Midterm 2

15-317: Constructive Logic

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Instructions

- This exam is closed-book, but one (two-sided) sheet of notes is permitted.
- You have 80 minutes to complete the exam.
- There are three problems and one extra-credit problem.
- The extra credit problem does **NOT** influence your score on the exam itself! It will be taken into account when determining final grades for the course. *Do not work on the extra credit problem until you have finished the exam itself!*

	Problem 1	Problem 2	Problem 3	Total	Extra Credit
Score					
Max	50	45	55	150	20
Grader					

1 Arithmetic (50 points)

Recall the rules for equality of natural numbers:

$$\frac{}{\mathsf{z} = \mathsf{z}} =I_{\mathsf{z}} \qquad \frac{n = m}{\mathsf{s}\, n = \mathsf{s}\, m} =I_{\mathsf{s}}$$

(no elim rule for z = z)

$$\frac{\mathsf{s}\, n = \mathsf{s}\, m}{n = m} =E_{\mathsf{ss}} \qquad \frac{\mathsf{s}\, n = \mathsf{z}}{C} =E_{\mathsf{sz}} \qquad \frac{\mathsf{z} = \mathsf{s}\, m}{C} =E_{\mathsf{zs}}$$

Task 1 (15 pts). Give analogous rules for a proposition $n \leq m$ meaning n is less than or equal to m:

Task 2 (35 pts). Give a natural deduction derivation of the following proposition:

$$\forall n{:}\mathsf{nat}.\forall m{:}\mathsf{nat}.\ (n \leq m)\ \supset\ \exists p{:}\mathsf{nat}.\ (n + p) = m$$

You may freely use computational equalities $0 + k = k$ and $(\mathsf{s}\ i) + k = \mathsf{s}\ (i + k)$.

2 Logic Programming (45 points)

2.1 Reverse

Recall the append relation:

```
append([],Ys,Ys).

append([X|Xs],Ys,[X|Zs]) :-
   append(Xs,Ys,Zs).
```

Task 1 (20 pts). Use `append` to define a relation `reverse(Xs,Ys)` that relates a list `Xs` to a list `Ys` containing the same elements in the opposite order. For example, `reverse([1,2,3], [3,2,1])` should hold.

Your relation should have the following mode:

```
reverse(+Xs,-Ys)
```

That is, the first argument is an input and the second argument is an output.

Fill in the holes in the following code:

```
reverse([], ____________________).

reverse([X|Xs], ____________________) :-
```

2.2 Mode

Task 2 (4 pts). What mode does `append` need to have to for reverse to have mode `reverse(+Xs,-Ys)`?

Task 3 (6 pts). Explain why `append` has this mode, and why your code for `reverse` has mode `(+Xs,-Ys)`.

2.3 Termination

Task 4 (7 pts). Does the following Prolog program terminate on all inputs with mode `a(+N1,+N2)`?

```
a(N,s(M)) :- a(N,M).
a(s(N),M) :- a(N,s(s(M))).
```

If so, state the termination order: what gets smaller at each call?

If not, find an input on which proof search fails to terminate, trace the sequence of subgoals considered, and explain why this loops.

Task 5 (8 pts). Does the following Prolog program terminate on all inputs with mode `b(+N1,+N2)`?

```
b(N,s(M)) :- b(s(N),M).
b(s(N),M) :- b(N,s(s(M))).
```

If so, state the termination order: what gets smaller at each call?

If not, find an input on which proof search fails to terminate, trace the sequence of subgoals considered, and explain why this loops.

3 Sequent Calculus

In this problem, we will consider a sequent calculus with rules for negation.

We consider two forms of sequent: $A_1 ... A_n \Longrightarrow A$ (as usual) and $A_1 \ldots A_n \Longrightarrow \#$ (if $A_1, \ldots, A_n$ are true, then contradiction). We write J for an arbitrary conclusion (A or $\#$).

The rules for negation are as follows:

$$\frac{\Gamma, A \Longrightarrow \#}{\Gamma \Longrightarrow \neg A} \neg R \qquad \frac{\Gamma, \neg A \Longrightarrow A}{\Gamma, \neg A \Longrightarrow J} \neg L$$

3.1 Identity

Identity: For all A and Γ, the sequent $\Gamma, A \Longrightarrow A$ is derivable.

Task 1 (20 pts). Prove the $\neg A$ case:

Assume $\Gamma, A \Longrightarrow A$ for all Γ
and show $\Gamma, \neg A \Longrightarrow \neg A$ for all Γ.

3.2 Cut

Cut: If $\Gamma \Longrightarrow A$ and $\Gamma, A \Longrightarrow J$ then $\Gamma \Longrightarrow J$.

Task 2 (20 pts). Prove the principal cut case for $\neg A$: Assume

$$\mathcal{D} = \dfrac{\begin{array}{c}\mathcal{D}_1\\ \Gamma, A \Longrightarrow \#\end{array}}{\Gamma \Longrightarrow \neg A}\ \neg R \qquad \text{and} \qquad \mathcal{E} = \dfrac{\begin{array}{c}\mathcal{E}_1\\ \Gamma, \neg A \Longrightarrow A\end{array}}{\Gamma, \neg A \Longrightarrow J}\ \neg L$$

and show $\Gamma \overset{\mathcal{F}}{\Longrightarrow} J$.

