Midterm 1

15-317: Constructive Logic

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Instructions

- This exam is closed-book, but one (two-sided) sheet of notes is permitted.
- There are five problems. You have 80 minutes to complete the exam.

	Problem 1	Problem 2	Problem 3	Problem 4	Problem 5	Total
Score						
Max	25	45	15	25	40	150
Grader						

1 Parsing (25 points)

Here's a variation on the Clue problem:

> *If Mr. Green committed the murder, then it was with the candle-stick in the billiard room or with the revolver in the hall. If it wasn't in the billiard room, then Miss Scarlet did it. Either it was done with the revolver, or in the hall. There was a murderer, and there was a murder weapon, and it happened somewhere. There was only one murderer.*

Write out the statements above in the language of first-order logic, using the following notation:

Objects:

$$
\begin{aligned}
green &= \text{Mr. Green} \\
scarlet &= \text{Mrs. Scarlet} \\
billiard &= \text{the billiard room} \\
hall &= \text{the hallway} \\
revolver &= \text{the revolver} \\
candle &= \text{the candle-stick}
\end{aligned}
$$

Atomic Propositions:

$$
\begin{aligned}
did(x) &= \text{the murderer was x} \\
with(x) &= \text{the murder weapon was x} \\
in(x) &= \text{the murder weapon was committed in x} \\
same(x, y) &= \text{x and y are the same person}
\end{aligned}
$$

(question continues on the next page)

Task 1 (5 pts).

If Mr. Green committed the murder, then it was with the candle-stick in the billiard room or with the revolver in the hall.

Task 2 (5 pts).

If it wasn't in the billiard room, then Miss Scarlet did it.

Task 3 (5 pts).

Either it was done with the revolver, or in the hall.

Task 4 (5 pts).

There was a murderer, and there was a murder weapon, and it happened somewhere.

Task 5 (5 pts).

There was only one murderer.

2 Natural Deduction and Proof Terms (45 points)

2.1 Natural Deduction

Consider the following proposition:

$$(A \supset B \supset C) \supset (A \supset B) \supset (A \supset C)$$

Here is a Tutch proof:

```
proof scomb : (A => B => C) => (A => B) => (A => C) =
begin
[(A => B => C);
 [A => B;
  [A;
   B => C;
   B;
   C];
  A => C];
 (A => B) => (A => C)];
(A => B => C) => (A => B) => (A => C);
end;
```

Task 1 (20 pts). Give the natural deduction proof corresponding to this Tutch proof. Label each inference rule used with the rule name and any variables discharged (e.g., $\supset I^u$).

2.2 Proof Terms

Say you have 2 rows with 2 pigeonholes in each row:

H_1	H_2
H_3	H_4

Then the *pigeonhole principle* says that if there are at least three pigeons in holes, then at least one row has both holes filled. For example, the pigeons might be arranged like this:

•	•
•	

We can formalize this as follows:

- Consider atomic propositions H_1, H_2, H_3, H_4, where H_i means "there is a pigeon in hole i".
- We can state "there are at least three pigeons in holes" by enumerating the four configurations in which this is true:

$$(H_1 \wedge H_2 \wedge H_3) \vee (H_1 \wedge H_2 \wedge H_4) \vee (H_1 \wedge H_3 \wedge H_4) \vee (H_2 \wedge H_3 \wedge H_4)$$

- We can state "at least one row has both holes filled" by enumerating the two ways this can be true:

$$(H_1 \wedge H_2) \vee (H_3 \wedge H_4)$$

(Question continues on the next page)

Task 1 (25 pts). Give a proof term for the pigeonhole principle:

$$(H_1 \wedge H_2 \wedge H_3) \vee (H_1 \wedge H_2 \wedge H_4) \vee (H_1 \wedge H_3 \wedge H_4) \vee (H_2 \wedge H_3 \wedge H_4)$$

$$\supset$$

$$(H_1 \wedge H_2) \vee (H_3 \wedge H_4)$$

Note that conjunction and disjunction associate to the right, so $A \wedge B \wedge C$ is shorthand for $A \wedge (B \wedge C)$.

For reference, there is a Tutch proof on the next page (but feel free to ignore this proof—it is not required that your proof term correspond to it).

```
proof pidg : ((H1 & (H2 & H3))
             | ((H1 & H2 & H4)
                | ((H1 & H3 & H4)
                   | (H2 & H3 & H4))))
            => ((H1 & H2) | (H3 & H4)) =
begin
[(H1 & H2 & H3) | (H1 & H2 & H4) | (H1 & H3 & H4) | (H2 & H3 & H4);
 [(H1 & H2 & H3);
  H1;
  H2 & H3;
  H2;
  H1 & H2;
  (H1 & H2) | (H3 & H4)];
 [(H1 & H2 & H4) | ((H1 & H3 & H4) | (H2 & H3 & H4));
  [(H1 & H2 & H4);
   H1;
   H2 & H4;
   H2;
   H1 & H2;
   (H1 & H2) | (H3 & H4)];
  [(H1 & H3 & H4) | (H2 & H3 & H4);
   [H1 & H3 & H4;
    H3 & H4;
    ((H1 & H2) | (H3 & H4))];
   [H2 & H3 & H4;
    H3 & H4;
    (H1 & H2) | (H3 & H4)];
   (H1 & H2) | (H3 & H4)];
  (H1 & H2) | (H3 & H4)];
 (H1 & H2) | (H3 & H4)];
(H1 & H2 & H3) | (H1 & H2 & H4) | (H1 & H3 & H4) | (H2 & H3 & H4)
  => ((H1 & H2) | (H3 & H4));
end;
```

