repeat the generalized rules from Figure 1.

As an example derivation, we show the sequent proof of $\Box(A \supset B) \supset (\Diamond A \supset \Diamond B)$. We silently omit propositions that are no longer needed. Furthermore, when Δ is empty or no longer needed, we may omit the leading "Δ;" and when γ is $C\ \mathit{right}$ we omit the trailing "$; \cdot$". In this notation, purely non-modal intuitionistic reasoning looks just as before.

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{}{A \Longrightarrow A}\ \mathsf{init} \quad \cfrac{}{B \Longrightarrow B}\ \mathsf{init}}{A \supset B, A \Longrightarrow B}\ \supset L}{A \supset B, A \Longrightarrow \cdot; B}\ \mathsf{poss}}{A \supset B; A \Longrightarrow \cdot; B}\ \mathsf{valid}}{A \supset B; \Diamond A \Longrightarrow \cdot; B}\ \Diamond L}{A \supset B; \Diamond A \Longrightarrow \Diamond B}\ \Diamond R}{\Box(A \supset B), \Diamond A \Longrightarrow \Diamond B}\ \Box L}{\Longrightarrow \Box(A \supset B) \supset (\Diamond A \supset \Diamond B)}\ \supset R \times 2$$

8 Properties of the Modal Sequent Calculus

The properties of the sequent calculus we developed in the preceding sections carry over to the modal sequent calculus. We write $\hat{\Delta}$ for the translation that replaces $A\,valid$ with $A\,lvalid$. We also write γ for $C\,right$ or $C\,rposs$.

Theorem 8 (From Verifications to Modal Sequent Calculus)

(i) *If* $\Delta; \Gamma \vdash C \uparrow$ *then* $\hat{\Delta}; \hat{\Gamma} \vdash C\,right$*.*

(ii) *If* $\Delta; \Gamma \vdash C \Uparrow$ *then* $\hat{\Delta}; \hat{\Gamma} \vdash C\,rposs$*.*

(iii) *If* $\Delta; \Gamma \vdash A \downarrow$ *and* $\hat{\Delta}; \hat{\Gamma}, A\,left \vdash \gamma$ *then* $\hat{\Delta}; \hat{\Gamma} \vdash \gamma$*.*

Proof: By induction on the first given deduction, as for Theorem 1. □

For the other direction, we write $\check{\Delta}$ for the translation from hypotheses $A\,lvalid$ to $A \Downarrow$.

Theorem 9 (From Modal Sequent Calculus to Verifications)

(i) *If* $\Delta; \Gamma \vdash C\,right$ *then* $\check{\Delta}; \check{\Gamma} \vdash C \uparrow$

(ii) *If* $\Delta; \Gamma \vdash C\,rposs$ *then* $\check{\Delta}; \check{\Gamma} \vdash C \Uparrow$

Proof: By induction on the given deduction, as for Theorem 3. We need a straightforward generalization of the substitution properties for uses (Theorem 2). □

The admissibility of cut comes in several flavors, which mutually depend on each other. This complicates the induction principle that we need.

Theorem 10 (Admissibility of Cut in Modal Sequent Calculus)

(i) *If* $\Delta; \Gamma \vdash A\,right$ *and* $\Delta; \Gamma, A\,left \vdash \gamma$ *then* $\Delta; \Gamma \vdash \gamma$*.*

(ii) *If* $\Delta; \bullet \vdash A\,right$ *and* $\Delta, A\,lvalid; \Gamma \vdash \gamma$ *then* $\Delta; \Gamma \vdash \gamma$*.*

(iii) *If* $\Delta; \Gamma \vdash A\,rposs$ *and* $\Delta; \bullet, A\,left \vdash C\,rposs$ *then* $\Delta; \Gamma \vdash C\,rposs$

Proof: By nested induction

1. on the cut formula A,
2. on the cut judgment, where $(A\,left) < (A\,lvalid)$,

3. on the structure of the two given derivations, where one may get smaller while the other one remains the same.

□

The identity theorem changes less, because we do not have any directly related judgments on the left and the right of a sequent except for $A\,\mathit{left}$ and $A\,\mathit{right}$.

