15-317 Homework 5

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Task 1

$\mathsf{even}(0)\ \mathsf{true}$ cannot be eliminated using the rules. There is also no way to introduce $\mathsf{odd}(0)\ \mathsf{true}$ using these rules. Consider the other cases:

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}\\ \mathsf{odd}(n)\ \mathsf{true}\end{matrix}}{\mathsf{even}(s\ n)\ \mathsf{true}}\,evI_S}{\mathsf{odd}(n)\ \mathsf{true}}\,evE_S \qquad \Longrightarrow_R \qquad \begin{matrix}\mathcal{D}\\ \mathsf{odd}(n)\ \mathsf{true}\end{matrix}$$

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}\\ \mathsf{even}(n)\ \mathsf{true}\end{matrix}}{\mathsf{odd}(s\ n)\ \mathsf{true}}\,oddI_S}{\mathsf{even}(n)\ \mathsf{true}}\,oddE_S \qquad \Longrightarrow_R \qquad \begin{matrix}\mathcal{D}\\ \mathsf{even}(n)\ \mathsf{true}\end{matrix}$$

Task 2

I will omit true to save space.

$$\dfrac{\dfrac{\dfrac{}{n : \mathsf{nat}} \quad \dfrac{\dfrac{}{\mathsf{even}(0)}\,evI_0}{\mathsf{even}(0) \lor \mathsf{odd}(0)}\,\lor I_L \quad \dfrac{\dfrac{}{\mathsf{even}(m) \lor \mathsf{odd}(m)}\,u \quad \dfrac{\dfrac{\dfrac{}{\mathsf{even}(m)}\,v}{\mathsf{odd}(s\ m)}\,oddI_S}{\mathsf{even}(s\ m) \lor \mathsf{odd}(s\ m)}\,\lor I_R \quad \dfrac{\dfrac{\dfrac{}{\mathsf{odd}(m)}\,w}{\mathsf{even}(s\ m)}\,evI_S}{\mathsf{even}(s\ m) \lor \mathsf{odd}(s\ m)}\,\lor I_L}{\mathsf{even}(s\ m) \lor \mathsf{odd}(s\ m)}\,\lor E^{v,w}}{\mathsf{even}(n) \lor \mathsf{odd}(n)}\,\mathsf{nat}E^{m,u}}{\forall x : \mathsf{nat}.\mathsf{even}(x) \lor \mathsf{odd}(x)}\,\forall I^n$$

Task 3

I will also omit the type nat in quantifiers. Also, let $J(a)$ be the judgment $(\mathsf{even}(a) \supset \exists m.(a = 2\cdot m)) \land (\mathsf{odd}(a) \supset \exists m.(a = 2\cdot m+1))\ \mathsf{true}$. Let $\mathcal{D}_1$ be the following derivation from the assumptions $x : \mathsf{nat}$ and $b : J(x)$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\mathsf{even}(s\ x)}\,a}{\mathsf{odd}(x)}\,evE_S \quad \dfrac{\dfrac{}{J(x)}\,b}{\mathsf{odd}(x) \supset \exists m.(x = 2\cdot m+1)}\,\land E_R}{\exists m.(x = 2\cdot m + 1)}\,\supset E \quad \dfrac{\dfrac{\dfrac{}{m' : \mathsf{nat}}}{s\ m' : \mathsf{nat}}\,\mathsf{nat}I_s \quad \dfrac{\dfrac{}{x = 2\cdot m' + 1}\,p}{s\ x = 2\cdot (s\ m')}\,eqI_S}{\exists m.(s\ x = 2\cdot m)}\,\exists I}{\exists m.(s\ x = 2\cdot m)}\,\exists E^{m',p}}{\mathsf{even}(s\ x) \supset \exists m.(s\ x = 2\cdot m)}\,\supset I^a$$

Let $\mathcal{D}_2$ be the following derivation from the assumptions $x : \mathsf{nat}$ and $b : J(x)$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\mathsf{odd}(s\ x)}\,a}{\mathsf{even}(x)}\,oddE_S \quad \dfrac{\dfrac{}{J(x)}\,b}{\mathsf{even}(x) \supset \exists m.(x = 2\cdot m)}\,\land E_L}{\exists m.(x = 2\cdot m)}\,\supset E \quad \dfrac{\dfrac{}{m' : \mathsf{nat}} \quad \dfrac{\dfrac{}{x = 2\cdot m'}\,p}{s\ x = 2\cdot m' + 1}\,eqI_S}{\exists m.(s\ x = 2\cdot m + 1)}\,\exists I}{\exists m.(s\ x = 2\cdot m + 1)}\,\exists E^{m',p}}{\mathsf{odd}(s\ x) \supset \exists m.(s\ x = 2\cdot m + 1)}\,\supset I^a$$

Let $\mathcal{D}$ be the following derivation from the assumptions $n : \mathsf{nat}$ and $w : J(n)$:

$$\dfrac{\begin{array}{c}[n/x, w/b]\mathcal{D}_1\\ \mathsf{even}(s\ n) \supset \exists m.(s\ n = 2\cdot m)\end{array} \quad \begin{array}{c}[n/x, w/b]\mathcal{D}_2\\ \mathsf{odd}(s\ n) \supset \exists m.(s\ n = 2\cdot m + 1)\end{array}}{J(s\ n)}\wedge I$$

The base case uses the fact that $2 \cdot 0$ is 0.

$$\dfrac{\dfrac{}{x' : \mathsf{nat}} \quad \dfrac{\dfrac{\dfrac{\dfrac{}{0 : \mathsf{nat}}\mathsf{nat}I_Z \quad \dfrac{}{0 = 2\cdot 0}eqI_Z}{\exists m.(0 = 2\cdot m)}\exists I}{\mathsf{even}(0) \supset \exists m.(0 = 2\cdot m)}\supset I^u \quad \dfrac{\dfrac{\dfrac{}{\mathsf{odd}(0)}v}{\exists m.(0 = 2\cdot m + 1)}oddE_0}{\mathsf{odd}(0) \supset \exists m.(0 = 2\cdot m + 1)}\supset I^v}{J(0)}\wedge I \quad \begin{array}{c}\dfrac{}{k : \mathsf{nat}} \quad \dfrac{}{J(k)}z\\ [k/n, z/w]\mathcal{D}\\ J(s\ k)\end{array}}{\dfrac{J(x')}{\forall x.J(x)}\forall I^{x'}}\mathsf{nat}E^{k,z}$$