

Assignment 2

The Simply Typed λ -Calculus

15-814: Types and Programming Languages
Frank Pfenning

Due Tuesday, September 22, 2020

This assignment is due on the above date and it must be submitted electronically as a PDF file on Canvas. Please use the attached template to typeset your assignment and make sure to include your full name and Andrew ID.

Please carefully read the [policies on collaboration and credit](http://www.cs.cmu.edu/~fp/courses/15814-f20/assignments.html) on the course web pages at <http://www.cs.cmu.edu/~fp/courses/15814-f20/assignments.html>.

1 Simple types

Task 1 (L4.1, 10 points) Fill in the blanks in the following typing judgments so the resulting judgment holds, or indicate there is no way to do so. You do not need to justify your answer or supply a typing derivation, and the types do not need to be “most general” in any sense. Remember that the function type constructor associates to the right, so that $\tau \rightarrow \sigma \rightarrow \rho = \tau \rightarrow (\sigma \rightarrow \rho)$.

(i) $\vdash y x : \alpha$

(ii) $\vdash x x :$

(iii) $\cdot \vdash$ $: (\alpha \rightarrow \alpha) \rightarrow \alpha$

(iv) $\cdot \vdash (\lambda z. z) (\lambda x. \lambda y. \lambda p. p x y) :$

(v) $\cdot \vdash \lambda f. \lambda g. \lambda x. (f x) (g x) : (\alpha \rightarrow$ $) \rightarrow (\alpha \rightarrow$ $) \rightarrow (\alpha \rightarrow$ $)$

2 Proof by Rule Induction

Since this is the first time we (that is, you) are proving theorems about judgments defined by rules, we ask you to be very explicit, as we were in the lectures and lecture notes. In particular:

- Explicitly state the overall structure of your proof: whether it proceeds by rule induction, and, if so, on the derivation of which judgment, or by structural induction, or by inversion, or just directly. If you need to split out a lemma for your proof, state it clearly and prove it separately. If you need to generalize your induction hypothesis, clearly state the generalized form.

- Explicitly list all cases in an induction proof. If a case is impossible, prove that it is impossible. Often, that's just inversion, but sometimes it is more subtle.
- Explicitly note any appeals to the induction hypothesis.
- Any appeals to inversion should be noted as such, as well as the rules that could have inferred the judgment we already know. This could lead to zero cases (a contradiction—the judgment could not have been derived), one case (there is exactly one rule whose conclusion matches our knowledge), or multiple cases, in which case your proof now splits into multiple cases.
- We recommend that you follow the line-by-line style of presentation where each line is justified by a short phrase. This will help you to check your proof and us to read and verify it.

Task 2 (L5.1, 10 points) Define multi-step reduction $e \longrightarrow^* e'$ by the following rules:

$$\frac{}{e \longrightarrow^* e} \text{ red}^*/\text{refl} \qquad \frac{e \longrightarrow e' \quad e' \longrightarrow^* e''}{e \longrightarrow^* e''} \text{ red}^*/\text{step}$$

Prove by rule induction that if $\Gamma \vdash e : \tau$ and $e \longrightarrow^* e'$ then $\Gamma \vdash e' : \tau$. Here (as in general in the course), you may use theorems we have proved in the course (lecture or notes).

Task 3 (L5.2, 40 points) Define a new single-step relation $e \mapsto e'$ which means that e reduces to e' by *leftmost-outermost reduction*, using a collection of inference rules. Recall that I claimed this strategy is *sound* (it only performs β -reductions) and *complete for normalization* (if e has a normal form, we can reach it by performing only leftmost-outermost reductions). Prove the following statements about your reduction judgment:

- If $e \mapsto e'$ then $e \longrightarrow e'$.
- \mapsto is *small-step deterministic*, that is, if $e \mapsto e_1$ and $e \mapsto e_2$ then $e_1 = e_2$.

You should interpret $=$ as α -equality, that is, the two terms differ only in the names of their bound variables (which we always take for granted). For each of the following statements, either indicate that they are true (without proof) or provide a counterexample.

- For all e , either $e \mapsto e'$ for some e' or e *normal*.
- There does not exist an e such that $e \mapsto e'$ for some e' and e *normal*.
- If $e \longrightarrow e'$ then $e \mapsto e'$.
- \longrightarrow is *small-step deterministic*.
- \longrightarrow is *big-step deterministic*, that is, if $e \longrightarrow^* e_1$ and $e \longrightarrow^* e_2$ where e_1 *normal* and e_2 *normal*, then $e_1 = e_2$.
- For arbitrary e and *normal* e' , $e \longrightarrow^* e'$ iff $e \mapsto^* e'$.