Lecture Notes on
Adjoint Ordered Types

15-417/817: HOT Compilation
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1 Introduction

We have introduced ordered types in Lecture 11 and discussed purely ordered type-checking in Lecture 13. Can we extend the adjoint point of view to fully integrate order? The day before this lecture Sophia Roshal and I conjectured the system described here, and since then it has been holding up (e.g., the proof of cut elimination seems to go through).

The key step, already taken by Kanovich et al. [2018, 2019], is to replace the rule of exchange

$$\frac{\Omega_L\ B_k\ A_m\ \Omega_R \vdash C_r}{\Omega_L\ A_m\ B_k\ \Omega_R \vdash C_r}\ \mathsf{exchange}$$

by two rules for *mobility* (under suitable conditions on mode m)

$$\frac{\Omega_L\ \Omega_M\ A_m\ \Omega_R \vdash C_r}{\Omega_L\ A_m\ \Omega_M\ \Omega_R \vdash C_r}\ \mathsf{move}^{\leftarrow} \qquad \frac{\Omega_L\ A_m\ \Omega_M\ \Omega_R \vdash C_r}{\Omega_L\ \Omega_M\ A_m\ \Omega_R \vdash C_r}\ \mathsf{move}^{\rightarrow}$$

The advantage of these rules is that they pertain only to one mode (m in the rules above) while being parametric in all other modes.

If we are looking at a single mode, then being able to move left is tantamount to being able to move right, because in $A_m\ B_m$ we can move A to the right of B, or B to the left of A in order to obtain exchange $B_m\ A_m$. However, if we have $A_m\ C_k\ B_m$ where k does not support any mobility, then we can obtain $C_k\ A_m\ B_m$ and $C_k\ B_m\ A_m$ with right mobility and $A_m\ B_m\ C_k$ and $B_m\ A_m\ C_k$ with left mobility. But these are not interchangeable unless we also have mobility in the other direction.

As far as we are aware, this form of directed mobility has not yet been considered, and so far we have not found any proof-theoretic reason why one might entail the other. But then again, we don't yet have any clear applications for this potentially expressive system.

As we will see, considerations for contraction are quite analogous.

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2 Mobility

We have two structural properties, ML and MR, both applied to antecedents.

$$\frac{(\mathsf{ML} \in \sigma(m)) \quad \Omega_L\, \Omega_M\, A_m\, \Omega_R \vdash C_r}{\Omega_L\, A_m\, \Omega_M\, \Omega_R \vdash C_r}\ \mathsf{move}^{\leftarrow} \qquad \frac{(\mathsf{MR} \in \sigma(m)) \quad \Omega_L\, A_m\, \Omega_M\, \Omega_R \vdash C_r}{\Omega_L\, \Omega_M\, A_m\, \Omega_R \vdash C_r}\ \mathsf{move}^{\rightarrow}$$

How does this work with cut?

$$\frac{(\Omega \geq m \geq r) \quad \Omega \vdash A_m \quad \Omega_L\, A_m\, \Omega_R \vdash C_r}{\Omega_L\, \Omega\, \Omega_R \vdash C_r}\ \mathsf{cut}$$

Either premise of the cut could end in a move. In the second premise, the moving formula could be in Ω_L or Ω_R or be A_m, and it could end up in several places. In most cases, we just push up the cut and we can mimic the move in the conclusion. We show the one mildly interesting case, when the cut formula A_m itself moves.

$$\frac{(\Omega \geq m \geq r) \quad \overset{\mathcal{D}}{\Omega \vdash A_m} \quad \dfrac{(\mathsf{MR} \in \sigma(m)) \quad \overset{\mathcal{E}'}{\Omega_L\, A_m\, \Omega_M\, \Omega_R \vdash C_r}}{\Omega_L\, \Omega_M\, A_m\, \Omega_R \vdash C_r}\ \mathsf{move}^{\rightarrow}}{\Omega_L\, \Omega_M\, \Omega\, \Omega_R \vdash C_r}\ \mathsf{cut}$$

$$\longrightarrow$$

$$\dfrac{\dfrac{\overset{\mathcal{D}}{\Omega \vdash A_m} \quad \overset{\mathcal{E}'}{\Omega_L\, A_m\, \Omega_M\, \Omega_R \vdash C_r}}{\Omega_L\, \Omega\, \Omega_M\, \Omega_R \vdash C_r}\ \mathsf{cut}}{\Omega_L\, \Omega_M\, \Omega\, \Omega_R \vdash C_r}\ \mathsf{move}^{\rightarrow} \times |\Omega|$$

Why is the move of Ω valid? We know $\Omega \geq m$ and also $\mathsf{MR} \in \sigma(m)$. Therefore, by monotonicity, $\mathsf{MR} \in \sigma(k)$ for all $B_k \in \Omega$, and Ω can mimic the right move of A_m.

