Assignment 5:
Arithmetic

15-317: Constructive Logic

Out: Thursday, October 9, 2008
Due: Thursday, October 16, 2008, before class

1 Tutch Proofs (10 pts)

In Tutch, the recursor `R(t,M0,x.u.M1(x,u))` is written as a primitive recursion schema:

```
rec t of f 0 => M0
       | f (s x) => M1(x,f(x)) end;
```

For example, here is a defintion of addition:

```
val plus : nat -> nat -> nat =
fn x => rec x of
   p 0 => fn y => y
 | p (s n) => fn y => s (p n y)
 end;
```

Here's an example of a proof that uses induction over natural numbers.

```
proof plus0R : (!m:nat. m = plus m 0) =
begin
[m : nat;

 % the 0 case
 0 = plus 0 0;

 % the s case; note that it binds two assumptions
 [x : nat , x = plus x 0;
  s x = plus (s x) 0];

 m = plus m 0];
(!m:nat. m = plus m 0);
end;
```

Task 1 (15 pts). Your task is to prove transitivity of equality. Hint: this will be longer than any Tutch proof you have done so far.

```
proof eqtrans : !x:nat. !y:nat. !z:nat. (x = y) => (y = z) => (x = z);
```

2 Proof terms (10 pts)

Here is an example proof term:

```
term plus0R : (!m:nat. m = plus m 0) =
fn m => rec m of
          p 0 => eq0
        | p (s x) => eqS (p x)
end;
```

Task 1 (10 pts). Prove the following:

```
term eqrefl : (!m:nat. m = m);
term plusSR : (!m:nat. !n:nat. s (plus m n) = plus m (s n));
```

Hint: you will need to use `eqrefl` as a lemma to prove `plusSR`.

The rules for equality are named as follows:

$$\frac{}{\texttt{eq0 : (0 = 0)}} \qquad \frac{\texttt{M : (t = t')}}{\texttt{eqSS M : (s t = s t')}} \qquad \frac{\texttt{M : (s t = s t')}}{\texttt{eqESS M : (t = t')}} \qquad \frac{\texttt{M : 0 = s t}}{\texttt{eqE0S M : J}} \qquad \frac{\texttt{M : s t = 0}}{\texttt{eqES0 M : J}}$$

3 Even and Odd (20 pts)

Consider the following rules for even and odd:

$$\frac{}{\mathsf{even}(0)\,\mathsf{true}}\ evI_0 \qquad \frac{\mathsf{odd}(n)\,\mathsf{true}}{\mathsf{even}(\mathsf{s}\,n)\,\mathsf{true}}\ evI_S \qquad \frac{\mathsf{even}(n)\,\mathsf{true}}{\mathsf{odd}(\mathsf{s}\,n)\,\mathsf{true}}\ oddI_S$$

$$\frac{\mathsf{odd}(0)\,\mathsf{true}}{J}\ oddE_0 \qquad \frac{\mathsf{even}(\mathsf{s}\,n)\,\mathsf{true}}{\mathsf{odd}(n)\,\mathsf{true}}\ evE_S \qquad \frac{\mathsf{odd}(\mathsf{s}\,n)\,\mathsf{true}}{\mathsf{even}(n)\,\mathsf{true}}\ oddE_S$$

Task 1 (5 pts). Prove that these rules are locally sound.

Task 2 (5 pts). Give a natural deduction derivation of the following:

$$(\forall x : \mathsf{nat}.\mathsf{even}(x) \vee \mathsf{odd}(x))\,\mathsf{true}$$

Be sure to label each inference with the rule used.

Task 3 (10 pts). Give a natural deduction derivation for the following:

$$(\forall x : \mathsf{nat}.(\mathsf{even}(x) \supset \exists m : \mathsf{nat}.(x = 2 * m)) \wedge (\mathsf{odd}(x) \supset \exists m : \mathsf{nat}.(x = 2 * m + 1)))\,\mathsf{true}$$

Be sure to label each inference with the rule used. State what properties of addition and multiplication you need to complete the derivation (e.g., $2m+2 = 2(m+1)$), but you do not need to give derivations for these equalities.

4 Handin Instructions

- To run Tutch with the requirements files, run

  ```
  /afs/andrew/course/15/317/bin/tutch -r hw05.req <your file>
  ```

  This uses the requirements file `/afs/andrew/course/15/317/req/hw05.req`.

- To submit your Tutch proofs, run

  ```
  /afs/andrew/course/15/317/bin/submit -r hw05.req <your file>
  ```

  To check the status of your submission, run `/afs/andrew/course/15/317/bin/status hw05`.

- Submit your written work at the beginning of class, or, if you wish to do an electronic handin, copy a PDF to

  ```
  /afs/andrew/course/15/317/submit/<yourid>/hw05.pdf
  ```