Assignment 10 Solutions:
Linear Logic

15-317: Constructive Logic

Out: Friday, November 28, 2008
Due: Friday, December 5, 2008

1 Linear Logic Derivations (20 points)

1.1 Distribute Tensor over Plus

Both directions work:

- $(A \otimes B) \oplus (A \otimes C) \Vdash A \otimes (B \oplus C)$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A \Vdash A}\,init \quad \dfrac{\dfrac{}{B \Vdash B}\,init}{B \Vdash B \oplus C}\,\oplus R_1}{A, B \Vdash A \otimes (B \oplus C)}\,\otimes R}{(A \otimes B) \Vdash A \otimes (B \oplus C)}\,\otimes L \qquad \dfrac{symmetric}{(A \otimes C) \Vdash A \otimes (B \oplus C)}}{(A \otimes B) \oplus (A \otimes C) \Vdash A \otimes (B \oplus C)}\,\otimes L$$

- $A \otimes (B \oplus C) \Vdash (A \otimes B) \oplus (A \otimes C)$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \Vdash A}\,init \quad \dfrac{}{B \Vdash B}\,init}{A, B \Vdash (A \otimes B)}\,\otimes R}{A, B \Vdash (A \otimes B) \oplus (A \otimes C)}\,\oplus R_1 \qquad \dfrac{symmetric}{A, C \Vdash (A \otimes B) \oplus (A \otimes C)}}{A, (B \oplus C) \Vdash (A \otimes B) \oplus (A \otimes C)}\,\oplus L}{A \otimes (B \oplus C) \Vdash (A \otimes B) \oplus (A \otimes C)}\,\otimes L$$

1.2 Distribute With over Plus

One direction works; the other does not:

- $A \& (B \oplus C) \Vdash (A \& B) \oplus (A \& C)$

  Not derivable. We can't use $\oplus R$ first because then we'd have to show (say)

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A \Vdash A}\,init}{A \& (B \oplus C) \Vdash A}\,\& L_1 \qquad \dfrac{\dfrac{???}{(B \oplus C) \Vdash B}\,STUCK}{A \& (B \oplus C) \Vdash B}\,\& L_2}{A \& (B \oplus C) \Vdash (A \& B)}\,\& R}{A \& (B \oplus C) \Vdash (A \& B) \oplus (A \& C)}\,\oplus R_1$$

but we can't get B from $(B \oplus C)$. (This attempt at a derivation doesn't work for the corresponding problem in regular constructive logic.)

However, if we try to work on the left first, then we must choose one of two paths:

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A}\,\mathit{init} \quad \dfrac{\vdots}{A \Vdash B}\,\mathit{STUCK}}{B \Vdash (A \& B)}\,\& R \qquad \dfrac{\vdots}{C \Vdash (A \& B) \oplus (A \& C)}}{A \Vdash (A \& B) \oplus (A \& C)}\,\oplus R_1}{A \& (B \oplus C) \Vdash (A \& B) \oplus (A \& C)}\,\& L_1
$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{B \Vdash A}\,\mathit{STUCK} \quad \dfrac{}{B \vdash B}\,\mathit{init}}{B \Vdash (A \& B)}\,\& R}{B \Vdash (A \& B) \oplus (A \& C)}\,\oplus R_1 \qquad \dfrac{\vdots}{C \Vdash (A \& B) \oplus (A \& C)}}{(B \oplus C) \Vdash (A \& B) \oplus (A \& C)}\,\oplus L}{A \& (B \oplus C) \Vdash (A \& B) \oplus (A \& C)}\,\& L_2
$$

In the first, we lose the proof of $B \oplus C$, and so we get stuck when we try to prove B; in the second, we lose the proof of A, and so get stuck when we try to prove that.

A formal proof of the non-derivability of this sequent would have to consider a few more cases and permutations of the inference rules used here, but they are analogous to these three.

- $(A \& B) \oplus (A \& C) \Vdash A \& (B \oplus C)$

  Proof:

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{A \Vdash A}\,\mathit{init}}{(A \& B) \Vdash A}\,\& L_1 \quad \dfrac{\dfrac{\dfrac{}{B \Vdash B}\,\mathit{init}}{B \Vdash (B \oplus C)}\,\oplus R_1}{(A \& B) \Vdash (B \oplus C)}\,\& L_2}{(A \& B) \Vdash A \& (B \oplus C)}\,\& R \qquad \dfrac{\mathit{symmetric}}{(A \& B) \oplus (A \& C) \Vdash A \& (B \oplus C)}}{(A \& B) \oplus (A \& C) \Vdash A \& (B \oplus C)}\,\oplus L
$$

  Why does this direction work but the opposite direction fails? In this part, the outer connectives are invertible (& on the right; $\oplus$ on the left), so we don't have to make any early choices. The above theorem puts & on the left and $\oplus$ on the right, so we need to make early choices on both sides.

1.3 Distribute Plus over Tensor

Neither direction is provable (but note that both directions work if you ignore linearity!).

- $A \oplus (B \otimes C) \Vdash (A \oplus B) \otimes (A \oplus C)$

  Apply $\oplus L$; in the left-hand branch, you essentially have to prove $A \otimes A$, which you cannot do because of linearity: you only have one copy of A, but the conclusion asks for two.

- $(A \oplus B) \otimes (A \oplus C) \Vdash A \oplus (B \otimes C)$

  Apply $\otimes L$ and $\oplus L$; in one of the four cases, you need to prove $A, A \Vdash A$, which you cannot do because of linearity: you need to use two copies of A, but you only can use one.

1.4 Distribute Plus over With

As above, one direction works but the other does not.

- $A \oplus (B \& C) \Vdash (A \oplus B) \& (A \oplus C)$

  Here the outer connectives are invertible, so it works:

$$\dfrac{\dfrac{\dfrac{}{A \Vdash A}\,init}{A \Vdash A \oplus B}\,\oplus R_1 \quad \dfrac{symmetric}{A \Vdash A \oplus C} \quad \dfrac{\dfrac{analogous}{B \Vdash A \oplus B}}{B \& C \Vdash A \oplus B}\,\& R_1 \quad \dfrac{symmetric}{B \& C \Vdash A \oplus C}}{A \oplus (B \& C) \Vdash (A \oplus B) \& (A \oplus C)}\,\oplus L, \& R$$

- $(A \oplus B) \& (A \oplus C) \Vdash A \oplus (B \& C)$

  Here we need to make early choices, so we get stuck. We can't make the choice on the right first: if we choose to prove A, then we get stuck after choosing something on the left and case-analyzing it using $\oplus L$ (because one case has a B or a C); similarly if we choose $B \& C$ (one case has an A).

  If you use $\& L_1$ first, you have no way of proving C, so you have no way of proving $B \& C$; symmetrically if you use $\& L_2$ (no way of proving B).