The cases for left moves as symmetric.

3 Contraction

The "local" contraction

$$\frac{\Omega_L\, A_m\, A_m\, \Omega_R \vdash C_r}{\Omega_L\, A_m\, \Omega_R \vdash C_r}\ \mathsf{contract}??$$

has a serious problem. Consider

$$\dfrac{(\Omega \geq m \geq r) \quad \begin{array}{c}\mathcal{D}\\ \Omega \vdash A_m\end{array} \quad \dfrac{\begin{array}{c}\mathcal{E}'\\ \Omega_L\, A_m\, A_m\, \Omega_R \vdash C_r\end{array}}{\Omega_L\, A_m\, \Omega_R \vdash C_r}\,\textsf{contract}{\color{red}??}}{\Omega_L\, \Omega\, \Omega_R \vdash C_r}\,\textsf{cut}$$

$$\longrightarrow$$

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ \Omega \vdash A_m\end{array} \quad \dfrac{\begin{array}{c}\mathcal{D}\\ \Omega \vdash A_m\end{array} \quad \begin{array}{c}\mathcal{E}'\\ \Omega_L\, A_m\, A_m\, \Omega_R \vdash C_r\end{array}}{\Omega_L\, \Omega\, A_m\, \Omega_R \vdash C_r}\,\textsf{cut}}{\Omega_L\, \Omega\, \Omega\, \Omega_R \vdash C_r}\,\textsf{cut}}{\Omega_L\, \Omega\, \Omega_R \vdash C_r}\,{\color{red}??}$$

The problem here is that we cannot always contract $\Omega\,\Omega$ to just Ω because the formulas we want to contract may not be adjacent.

This, by itself, does not imply that cut elimination fails, only that the particular reduction we chose fails. Kanovich et al. [2019] give a counterexample showing that cut elimination actually fails when contraction is local. Here is another one. Consider atomic propositions p, q, and r, all of the same mode m that admits local contraction. Then there are easy proofs of

$$\begin{array}{c}\vdots\\ p\, q \vdash p \bullet q\end{array} \quad \text{and} \quad \dfrac{\begin{array}{c}\vdots\\ (p \bullet q)\, (p \bullet q)\, (p \bullet q \rightarrowtail (p \bullet q \rightarrowtail r)) \vdash r\end{array}}{(p \bullet q)\, (p \bullet q \rightarrowtail (p \bullet q \rightarrowtail r)) \vdash r}\,\textsf{contract}$$

but there is no cut-free proof of

$$\begin{array}{c}{\color{red}??}\\ p\, q\, (p \bullet q \rightarrowtail (p \bullet q \rightarrowtail r)) \vdash r\end{array}$$

One can apply $\rightarrowtail L$, but that will consume p and q. We can apply local contraction to p before, but q will not be available in the second premise. We can also apply local contraction to q, but then $\rightarrowtail L$ can not be successfully applied because the in the first premise of the rule we need to prove $p \bullet q$.

These problems disappear when contraction is not local. In fact, there are two forms of non-local contraction, with two corresponding structural properties.

$$\dfrac{(\textsf{CL} \in \sigma(m)) \quad \Omega_L\, A_m\, \Omega_M\, A_m\, \Omega_R \vdash C_r}{\Omega_L\, A_m\, \Omega_M\, \Omega_R \vdash C_r}\,\textsf{contract}^{\leftarrow}$$

$$\dfrac{(\textsf{CR} \in \sigma(m)) \quad \Omega_L\, A_m\, \Omega_M\, A_m\, \Omega_R \vdash C_r}{\Omega_L\, \Omega_M\, A_m\, \Omega_R \vdash C_r}\,\textsf{contract}^{\rightarrow}$$

Now the cut reduction from before works correctly, even if no obvious measure decreases.

$$\dfrac{(\Omega \geq m \geq r) \quad \overset{\mathcal{D}}{\Omega \vdash A_m} \quad \dfrac{(\mathsf{CL} \in \sigma(m)) \quad \overset{\mathcal{E}'}{\Omega_L\, A_m\, \Omega_M\, A_m\, \Omega_R \vdash C_r}}{\Omega_L\, A_m\, \Omega_M\, \Omega_R \vdash C_r}\ \mathsf{contract}^{\leftarrow}}{\Omega_L\, \Omega\, \Omega_M\, \Omega_R \vdash C_r}\ \mathsf{cut}$$

$$\longrightarrow$$

$$\dfrac{\dfrac{\overset{\mathcal{D}}{\Omega \vdash A_m} \quad \dfrac{\overset{\mathcal{D}}{\Omega \vdash A_m} \quad \overset{\mathcal{E}'}{\Omega_L\, A_m\, \Omega_M\, A_m\, \Omega_R \vdash C_r}}{\Omega_L\, \Omega\, \Omega_M\, A_m\, \Omega_R \vdash C_r}\ \mathsf{cut}}{\Omega_L\, \Omega\, \Omega_M\, \Omega\, \Omega_R \vdash C_r}\ \mathsf{cut}}{\Omega_L\, \Omega\, \Omega_M\, \Omega_R \vdash C_r}\ \mathsf{contract}^{\leftarrow} \times |\Omega|$$

The contractions at the end are valid due to monotonicity, since $\Omega \geq m$ and $\mathsf{CL} \in \sigma(m)$.