3 Quantifiers (15 points)

Consider the following proof:

$$
\dfrac{\dfrac{\dfrac{}{(\exists x:\tau.A(x)\supset B(x))}\,u \quad \dfrac{\dfrac{}{(\exists x:\tau.B(x)\supset C(x))}\,v \quad \dfrac{\dfrac{}{x:\tau} \quad \dfrac{\dfrac{\dfrac{}{B(x)\supset C(x)}\,w_2 \quad \dfrac{\dfrac{}{A(x)\supset B(x)}\,w_1 \quad \dfrac{}{A(x)}\,w_3}{B(x)}\,{\supset}E}{C(x)}\,{\supset}E}{A(x)\supset C(x)}\,{\supset}I^{w_3}}{(\exists x:\tau.A(x)\supset C(x))}\,\exists I}{(\exists x:\tau.A(x)\supset C(x))}\,\exists E^{x,w_2}}{(\exists x:\tau.A(x)\supset C(x))}\,\exists E^{x,w_1}}{\dfrac{(\exists x:\tau.B(x)\supset C(x))\supset(\exists x:\tau.A(x)\supset C(x))}{(\exists x:\tau.A(x)\supset B(x))\supset(\exists x:\tau.B(x)\supset C(x))\supset(\exists x:\tau.A(x)\supset C(x))}\,{\supset}I^{u}}\,{\supset}I^{v}
$$

Task 1 (15 pts). This proof is incorrect. Circle the labels of the rule(s) whose applications are incorrect. Explain what is incorrect about each of these rule applications.

4 Proof Irrelevance (25 points)

Suppose we have

- Types nat and list
- An atomic proposition $\mathsf{append}(l_1, l_2, l_3)$ meaning that appending the lists l_1 and l_2 gives the list l_3.
- An atomic proposition $\mathsf{length}(l, n)$ meaning that the length of l is n.
- An atomic proposition $n > m$ meaning that the length of n is greater than m.

A function partitioning a list can be specified as follows:

$$\forall l : \mathsf{list}.\forall n : \mathsf{nat}.\ (\exists m : \mathsf{nat}.\mathsf{length}(l, m) \wedge (n > m)) \quad \vee \quad (\exists l_1 : \mathsf{list}.\exists l_2 : \mathsf{list}.\mathsf{append}(l_1, l_2, l) \wedge \mathsf{length}(l_1, n))$$

Task 1 (7 pts). Add proof-irrelevance brackets to this proposition in the correct places to give the following computational contents:

When given a list l and a number n, return $\mathsf{inr}(l_1, l_2, [])$ when n is less than or equal to the length of the list, and $\mathsf{inl}[]$ when n is greater than the length of the list.

Task 2 (7 pts). Add proof-irrelevance brackets to this proposition in the correct places to give the following computational contents:

When given a list l and a number n, return inr of a list containing the first n elements of l when n is less than or equal to the length of the list, or $\mathsf{inl}[]$ otherwise.

Task 2 (11 pts). Explain the computational contents of the following proposition:

$$\forall l : \mathsf{list}.\forall n : \mathsf{nat}. \quad [\exists m : \mathsf{nat}.\mathsf{length}(l, m) \wedge (n > m)]$$

$$\vee$$

$$[\exists l_1 : \mathsf{list}.\exists l_2 : \mathsf{list}.\mathsf{append}(l_1, l_2, l) \wedge \mathsf{length}(l_1, n)]$$

5 Decidability (40 points)

Recall the rules for constructive logic with the judgements $A\ \mathsf{false}$ and $\#$:

$$\frac{\#}{J}\ \#E \qquad \frac{\begin{array}{c}\overline{A\ \mathsf{true}}\ u\\ \vdots\\ \#\end{array}}{A\ \mathsf{false}}\ fI^u \qquad \frac{A\ \mathsf{false}\quad A\ \mathsf{true}}{\#}\ fE \qquad \frac{A\ \mathsf{false}}{\neg A\ \mathsf{true}}\ \neg I \qquad \frac{\neg A\ \mathsf{true}\qquad \begin{array}{c}\overline{A\ \mathsf{false}}\ k\\ \vdots\\ J\end{array}}{J}\ \neg E^k$$

We can define a connective ΔA meaning that A is decidable:

$$\frac{A\ \mathsf{true}}{\Delta A\ \mathsf{true}}\ \Delta I_L \qquad \frac{A\ \mathsf{false}}{\Delta A\ \mathsf{true}}\ \Delta I_R \qquad \frac{\Delta A\ \mathsf{true}\qquad \begin{array}{c}\overline{A\ \mathsf{true}}\ u\\ \vdots\\ J\end{array}\qquad \begin{array}{c}\overline{A\ \mathsf{false}}\ k\\ \vdots\\ J\end{array}}{J}\ \Delta E^{u,k}$$

Task 1 (15 pts). Prove that ΔA is locally sound:

Task 2 (10 pts). Prove that ΔA is locally complete:

Task 3 (15 pts). For this task, we will work in classical logic, which means you may also use the following rule:

$$\dfrac{\begin{array}{c}\overline{A\,\mathsf{false}}\ k\\ \vdots\\ \#\end{array}}{A\,\mathsf{true}}\ DNE^k$$

Give a natural deduction derivation of ΔA for an arbitrary A: