Constructive Logic (15-317), Fall 2009
Assignment 4: Classical and Intuitionistic Logic

William Lovas (`wlovas@cs`)

Out: Thursday, September 24, 2009
Due: Thursday, October 1, 2009 (before class)

The goal of this assignment is to give you a bit of intuition about the relationship between classical and intuitionistic reasoning. You'll explore the boundaries between the two, seeing just where intuitionistic reasoning makes more distinctions, and you'll work with the double-negation translation that embeds classical logic into intuitionistic logic.

This assignment is a bit shorter than usual to allow you time to prepare for the midterm on Thursday, October 1. If you hand this homework in during lecture on Tuesday, we will grade and return it the following day in recitation so you can have feedback before the midterm.

The Tutch portion of your work (Section 1) should be submitted electronically using the command

```
$ /afs/andrew/course/15/317/bin/submit -r hw04 <files...>
```

from any Andrew server. You may check the status of your submission by running the command

```
$ /afs/andrew/course/15/317/bin/status hw04
```

If you have trouble running either of these commands, email William.

The written portion of your work (Section 2) should be submitted at the beginning of class. If you are familiar with LaTeX, you are encouraged to use this document as a template for typesetting your solutions, but you may alternatively write your solutions *neatly* by hand.

1 Tutch Proofs (28 points)

Tutch allows you to do classical proofs by employing a non-judgemental analogue of our PBC^k rule:

$$\dfrac{\begin{array}{c}\neg A\ \mathit{true}\\ \vdots\\ \bot\ \mathit{true}\end{array}}{A\ \mathit{true}}\ PBC^u_{\mathit{Tutch}}$$

Classical proofs must be declared with `classical proof`. Here's an example, proving double-negation elimination:

```
classical proof dne : ~~A => A =
begin
  [ ~~A;
    % prove A by contradiction:
    [ ~A;
      F ];
    A ];
  ~~A => A
end;
```

Task 1 (28 pts). Which of the following propositions are provable intuitionistically? Classically? Prove them in Tutch intuitionistically if possible, classically if necessary.

```
proof deMorganNotOr : ~(A | B) => ~A & ~B
proof deMorganOrNot : (~A | ~B) => ~(A & B)
proof deMorganNotAnd : ~(A & B) => ~A | ~B
proof deMorganAndNot : (~A & ~B) => ~(A | B)

proof deMorganNotEx : ~(?x:t. A(x)) => !x:t. ~A(x)
proof deMorganExNot : (?x:t. ~A(x)) => ~!x:t. A(x)
proof deMorganNotAll : ~(!x:t. A(x)) => ?x:t. ~A(x)
proof deMorganAllNot : (!x:t. ~A(x)) => ~?x:t. A(x)
```

On Andrew machines, you can check your progress against the requirements file `/afs/andrew/course/15/317/req/hw04.req` by running the command

```
$ /afs/andrew/course/15/317/bin/tutch -r hw04 <files...>
```

2 Double-Negation Translation (12 points)

Recall from lecture the double-negation translation A^* embedding classical logic into intuitionistic logic:

$$\begin{aligned} P^* &= \neg\neg P \qquad (P \text{ atomic}) \\ \top^* &= \top \\ (A \wedge B)^* &= A^* \wedge B^* \\ \bot^* &= \bot \\ (A \vee B)^* &= \neg\neg(A^* \vee B^*) \\ (\neg A)^* &= \neg(A^*) \end{aligned}$$

It is a theorem that if A *true* is derivable classically, then A^* *true* is derivable intuitionistically. The key lemma in the proof of this theorem is that double-negation elimination holds on the target of the translation:

Lemma: For every A, the rule

$$\frac{\neg\neg A^* \; true}{A^* \; true} \; DNE_A$$

is intuitionistically derivable.

This lemma can be proven by structural induction on A.

Task 2 (6 pts)**.** Prove the case for conjunction. That is, show that the rule

$$\frac{\neg\neg(A \wedge B)^* \; true}{(A \wedge B)^* \; true} \; DNE_{A\wedge B}$$

is derivable, assuming the following new rules:

$$\frac{\neg\neg A^* \; true}{A^* \; true} \; I.H._A \qquad \frac{\neg\neg B^* \; true}{B^* \; true} \; I.H._B$$

Task 3 (6 pts)**.** Prove the case for disjunction. That is, show that the rule

$$\frac{\neg\neg(A \vee B)^* \; true}{(A \vee B)^* \; true} \; DNE_{A\vee B}$$

is derivable.

For both problems, you may choose to treat negation $\neg A$ as a notational definition $A \supset \bot$ or using the direct definition given in the Lecture Notes on Classical Logic, whichever you are more comfortable with.