

Abstract Syntax

Types	$\tau ::= \tau_1 \rightarrow \tau_2 \mid \forall \alpha. \tau \mid \alpha \mid \tau_1 \times \tau_2 \mid 1 \mid \tau_1 + \tau_2 \mid 0 \mid \mu \alpha. \tau$	
Terms	$e ::= x \mid \lambda x. e \mid e_1 e_2 \quad (\rightarrow)$ $\quad \mid \Lambda \alpha. e \mid e[\tau] \quad (\forall)$ $\quad \mid \langle e_1, e_2 \rangle \mid \text{case } e (\langle x_1, x_2 \rangle \Rightarrow e') \quad (\times)$ $\quad \mid \langle \rangle \mid \text{case } e (\langle \rangle \Rightarrow e') \quad (1)$ $\quad \mid \mathbf{l} \cdot e \mid \mathbf{r} \cdot e \mid \text{case } e (\mathbf{l} \cdot x_1 \Rightarrow e_1 \mid \mathbf{r} \cdot x_2 \Rightarrow e_2) \quad (+)$ $\quad \mid \text{case } e () \quad (0)$ $\quad \mid \text{fold } e \mid \text{unfold } e \quad (\mu)$ $\quad \mid \text{fix } f. e$	
Contexts	$\Gamma ::= \cdot \mid \Gamma, \alpha \text{ type} \mid \Gamma, x : \tau$	(all variables distinct)

Judgments

$\Gamma \text{ ctx}$	Γ is a valid context	
$\Gamma \vdash \tau \text{ type}$	τ is a valid type	presupposes $\Gamma \text{ ctx}$
$\Gamma \vdash e : \tau$	expression e has type τ	presupposes $\Gamma \text{ ctx}$, ensures $\Gamma \vdash \tau \text{ type}$
$e \text{ value}$	expression e is a value	presupposes $\cdot \vdash e : \tau$ for some τ
$e \mapsto e'$	expression e steps to e'	presupposes $\cdot \vdash e : \tau$ for some τ

Theorems

Preservation. If $\cdot \vdash e : \tau$ and $e \mapsto e'$ then $\cdot \vdash e' : \tau$.

Progress. If $\cdot \vdash e : \tau$ then either $e \mapsto e'$ for some e' or $e \text{ value}$.

Finality of Values. There is no $\cdot \vdash e : \tau$ such that $e \mapsto e'$ for some e' and $e \text{ value}$.

Sequentiality. If $e \mapsto e_1$ and $e \mapsto e_2$ then $e_1 = e_2$.

Canonical Forms. Assume $\cdot \vdash v : \tau$ and $v \text{ value}$.

- (i) If $\tau = \tau_1 \rightarrow \tau_2$ then $v = \lambda x. e_2$ for some e_2
- (ii) If $\tau = \forall \alpha. \tau'$ then $v = \Lambda \alpha. e'$ for some e'
- (iii) If $\tau = \tau_1 \times \tau_2$ then $v = \langle v_1, v_2 \rangle$ for some $v_1 \text{ value}$ and $v_2 \text{ value}$
- (iv) If $\tau = 1$ then $v = \langle \rangle$
- (v) If $\tau = \tau_1 + \tau_2$ then $v = \mathbf{l} \cdot v_1$ for some $v_1 \text{ value}$ or $v = \mathbf{r} \cdot v_2$ for some $v_2 \text{ value}$
- (vi) If $\tau = 0$ then we have a contradiction
- (vii) If $\tau = \mu \alpha. \tau'$ then $v = \text{fold } v'$ for some $v' \text{ value}$

Contexts Γ

$\frac{}{(\cdot) \text{ ctx}} \text{ ctx/emp}$	$\frac{\Gamma \text{ ctx}}{(\Gamma, \alpha \text{ type}) \text{ ctx}} \text{ ctx/tpvar}$	$\frac{\Gamma \text{ ctx} \quad \Gamma \vdash \tau \text{ type}}{(\Gamma, x : \tau) \text{ ctx}} \text{ ctx/var}$
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Functions $\tau_1 \rightarrow \tau_2$