Theorem 11 (Identity for Modal Sequent Calculus) *$\Delta; \Gamma, A\,\mathit{left} \vdash A\,\mathit{right}$ for any Δ, Γ and A.*

Proof: By induction on the structure of A. □

Again, exploiting cut and identity, we can show that natural deductions and sequent calculus derivations prove the same theorems. Here we write $\bar{\Delta}$ and $\bar{\Gamma}$ for the translation of hypotheses $A\,\mathit{valid}$ and $A\,\mathit{true}$ to $A\,\mathit{lvalid}$ and $A\,\mathit{left}$, respectively.

Theorem 12 (From Natural Deductions to Modal Sequent Calculus)

(i) If $\Delta; \Gamma \vdash A\,\mathit{true}$ then $\bar{\Delta}; \bar{\Gamma} \vdash A\,\mathit{right}$.

(ii) If $\Delta; \Gamma \vdash A\,\mathit{poss}$ then $\bar{\Delta}; \bar{\Gamma} \vdash A\,\mathit{rposs}$.

Proof: By induction on the structure of the given natural deduction, exploiting cut and identity. □

We will not restate Corollary 7, but it follows directly from the theorems above: Every true proposition has a verification.

9 Unprovable Propositions

In the sequent calculus, all inference rules construct a proof bottom-up, from the conclusion to the premises. Moreover, all rules have the subformula property: the sequent in the premises is constructed only from subformulas of the sequent in the conclusion. These two properties combine to make it a powerful tool for showing that certain propositions cannot be proven.

In these examples, we treat the propositional variables $A, B, C, \ldots$ as atomic propositions. When we say "*by inversion*" we mean that we find zero or more possible rules that could conclude with a given sequent and distinguish these finitely many cases.

$\not\vdash \bot$ *true*. Since sequent proofs are complete for truth, it suffices to show that there cannot be a sequent proof for arbitrary atomic propositions A. This reasoning applies for the following examples as well.

$\Longrightarrow \bot$	Assumption
Contradiction	By inversion (no possible rule)

$\not\vdash A \vee (A \supset \bot)$ *true*.

$\Longrightarrow A \vee (A \supset \bot)$	Assumption
$\Longrightarrow A$ or $\Longrightarrow A \supset \bot$	By inversion ($\vee R_1$ or $\vee R_2$)
$\quad \Longrightarrow A$	First case
$\quad$ Contradiction	By inversion (no possible rule)
$\quad \Longrightarrow A \supset \bot$	Second case
$\quad A \Longrightarrow \bot$	By inversion ($\supset R$)
$\quad$ Contradiction	By inversion (no possible rule)

$\not\vdash A \supset \Box A$ *true*.

$\Longrightarrow A \supset \Box A$	Assumption
$A \Longrightarrow \Box A$	By inversion ($\supset R$)
$\bullet \Longrightarrow A$	By inversion ($\Box R$)
Contradiction	By inversion (no possible rule)

$\not\vdash \Diamond A \supset A$ *true*.

$\Longrightarrow \Diamond A \supset A$	Assumption
$\Diamond A \Longrightarrow A$	By inversion ($\supset R$)
Contradiction	By inversion (no possible rule)

$\not\vdash \Diamond\bot \supset \bot$.

$\Longrightarrow \Diamond\bot \supset \bot$	Assumption
$\Diamond\bot \Longrightarrow \bot$	By inversion ($\supset R$)
Contradiction	By inversion (no possible rule)

$\not\vdash \Diamond(A \vee B) \supset (\Diamond A \vee \Diamond B)$.

$\Longrightarrow \Diamond(A \vee B) \supset (\Diamond A \vee \Diamond B)$ — Assumption
$\Diamond(A \vee B) \Longrightarrow \Diamond A \vee \Diamond B$ — By inversion ($\supset R$)
$\Diamond(A \vee B) \Longrightarrow \Diamond A$ or $\Diamond(A \vee B) \Longrightarrow \Diamond A$ — By inversion ($\vee R_1$ or $\vee R_2$)

$\Diamond(A \vee B) \Longrightarrow \Diamond A$ — First case
$\Diamond(A \vee B) \Longrightarrow \cdot; A$ — By inversion ($\Diamond R$)

$\Diamond(A \vee B) \Longrightarrow A$ or $A \vee B \Longrightarrow \cdot; A$ — By inversion (poss or $\Diamond R$)

$\Diamond(A \vee B) \Longrightarrow A$ — First subcase
Contradiction — By inversion (no possible rule)

