Constructive Logic (15-317), Fall 2009
Assignment 9: Proving Metatheorems in Twelf

William Lovas (`wlovas@cs`)

Out: Thursday, November 12, 2009
Due: Thursday, November 19, 2009 (before class)

In this assignment, you'll review an important metatheorem, the Identity Theorem, and see how its proof is represented in Twelf as a well-moded total logic program. You should submit the written portion of your work (Section 1) at the beginning of class, and you should submit your Twelf code (Section 2) by copying it to the directory

```
/afs/andrew/course/15/317/submit/<userid>/hw09
```

where `<userid>` is replaced with your Andrew ID. Submit a file with the same name as the starter code, `identity.elf`, which is available from the course website.

Some tutorial notes are available via the web on using Twelf with Emacs and using Twelf without Emacs. These are part of the Twelf Wiki, a resource that you may find helpful while learning Twelf.

1 Identity Theorem on Paper (10 points)

In Figure 1, you will find the rules we gave earlier for the classical sequent calculus. Recall that a sequent $\Gamma \;\#\; \Delta$ can be interpreted in two ways:

1. If every proposition in Γ is true and every proposition in Δ is false, then they are in contradiction.
2. If every proposition in Γ is true, then at least one proposition in Δ is true.

Under the first interpretation, the Identity Theorem says that if A is both true and false, we have a contradiction. Under the second, it says that A follows from itself.

Task 1 (10 pts). Prove the Identity Theorem for the classical sequent calculus: for any proposition A, we can derive $A \;\#\; A$.

$$\frac{}{\Gamma, P \;\#\; P, \Delta}\ \text{init}$$

$$\frac{\Gamma \;\#\; A, B, \Delta}{\Gamma \;\#\; A \vee B, \Delta}\ \vee R \qquad\qquad \frac{\Gamma, A \;\#\; \Delta \quad \Gamma, B \;\#\; \Delta}{\Gamma, A \vee B \;\#\; \Delta}\ \vee L$$

$$\frac{\Gamma, A \;\#\; \Delta}{\Gamma \;\#\; \neg A, \Delta}\ \neg R \qquad\qquad \frac{\Gamma \;\#\; A, \Delta}{\Gamma, \neg A \;\#\; \Delta}\ \neg L$$

Figure 1: Classical sequent calculus.

$$\frac{\Gamma, A \;\#\; B, \Delta}{\Gamma \;\#\; A \supset B, \Delta}\ \supset R \quad = \quad \dfrac{\dfrac{\Gamma, A \;\#\; B, \Delta}{\Gamma \;\#\; \neg A, B, \Delta}\ \neg R}{\Gamma \;\#\; \neg A \vee B, \Delta}\ \vee R$$

$$\frac{\Gamma \;\#\; A, \Delta \quad \Gamma, B \;\#\; \Delta}{\Gamma, A \supset B \;\#\; \Delta}\ \supset L \quad = \quad \dfrac{\dfrac{\Gamma \;\#\; A, \Delta}{\Gamma, \neg A \;\#\; \Delta}\ \neg L \quad \Gamma, B \;\#\; \Delta}{\Gamma, \neg A \vee B \;\#\; \Delta}\ \vee L$$

Figure 2: Derived rules for classical implication.

2 Identity Theorem in Twelf (30 points)

On the course website, you will find a file of starter code `identity.elf` defining the syntax of propositions and the rules of inference. Recall that we represent propositions A in Γ as hypotheses `true A` and propositions B in Δ as hypotheses `false B`.

Task 2 (10 pts). Classically, $A \supset B \triangleq \neg A \vee B$. Encode implication and its derived inference rules (see Figure 2) as notational definitions in Twelf. (For Twelf examples of defined connectives and derived rules, see the code for Lecture 19: prop.elf, verif.elf.)

The Identity Theorem can be stated as a judgement relating each proposition to a contradiction under two new hypotheses:

```
identity : {A:prop} (true A -> false A -> contra) -> type.
%mode idenity +A -D.

%worlds () (identity A D).
%total A (identity A D).
```

Task 3 (20 pts). Prove the Identity Theorem in Twelf by giving clauses defining the `identity` relation as a well-moded total logic program.