$\frac{\Gamma \vdash \tau_1 \text{ type} \quad \Gamma \vdash \tau_2 \text{ type}}{\Gamma \vdash \tau_1 \rightarrow \tau_2 \text{ type}} \text{ tp/arrow}$	
$\frac{x : \tau \in \Gamma}{\Gamma \vdash x : \tau} \text{ tp/var}$	
$\frac{\Gamma \vdash \tau_1 \text{ type} \quad \Gamma, x_1 : \tau_1 \vdash e_2 : \tau_2}{\Gamma \vdash \lambda x_1 : \tau_1. e_2 : \tau_1 \rightarrow \tau_2} \text{ tp/lam}$	$\frac{\Gamma \vdash e_1 : \tau_2 \rightarrow \tau_1 \quad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash e_1 e_2 : \tau_1} \text{ tp/app}$
$\frac{}{\lambda x. e \text{ value}} \text{ val/lam}$	$\frac{e_2 \text{ value}}{(\lambda x. e_1) e_2 \mapsto [e_2/x]e_1} \text{ step/app/lam}$
$\frac{e_1 \mapsto e'_1}{e_1 e_2 \mapsto e'_1 e_2} \text{ step/app}_1$	$\frac{e_1 \text{ value} \quad e_2 \mapsto e'_2}{e_1 e_2 \mapsto e_1 e'_2} \text{ step/app}_2$

Polymorphic Types $\forall \alpha. \tau$

$\frac{\alpha \text{ type} \in \Gamma}{\Gamma \vdash \alpha \text{ type}} \text{ tp/tpvar}$	$\frac{\Gamma, \alpha \text{ type} \vdash \tau \text{ type}}{\Gamma \vdash \forall \alpha. \tau \text{ type}} \text{ tp/forall}$
$\frac{\Gamma, \alpha \text{ type} \vdash e : \tau}{\Gamma \vdash \Lambda \alpha. e : \forall \alpha. \tau} \text{ tp/tplam}$	$\frac{\Gamma \vdash e : \forall \alpha. \tau \quad \Gamma \vdash \sigma \text{ type}}{\Gamma \vdash e[\sigma] : [\sigma/\alpha]\tau} \text{ tp/tpapp}$
$\frac{}{\Lambda \alpha. e \text{ value}} \text{ val/tplam}$	$\frac{}{(\Lambda \alpha. e) [\tau] \mapsto [\tau/\alpha]e} \text{ step/tpapp/tplam}$
	$\frac{e \mapsto e'}{e[\tau] \mapsto e'[\tau]} \text{ step/tpapp}$

Pairs $\tau_1 \times \tau_2$

$\frac{\Gamma \vdash \tau_1 \text{ type} \quad \Gamma \vdash \tau_2 \text{ type}}{\Gamma \vdash \tau_1 \times \tau_2 \text{ type}} \text{ tp/prod}$	
$\frac{\Gamma \vdash e_1 : \tau_1 \quad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash \langle e_1, e_2 \rangle : \tau_1 \times \tau_2} \text{ tp/pair}$	$\frac{\Gamma \vdash e : \tau_1 \times \tau_2 \quad \Gamma, x_1 : \tau_1, x_2 : \tau_2 \vdash e' : \tau'}{\Gamma \vdash \text{case } e (\langle x_1, x_2 \rangle \Rightarrow e') : \tau'} \text{ tp/casep}$
$\frac{e_1 \text{ value} \quad e_2 \text{ value}}{\langle e_1, e_2 \rangle \text{ value}} \text{ val/pair}$	
$\frac{v_1 \text{ value} \quad v_2 \text{ value}}{\text{case } \langle v_1, v_2 \rangle (\langle x_1, x_2 \rangle \Rightarrow e_3) \mapsto [v_1/x_1][v_2/x_2]e_3} \text{ step/casep/pair}$	
$\frac{e_1 \mapsto e'_1}{\langle e_1, e_2 \rangle \mapsto \langle e'_1, e_2 \rangle} \text{ step/pair}_1$	$\frac{v_1 \text{ value} \quad e_2 \mapsto e'_2}{\langle v_1, e_2 \rangle \mapsto \langle v_1, e'_2 \rangle} \text{ step/pair}_2$
$\frac{e_0 \mapsto e'_0}{\text{case } e_0 (\langle x_1, x_2 \rangle \Rightarrow e_3) \mapsto \text{case } e'_0 (\langle x_1, x_2 \rangle \Rightarrow e_3)} \text{ step/casep}_0$	

Unit 1

$\frac{}{\Gamma \vdash 1 \text{ type}} \text{ tp/one}$	$\frac{}{\Gamma \vdash \langle \rangle : 1} \text{ tp/unit}$	$\frac{\Gamma \vdash e : 1 \quad \Gamma \vdash e' : \tau'}{\Gamma \vdash \text{case } e (\langle \rangle \Rightarrow e') : \tau'} \text{ tp/caseu}$
$\frac{}{\langle \rangle \text{ value}} \text{ val/unit}$		
$\frac{}{\text{case } \langle \rangle (\langle \rangle \Rightarrow e) \mapsto e} \text{ step/caseu/unit}$		
$\frac{e_0 \mapsto e'_0}{\text{case } e_0 (\langle \rangle \Rightarrow e_1) \mapsto \text{case } e'_0 (\langle \rangle \Rightarrow e_1)} \text{ step/caseu}_0$		