$A \vee B \Longrightarrow \cdot; A$ — Second subcase
$A \vee B, B \Longrightarrow \cdot; A$ — By inversion, first subsubcase ($\vee L$)
$B \Longrightarrow \cdot; A$ — By strengthening (see below)
$B \Longrightarrow A$ — By inversion
Contradiction — By inversion (no possible rule)

$A \vee B \Longrightarrow A$ — Second subsubcase (poss)
$A \vee B, B \Longrightarrow A$ — By inversion ($\vee L$)
$B \Longrightarrow A$ — By strengthening (see below)
Contradiction — By inversion (no possible rule)

$\Diamond(A \vee B) \Longrightarrow \Diamond B$ — Second case
Symmetric to first case

In this last proof we used the following instance of strengthening: If $A \vee B, B \Longrightarrow \gamma$ then $B \Longrightarrow \gamma$. The proof goes as follows:

$B \Longrightarrow A \vee B$ — By init and $\vee R_2$
$A \vee B, B \Longrightarrow \gamma$ — Given
$B \Longrightarrow \gamma$ — By cut

$$\frac{}{\Delta; \Gamma, P \Longrightarrow P; \cdot}\ \mathsf{init}$$

$$\frac{\Delta; \Gamma \Longrightarrow A; \cdot \quad \Delta; \Gamma \Longrightarrow B; \cdot}{\Delta; \Gamma \Longrightarrow A \wedge B; \cdot}\ \wedge R \qquad \frac{\Delta; \Gamma, A \wedge B, A \Longrightarrow \gamma}{\Delta; \Gamma, A \wedge B \Longrightarrow \gamma}\ \wedge L_1$$

$$\frac{\Delta; \Gamma, A \wedge B, B \Longrightarrow \gamma}{\Delta; \Gamma, A \wedge B \Longrightarrow \gamma}\ \wedge L_2$$

$$\frac{\Delta; \Gamma, A \Longrightarrow B; \cdot}{\Delta; \Gamma \Longrightarrow A \supset B; \cdot}\ \supset R \qquad \frac{\Delta; \Gamma, A \supset B \Longrightarrow A; \cdot \quad \Delta; \Gamma, A \supset B, B \Longrightarrow \gamma}{\Delta; \Gamma, A \supset B \Longrightarrow \gamma}\ \supset L$$

$$\frac{\Delta; \Gamma \Longrightarrow A; \cdot}{\Delta; \Gamma \Longrightarrow A \vee B; \cdot}\ \vee R_1 \qquad \frac{\Delta; \Gamma, A \vee B, A \Longrightarrow \gamma \quad \Delta; \Gamma, A \vee B, B \Longrightarrow \gamma}{\Delta; \Gamma, A \vee B \Longrightarrow \gamma}\ \vee L$$

$$\frac{\Delta; \Gamma \Longrightarrow B; \cdot}{\Delta; \Gamma \Longrightarrow A \vee B; \cdot}\ \vee R_2$$

$$\frac{}{\Delta; \Gamma \Longrightarrow \top; \cdot}\ \top R \qquad \text{no } \top L \text{ rule}$$

$$\text{no } \bot R \text{ rule} \qquad \frac{}{\Delta; \Gamma, \bot \Longrightarrow \gamma}\ \bot L$$

$$\frac{(\Delta, A); (\Gamma, A) \Longrightarrow \gamma}{(\Delta, A); \Gamma \Longrightarrow \gamma}\ \mathsf{valid}$$

$$\frac{\Delta; \bullet \Longrightarrow A; \cdot}{\Delta; \Gamma \Longrightarrow \Box A; \cdot}\ \Box R \qquad \frac{(\Delta, A); (\Gamma, \Box A) \Longrightarrow \gamma}{\Delta; (\Gamma, \Box A) \Longrightarrow \gamma}\ \Box L$$

$$\frac{\Delta; \Gamma \Longrightarrow A; \cdot}{\Delta; \Gamma \Longrightarrow \cdot; A}\ \mathsf{poss}$$

$$\frac{\Delta; \Gamma \Longrightarrow \cdot; A}{\Delta; \Gamma \Longrightarrow \Diamond A; \cdot}\ \Diamond R \qquad \frac{\Delta; (\bullet, A) \Longrightarrow \cdot; C}{\Delta; (\Gamma, \Diamond A) \Longrightarrow \cdot; C}\ \Diamond L$$

Figure 2: Intuitionistic Modal Sequent Calculus

Exercises

Exercise 1 *We can annotate modal verifications and uses with the proof terms introduced in Lecture 4. This will induce several syntactic classes of terms:*

Canonical terms N *such that* $\Delta; \Gamma \vdash N : A \uparrow$
Atomic terms R *such that* $\Delta; \Gamma \vdash R : A \downarrow$
Normal expressions E *such that* $\Delta; \Gamma \vdash E \div A \uparrow$.

