Assignment 7
Parametricity

15-814: Types and Programming Languages
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Due Sat Nov 8, 2025
80 pts

This assignment is due on the above date and it must be submitted electronically on Gradescope. Please use the attached template to typeset your assignment and make sure to include your full name and Andrew ID. For the written problems, you may also submit handwritten answers that have been scanned and are **easily legible**.

Please carefully read the policies on collaboration and credit on the course web pages at `http://www.cs.cmu.edu/~fp/courses/15814-f25/assignments.html`.

You should hand in one file:

- `hw07.pdf` with the written answers to the questions.

1 Subtyping Revisited [40 pts]

In Lecture 12 we justified subtyping *syntactically*, using $v : \tau$ and $e : \tau$. In this problem we explore a *semantic* justification using unary logical relations.

Task 1 (5 pts) Give a corresponding *semantic definition of subtyping*, either referencing $[\![-]\!]$ and $[-]$ or defining it in a way that reflects the spirit of logical relations.

Task 2 (10 pts) Prove the *soundness* of the syntactic rules with respect to their semantic interpretation. This should be in some way analogous to proving the fundamental theorem which states that syntactic typing ($e : \tau$) implies semantic typing ($e \in [\![\tau]\!]$).

You may limit your proof to the cases for eager products and functions.

Task 3 (5 pts) Extend the semantic definition to encompass parametric polymorphism, or explain why it already does.

Task 4 (5 pts) Extend the syntactic rules for subtyping to parametric polymorphism.

Task 5 (15 pts) Show the soundness of your new rule(s) by proving that they satisfy the semantic definition.

2 Existential Types Revisited [30 pts]

In this problem we explore the definition of a unary logic relation for existential types.

Task 6 (5 pts) Give a definition of the unary logical relation for $v \in [\exists\alpha.\,\tau]$.

Task 7 (10 pts) Show the case for the constructor for $\exists\alpha.\,\tau$ in the proof of the fundamental theorem.

Task 8 (10 pts) Show the case for the destructor for $\exists\alpha.\,\tau$ in the proof of the fundamental theorem.

Task 9 (5 pts) Give some examples and counterexamples to membership in $[\exists\alpha.\,\tau]$

In the proofs, you may assume simple properties of substitution and evaluation, as well as lemmas such as closure under reverse evaluation (essentially: the properties we used in our proof for parametric polymorphism). If you need additional lemmas or properties, please state them.

3 Theorems for Free [10 pts]

Task 10 (10 pts) Derive a Wadler-style "theorem for free" knowing that $f : \forall\alpha.\,\forall\beta.\,\alpha \to \beta \to \alpha$