Sums $\tau_1 + \tau_2$

$\frac{\Gamma \vdash \tau_1 \text{ type} \quad \Gamma \vdash \tau_2 \text{ type}}{\Gamma \vdash \tau_1 + \tau_2 \text{ type}} \text{ tp/sum}$	
$\frac{\Gamma \vdash e_1 : \tau_1 \quad \Gamma \vdash \tau_2 \text{ type}}{\Gamma \vdash \mathbf{l} \cdot e_1 : \tau_1 + \tau_2} \text{ tp/left}$	$\frac{\Gamma \vdash \tau_1 \text{ type} \quad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash \mathbf{r} \cdot e_2 : \tau_1 + \tau_2} \text{ tp/right}$
$\frac{\Gamma \vdash e : \tau_1 + \tau_2 \quad \Gamma, x_1 : \tau_1 \vdash e_1 : \sigma \quad \Gamma, x_2 : \tau_2 \vdash e_2 : \sigma}{\Gamma \vdash \text{case } e (\mathbf{l} \cdot x_1 \Rightarrow e_1 \mid \mathbf{r} \cdot x_2 \Rightarrow e_2) : \sigma} \text{ tp/cases}$	
$\frac{e_1 \text{ value}}{\mathbf{l} \cdot e_1 \text{ value}} \text{ val/left}$	$\frac{e_2 \text{ value}}{\mathbf{r} \cdot e_2 \text{ value}} \text{ val/right}$
$\frac{v_1 \text{ value}}{\text{case } \mathbf{l} \cdot v_1 (\mathbf{l} \cdot x_1 \Rightarrow e_1 \mid \mathbf{r} \cdot x_2 \Rightarrow e_2) \mapsto [v_1/x_1]e_1} \text{ step/cases/left}$	
$\frac{v_2 \text{ value}}{\text{case } \mathbf{r} \cdot v_2 (\mathbf{l} \cdot x_1 \Rightarrow e_1 \mid \mathbf{r} \cdot x_2 \Rightarrow e_2) \mapsto [v_2/x_2]e_2} \text{ step/cases/right}$	
$\frac{e_1 \mapsto e'_1}{\mathbf{l} \cdot e_1 \mapsto \mathbf{l} \cdot e'_1} \text{ step/left}$	$\frac{e_2 \mapsto e'_2}{\mathbf{r} \cdot e_2 \mapsto \mathbf{r} \cdot e'_2} \text{ step/right}$
$\frac{e_0 \mapsto e'_0}{\text{case } e_0 (\mathbf{l} \cdot x_1 \Rightarrow e_1 \mid \mathbf{r} \cdot x_2 \Rightarrow e_2) \mapsto \text{case } e'_0 (\mathbf{l} \cdot x_1 \Rightarrow e_1 \mid \mathbf{r} \cdot x_2 \Rightarrow e_2)} \text{ step/cases}_0$	

Empty Type 0

$\frac{}{\Gamma \vdash 0 \text{ type}} \text{ tp/zero}$	(no constructor)	$\frac{\Gamma \vdash e_0 : 0 \quad \Gamma \vdash \tau \text{ type}}{\Gamma \vdash \text{case } e_0 () : \tau} \text{ tp/casez}$
$\frac{e_0 \mapsto e'_0}{\text{case } e_0 () \mapsto \text{case } e'_0 ()} \text{ step/casez}_0$		
(no values)		

Recursive Types $\mu\alpha.\tau$

$\frac{\Gamma, \alpha \text{ type} \vdash \tau \text{ type}}{\Gamma \vdash \mu\alpha.\tau \text{ type}} \text{ tp/mu}$	
$\frac{\Gamma \vdash e : [\mu\alpha.\tau/\alpha]\tau}{\Gamma \vdash \text{fold } e : \mu\alpha.\tau} \text{ tp/fold}$	$\frac{\Gamma \vdash e : \mu\alpha.\tau}{\Gamma \vdash \text{unfold } e : [\mu\alpha.\tau/\alpha]\tau} \text{ tp/unfold}$
$\frac{e \text{ value}}{\text{fold } e \text{ value}} \text{ val/fold}$	$\frac{v \text{ value}}{\text{unfold } (\text{fold } v) \mapsto v} \text{ step/unfold/fold}$
$\frac{e \mapsto e'}{\text{fold } e \mapsto \text{fold } e'} \text{ step/fold}$	$\frac{e \mapsto e'}{\text{unfold } e \mapsto \text{unfold } e'} \text{ step/unfold}_0$

Recursion

$\frac{\Gamma, f : \tau \vdash e : \tau}{\Gamma \vdash \text{fix } f : \tau. e : \tau} \text{ tp/fix}$	$\frac{}{\text{fix } f. e \mapsto [\text{fix } f. e / f]e} \text{ step/fix}$
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