As explained in the last lecture, the lower of the two cuts does not obviously have a smaller measure that could be used in an induction proof. Instead, we have to generalize to *multicut* (almost exactly like Gentzen's *mix*) that can cut out multiple copies of the same formula A_m at once.

4 Weakening

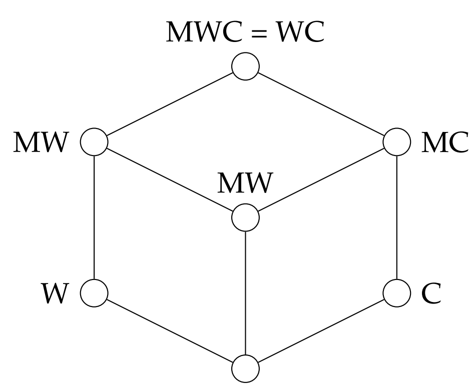
Weakening allows us to add an antecedent at an arbitrary place.

$$\dfrac{(\mathsf{W} \in \sigma(m)) \quad \Omega_L\, \Omega_R \vdash C_r}{\Omega_L\, A_m\, \Omega_R \vdash C_r}\ \mathsf{weaken}$$

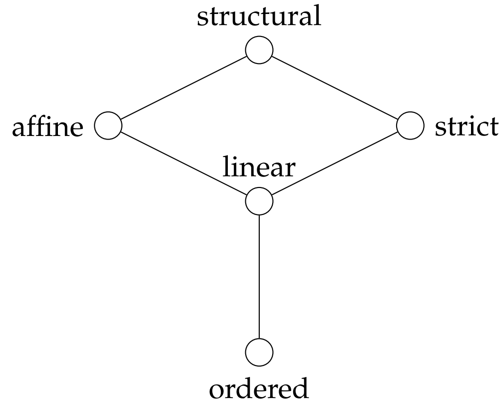
We can now observe that there are some redundancies. For example, if a mode m admits weakening and right contraction, then this implies right mobility (rule $\mathsf{move}^{\rightarrow}$):

$$\dfrac{(\mathsf{CR} \in \sigma(m)) \quad \dfrac{(\mathsf{W} \in \sigma(m)) \quad \Omega_L\, A_m\, \Omega_M\, \Omega_R \vdash C_r}{\Omega_L\, A_m\, \Omega_M\, A_m\, \Omega_R \vdash C_r}\ \mathsf{weaken}}{\Omega_L\, \Omega_M\, A_m\, \Omega_R \vdash C_r}\ \mathsf{contract}^{\rightarrow}$$

If we pair left and right mobility as M and left and right contraction as C, we obtain the following picture. The "hidden" node WC is the same as MWC.



We can find a subpicture when considering weakening and contraction only in the presence of mobility.



5 Other Inference Systems

Once we have the sequent calculus, we can use the techniques from earlier lectures to derive rules for adjoint natural deduction [Jang et al., 2024] and the semi-axiomatic sequent calculus. Before that, it might be helpful to develop an implicit form of the sequent calculus where the structural rules are baked into the other rules.

We consider one such calculus, fashioned after the additive typing for ND and Sax. This is an alternative approach to the one taken in Lecture 13. The idea is to define a relation $\Gamma \vdash \Omega_1 \gg \Omega_2$ where Ω_2 arises from an arbitrary collection of structural rules applied to Ω_1, knowing that Γ contains all variables lexically in scope (in no particular order). We define it by the following rules:

$$\frac{(\mathsf{ML} \in \sigma(m))}{\Gamma \vdash \Omega_L\, \Omega_M\, (x : A_m)\, \Omega_R \gg \Omega_L\, (x : A_m)\, \Omega_M\, \Omega_R}\ \mathsf{move}^{\leftarrow}$$

$$\frac{(\mathsf{ML} \in \sigma(m))}{\Gamma \vdash \Omega_L\, (x : A_m)\, \Omega_M\, \Omega_R \gg \Omega_L\, \Omega_M\, (x : A_m)\, \Omega_R}\ \mathsf{move}^{\rightarrow}$$

$$\frac{(\mathsf{CL} \in \sigma(m))}{\Gamma \vdash \Omega_L\, (x : A_m)\, \Omega_M\, (x : A_m)\, \Omega_R \gg \Omega_L\, (x : A_m)\, \Omega_M\, \Omega_R}\ \mathsf{contract}^{\leftarrow}$$