- Clearly state what gets smaller at each use of the induction hypothesis.
- You may use the following weakening principles:
  - **Left Weakening:** If $\Gamma \Longrightarrow J$ then $\Gamma, A \Longrightarrow J$.
  - **Right Weakening:** If $\Gamma \Longrightarrow \#$ then $\Gamma \Longrightarrow C$.

3.3 Admissible Rules

Task 3 (5 pts). Show that the rule of *contraction*, which states that assumptions may be duplicated (when reading a proof from bottom to top), is admissible. Prove the following:

$$\text{If } \Gamma, A, A \Longrightarrow J \text{ then } \Gamma, A \Longrightarrow J.$$

You may use weakening, identity, and cut in your proof.

Recall that a rule is called *invertible* if the conclusion implies the premises.

Task 4 (5 pts). Is the rule $\neg R$ invertible?

If so, show that the conclusion implies the premise (you may use identity and cut and weakening to prove this).

If not, find a particular Γ and A for which the conclusion is true but the premise is not.

Task 5 (5 pts). Is the rule $\neg L$ invertible?

If so, show that the conclusion implies the premise (you may use identity and cut and weakening to prove this).

If not, find a particular Γ and A for which the conclusion is true but the premise is not.

4 EXTRA CREDIT: Inversion

Recall the sequent calculus rules:

$$\frac{P \in \Gamma}{\Gamma \Longrightarrow P}\ init$$

$$\frac{\Gamma \Longrightarrow A \quad \Gamma \Longrightarrow B}{\Gamma \Longrightarrow A \wedge B}\ \wedge R \qquad \frac{\Gamma, A \wedge B, A, B \Longrightarrow C}{\Gamma, A \wedge B \Longrightarrow C}\ \wedge L$$

$$\frac{\Gamma \Longrightarrow A}{\Gamma \Longrightarrow A \vee B}\ \vee R_1 \quad \frac{\Gamma \Longrightarrow B}{\Gamma \Longrightarrow A \vee B}\ \vee R_2 \quad \frac{\Gamma, A \vee B, A \Longrightarrow C \quad \Gamma, A \vee B, B \Longrightarrow C}{\Gamma, A \vee B \Longrightarrow C}\ \vee L$$

$$\frac{\Gamma, A \Longrightarrow B}{\Gamma \Longrightarrow A \supset B}\ \supset R \qquad \frac{\Gamma, A \supset B \Longrightarrow A \quad \Gamma, A \supset B, B \Longrightarrow C}{\Gamma, A \supset B \Longrightarrow C}\ \supset L$$

Consider the following sequent:

$$[] \Longrightarrow ((P \wedge Q) \vee (P \wedge R)) \supset (P \wedge (Q \vee R))$$

Task 1 (10 pts). How many proofs of this sequent are there? If there are finitely many, count them. If there are infinitely many, explain why.

The following rules describe the left-inversion strategy that you implemented in Homework 7. The context Γ can contains only atoms P and implications $A \supset B$.

Rules for $\Gamma \Longrightarrow A$

$$\frac{P \in \Gamma}{\Gamma \Longrightarrow P}\ init \qquad \frac{\Gamma \Longrightarrow A \quad \Gamma \Longrightarrow B}{\Gamma \Longrightarrow A \wedge B}\ \wedge R \qquad \frac{\Gamma \Longrightarrow A}{\Gamma \Longrightarrow A \vee B}\ \vee R_1 \qquad \frac{\Gamma \Longrightarrow B}{\Gamma \Longrightarrow A \vee B}\ \vee R_2$$

$$\frac{\Gamma; A \Longrightarrow B}{\Gamma \Longrightarrow A \supset B}\ \supset R \qquad \frac{\Gamma, A \supset B \Longrightarrow A \quad \Gamma, A \supset B; B \Longrightarrow C}{\Gamma, A \supset B \Longrightarrow C}\ \supset L$$

Rules for inversion $\Gamma; \Delta \Longrightarrow A$

$$\frac{\Gamma; A, B, \Delta \Longrightarrow C}{\Gamma; A \wedge B, \Delta \Longrightarrow C}\ \wedge L_{inv} \qquad \frac{\Gamma; A, \Delta \Longrightarrow C \quad \Gamma; B, \Delta \Longrightarrow C}{\Gamma; \Delta, A \vee B \Longrightarrow C}\ \vee L_{inv}$$

$$\frac{\Gamma, P; \Delta \Longrightarrow C}{\Gamma; P, \Delta \Longrightarrow C}\ PL_{inv} \qquad \frac{\Gamma, A \supset B; \Delta \Longrightarrow C}{\Gamma; A \supset B, \Delta \Longrightarrow C}\ \supset L_{inv} \qquad \frac{\Gamma \Longrightarrow C}{\Gamma; [] \Longrightarrow C}\ \texttt{[]} L_{inv}$$

Task 2 (10 pts). How many proofs of

$$[] \Longrightarrow ((P \wedge Q) \vee (P \wedge R)) \supset (P \wedge (Q \vee R))$$

are there?

If there are finitely many, show all proofs. If there are infinitely many, explain why.