For arbitrary proof terms M, we used type annotations as, for example, in $\lambda x{:}A.\, M$ to make sure that every proof term has a unique type. For canonical terms these are no longer necessary.

(i) Give a syntactic characterization of the forms of N, R, and E.

(ii) Prove that even without any type annotations, if Γ, Δ, N, and A are given, the derivation of $\Gamma; \Delta \vdash N : A \uparrow$ is uniquely determined (if it exists), even without any type annotations in terms. Make sure to appropriately generalize the induction hypothesis to include the other judgment forms.

Exercise 2 *One way to assign proof terms to the sequent calculus is suggested by the translation from verifications to sequent deductions (Theorem 1). A (non-modal) sequent annotated with proof terms has the form*

$$M_1{:}A_1, \ldots, M_n{:}A_1 \Longrightarrow N : C$$

where $M_i{:}A_i \downarrow$ and $N : C \uparrow$.

(i) Restate Theorem 1 with proof terms.

(ii) Show the cases for $\supset I$, $\downarrow\uparrow$, $\supset E$, and hypotheses in the proof.

(iii) Generalize the theorem to include the judgments of validity and possibility.

(iv) Show the cases for valid hypotheses, $\Box I$, and $\Box E$.

(v) Show the cases for poss, *$\Diamond I$, and $\Diamond E$.*

(vi) Is it the case that for every term N such that $\vdash N : C \uparrow$ there is a sequent derivation such that $\Longrightarrow N : C$? In other words, is the sequent calculus complete with respect to all proofs, or just with respect to provability?

Exercise 3 *Once the admissibility of cut is established, we can add it as an inference rule to obtain a sequent calculus with an explicit rule of cut. Writing $\Gamma \overset{+}{\Longrightarrow} A$ for sequent calculus with the cut rule, prove that $\Gamma \overset{+}{\Longrightarrow} C$ iff $\Gamma \Longrightarrow A$.*

Exercise 4 *We have presented the sequent calculus in a form that is directly motivated by verifications. From the perspective of provability, however, there are some redundancies. For example, in the $\vee L$ rule, the assumption $A \vee B$ is redundant in both premises because the new assumption A or B, respectively, is stronger. We exploited this observation in the form of a strengthening principle to show the unprovability of $\Diamond(A \vee B) \supset (\Diamond A \vee \Diamond B)$.*

Rewrite the sequent calculus rules to exploit redundancy as much as possible. In which rules do we need to preserve the principal formula of the left rule?

Exercise 5 *We call a connective* negative *if its right rule in the sequent calculus can* always *be applied immediately without losing provability. For example, conjunction is negative because $\Delta; \Gamma \Longrightarrow A \wedge B$ iff $\Delta; \Gamma \Longrightarrow A$ and $\Delta; \Gamma \Longrightarrow B$. On the other hand, $\Box A$ is not negative, because the proof of $\cdot; \Box(A \wedge B) \Longrightarrow \Box A$ must start with the left rule.*

Determine which connectives among $\wedge$, $\supset$, $\vee$, $\top$, $\bot$, $\Box$, $\Diamond$ are negative.

By extension, a judgment *is negative if it does not occur on the right because it can always immediately be decomposed according to its definition. A valid is a negative judgment. Demonstrate that A poss is not negative.*

Exercise 6 *We call a connective* positive *if its left rule in the sequent calculus can* always *be applied immediately without losing provability, while at the same time not carrying the principal formula to any premises. For example, $A \vee B$ is positive, while $\Diamond A$ is not positive.*

Determine which connectives among $\wedge$, $\supset$, $\vee$, $\top$, $\bot$, $\Box$, and $\Diamond$ are positive.