$$\frac{(\mathsf{CR} \in \sigma(m))}{\Gamma \vdash \Omega_L\, (x : A_m)\, \Omega_M\, (x : A_m)\, \Omega_R \gg \Omega_L\, \Omega_M\, (x : A_m)\, \Omega_R}\ \mathsf{contract}^{\rightarrow}$$

$$\frac{(\mathsf{W} \in \sigma(m)) \quad x : A_m \in \Gamma}{\Gamma \vdash \Omega_L\, \Omega_R \gg \Omega_L\, (x : A_m)\, \Omega_R}\ \mathsf{weaken}$$

$$\frac{}{\Gamma \vdash \Omega \gg \Omega}\ \mathsf{refl} \qquad \frac{\Gamma \vdash \Omega_1 \gg \Omega_2 \quad \Gamma \vdash \Omega_2 \gg \Omega_3}{\Gamma \vdash \Omega_1 \gg \Omega_3}\ \mathsf{trans}$$

We design the additive system so that for

$$\Gamma \vdash M : C \,/\, \Omega \quad \text{and} \quad \Gamma \vdash \Omega \gg \Omega'$$

we have

$$\Omega' \vdash M : C$$

Since this work is very much in progress, we only show a few rules. We omit the proof terms since we haven't explicated any proof terms for the sequent calculus. However, it should be clear what they express, and how they would relate to ND and Sax.

$$\frac{x : A_m \in \Gamma}{\Gamma \vdash A_m \;/\; (x : A_m)}\ \mathsf{id}$$

This rule is sound in the sense sketched above, because

$$\frac{}{x : A_m \vdash A_m}\ \mathsf{id}$$

and whenever $\Gamma \vdash (x : A_m) \gg \Omega$ we can derive $\Omega \vdash A_m$ by the structural rules of the adjoint sequent calculus.

The right rule is entirely straightforward.

$$\frac{\Gamma \vdash A \;/\; \Omega_1 \quad \Gamma \vdash B \;/\; \Omega_2}{\Gamma \vdash A \otimes B \;/\; \Omega_1\, \Omega_2}\ \otimes R$$

If there are some antecedents C_m in Γ that are used in both premises, they will both show up in the conclusion $\Omega_1\,\Omega_2$. They can be contracted to a single use only if mode m admits contraction (either left or right).

In contrast, the left rule is tricky to interpret.

$$\frac{z : A_m \otimes B_m \in \Gamma \quad \Gamma, x : A_m, y : B_m \vdash C_r \;/\; \Omega \quad \Gamma \vdash \Omega \gg \Omega_L\,(x : A_m)\,(y : B_m)\,\Omega_R}{\Gamma \vdash C_r \;/\; \Omega_L\,(z : A_m \otimes B_m)\,\Omega_R}\ \otimes L$$

We note, for example, that if m admits weakening, then in the $\gg$ judgment x and y can be added to the output context. Also, if z is used in Ω, then it will appear either in Ω_L or Ω_R or both. In that case we will only be able to contract to a single use of z if, well, it admits contraction.

The intended implementation of such a system is via saturation: we generate all possible Ω' such that $\Gamma \vdash A \;/\; \Omega$ and $\Gamma \vdash \Omega \gg \Omega'$. In certain places like $\otimes L$, many of these will be ruled out because we filter out all except those where x and y are next to each other in the given order. On the other hand, rules such as $\otimes R$ could be prolific because of all the possible permutations and contractions of $\Omega_1\,\Omega_2$ have to be considered.

Besides efficiency, the biggest obstacle is the rule of weakening because we could be adding arbitrarily many copies of A_m from Γ as long as m admits weakening. We conjecture that this can be done *lazily*, that is, only in places where weakening may help a rule to be applicable. That is the case in rules such as $\otimes L$ where we might weaken by x and/or y, and also when combining the branches of a $\oplus L$ or $\& R$. These are exactly the places where in the adjoint system for ND or Sax (when assuming left and right mobility for all modes) we apply $\Omega \setminus x_m$ or $\Omega_1 \sqcup \Omega_2$. Similarly, we conjecture that contraction is needed only in places where the prior adjoint system would apply the join operation $\Omega_1 \,;\, \Omega_2$. Notice that various independence conditions are not directly connected to structural properties, but narrow the set of legal output contexts by filtering those violating independence.

When there are no viable output contexts, adjoint ordered type-checking fails. The other case it could fail is we check a definition

$$\mathbf{defn}\ F[\Omega] : C_r = M$$

where we check

$$\Omega \vdash M : C_r \;/\; \Omega'$$

and then have to verify that

$$\Omega \vdash \Omega' \gg \Omega$$

holds.

References

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