By extension, a judgment *is positive if it does not occur on the left because it can always be immediately decomposed according to its definition. A poss is a positive judgment. Demonstrate that A valid is not positive.*

Exercise 7 *As explained in Exercises 5 and 6, the judgment A valid is negative and A poss is positive so validity never appears on the right and possibility never on the left. In this exercise we explore the consequences of nevertheless allowing these judgments.*

(i) Formulate a system of verifications and uses that allows verifications for validity $A \Uparrow$ and uses of possibility $A \Downarrow$.

(ii) Give a proof term assignment as well as local reductions and expansions.

(iii) Design a corresponding sequent calculus.

(iv) Write out the expected cut principles.

(v) Give the induction order and show the new cases for the admissibility of cut.

(vi) Discuss the merits and demerits of this system when compared to the one present in lecture.

Exercise 8 *Re-examine the proposed rule*

$$\frac{\Gamma, A\downarrow \vdash C\downarrow \quad \Gamma, B\downarrow \vdash C\downarrow}{\Gamma \vdash A \vee B\downarrow} \vee E$$

from Exercise L1.9 in view of the connection between verifications and the sequent calculus.

Exercise 9 *Re-examine the connective $A \otimes B$ from exercise L2.9, which was defined by its elimination rule.*

(i) Give the corresponding left and right rules of the sequent calculus.

(ii) Give the new cases in the translation from verifications to sequent calculus (Theorem 1).

(iii) Give the new cases in the translation from the sequent calculus to verifications (Theorem 3).

(iv) Is $A \otimes B$ negative (see Exercise 5)?

(v) Is $A \otimes B$ positive (see Exercise 6)?

(vi) There is a proposition logically equivalent to $A \otimes B$. State and prove the equivalence. Any observations regarding (iv) and (v)?

Exercise 10 *Gentzen's original sequent calculus permitted right and left rules for negation, taking advantage of a sequent form with an empty right-hand side. From the judgmental perspective, we can obtain this using a new judgment of* contradiction, *written "contra" for natural deduction and "empty" for the sequent calculus. We then have the rules*

$$\frac{\Delta; \Gamma, A\ \mathit{left} \vdash \mathit{empty}}{\Delta; \Gamma \vdash \neg A\ \mathit{right}} \neg R \qquad \frac{\Delta; \Gamma, \neg A\ \mathit{left} \vdash A\ \mathit{right}}{\Delta; \Gamma, \neg A\ \mathit{left} \vdash \gamma} \neg L$$

The notation γ in this an all other rules now includes C right, C rposs, and empty.

In the short-hand form of sequents, we write "$A; \cdot$" for A right, "$\cdot; A$" for A rposs and "$\cdot; \cdot$" for empty, so the rules above could be written as

$$\frac{\Delta; \Gamma, A \Longrightarrow \cdot; \cdot}{\Delta; \Gamma \Longrightarrow \neg A; \cdot} \neg R \qquad \frac{\Delta; \Gamma \Longrightarrow A; \cdot}{\Delta; \Gamma, \neg A \Longrightarrow \gamma} \neg L$$

(i) Give the corresponding judgments and rules for natural deduction.

(ii) Extend the definition of verifications and uses.

(iii) Present the new cases in the translation from verifications to sequent proofs (Theorem 8).

(iv) Present the new cases in the translation from sequent proofs to verifications (Theorem 9).

(v) Present the new principal case(s) in the proof of the admissibility of cut (Theorem 10).

(vi) Present the new case(s) in the proof of identity (Theorem 11).

(vii) Present the new cases in the translation from natural deduction to sequent calculus (Theorem 12).

(viii) Prove that $\vdash \neg A \equiv (A \supset \bot)$.

(ix) Prove that $\not\vdash \neg\neg A \supset A$.

Exercise 11 *We define* $\star A = \neg(\neg A)$*, where you may chose to view* $\neg A$ *either as a notational definition or a new connective as explained in Exercise 10. Which of the following characteristic axioms of modal logic hold for* $\star A$*?*

(i) $\vdash A \supset \star A$

(ii) $\vdash \star A \supset A$

(iii) $\vdash \star A \supset \star\star A$

(iv) $\vdash \star\star A \supset \star A$

(v) $\vdash \star(A \supset B) \supset (\star A \supset \star B)$

Exercise 12 *Using the sequent calculus, show that the following are not provable.*

(i) $\Diamond A \wedge \Diamond B \supset \Diamond(A \wedge B)$

(ii) $\Box\neg A \supset \neg\Diamond A$

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