Assignment 2:
Proof Terms

15-317: Constructive Logic

Out: Thursday, September 11, 2008
Due: Thursday, September 18, 2008, before class

1 Biconditional (15 pts)

In this problem, you will give a direct definition of "A iff B", which means "A implies B and B implies A".

Here is the intro rule:

$$\dfrac{\begin{array}{cc} \overline{A\ \mathsf{true}}^{\,u} & \overline{B\ \mathsf{true}}^{\,v} \\ \vdots & \vdots \\ B\ \mathsf{true} & A\ \mathsf{true} \end{array}}{A \equiv B\ \mathsf{true}} \equiv I^{u,v}$$

Task 1 (3 pts). Give the elimination rule(s).

Solution.

$$\dfrac{A \equiv B\ \mathsf{true} \quad A\ \mathsf{true}}{B\ \mathsf{true}} \equiv E_L \qquad \dfrac{A \equiv B\ \mathsf{true} \quad B\ \mathsf{true}}{A\ \mathsf{true}} \equiv E_R$$

It is important that you don't mention $\supset$ in these rules, as that violates orthogonality of the connectives.

Task 2 (4 pts). Annotate the introduction and elimination rules with proof terms. (See Lecture 4 for examples.)

Solution.

$$\dfrac{\begin{array}{cc} \overline{u : A\ \mathsf{true}} & \overline{v : B\ \mathsf{true}} \\ \vdots & \vdots \\ M : B\ \mathsf{true} & N : A\ \mathsf{true} \end{array}}{\mathsf{iffi}(u.M, v.N) : A \equiv B\ \mathsf{true}}$$

$$\dfrac{M : A \equiv B\ \mathsf{true} \quad N : A\ \mathsf{true}}{\mathsf{iffel}(M, N) : B\ \mathsf{true}} \qquad \dfrac{M : A \equiv B\ \mathsf{true} \quad N : B\ \mathsf{true}}{\mathsf{iffer}(M, N) : A\ \mathsf{true}}$$

Task 3 (4 pts). Using these proof terms, give the local reduction $\Rightarrow_R$ and local expansion $\Rightarrow_E$ rules. (See Lecture 5 for examples.)

Solution.

$$
\begin{array}{rcl}
\mathsf{iffel}(\mathsf{iffi}(u.M, v.N), P) & \Rightarrow_R & [P/u]M \\
\mathsf{iffer}(\mathsf{iffi}(u.M, v.N), P) & \Rightarrow_R & [P/u]N \\
M : A \equiv B & \Rightarrow_E & \mathsf{iffi}(u.\mathsf{iffel}(M, u), v.\mathsf{iffer}(M, v))
\end{array}
$$

Task 4 (4 pts). Show the cases of the subject reduction/expansion theorems for these rules. (See Lecture 5 for examples.)

Recall that these theorems are stated as follows:

- Reduction: If $\Gamma \vdash M : A$ and $M \Rightarrow_R M'$ then $\Gamma \vdash M' : A$.
- Expansion: If $\Gamma \vdash M : A$ and $M \Rightarrow_E M'$ then $\Gamma \vdash M' : A$.

You may use the following substitution lemma:

If $\Gamma, x : A \vdash M : B$ and $\Gamma \vdash N : A$ then $\Gamma \vdash [N/x]M : B$

Solution.

Case:

$$\mathsf{iffel}(\mathsf{iffi}(u.M, v.N), P) \Rightarrow_R [P/u]M$$

By assumption $\Gamma \vdash \mathsf{iffel}(\mathsf{iffi}(u.M, v.N), P) : B$. By inversion, only the iffel rule can have applied, so $\Gamma \vdash \mathsf{iffi}(u.M, v.N) : A \equiv B$ and $\Gamma \vdash P : A$ for some A. By inversion again, only the iffi rule can have applied, so $\Gamma, u : A \vdash M : B$ and $\Gamma, v : B \vdash N : A$. By the substitution lemma, $\Gamma \vdash [P/u]M : B$.

Case:

$$\mathsf{iffer}(\mathsf{iffi}(u.M, v.N), P) \Rightarrow_R [P/v]N$$

By assumption $\Gamma \vdash \mathsf{iffel}(\mathsf{iffi}(u.M, v.N), P) : A$. By inversion, only the iffel rule can have applied, so $\Gamma \vdash \mathsf{iffi}(u.M, v.N) : A \equiv B$ and $\Gamma \vdash P : B$ for some B. By inversion again, only the iffi rule can have applied, so $\Gamma, u : A \vdash M : B$ and $\Gamma, v : B \vdash N : A$. By the substitution lemma, $\Gamma \vdash [P/v]N : B$.

Case:

$$M : A \equiv B \Rightarrow_E \mathsf{iffi}(u.\mathsf{iffel}(M, u), v.\mathsf{iffer}(M, v))$$

By assumption, $\Gamma \vdash M : A \equiv B$. By weakening (you can always add extra unused assumptions to Γ), $\Gamma, u : A \vdash M : A \equiv B$ and $\Gamma, v : B \vdash M : A \equiv B$.

Thus, we can apply rules as follows to get what we need to show:

$$
\dfrac{
\dfrac{\dfrac{\vdots}{\Gamma, u : A \vdash M : A \equiv B} \quad \dfrac{}{\Gamma, u : A \vdash u : A}}{\Gamma, u : A \vdash \mathsf{iffel}(M, u) : B}
\quad
\dfrac{\dfrac{\vdots}{\Gamma, v : B \vdash M : A \equiv B} \quad \dfrac{}{\Gamma, v : B \vdash v : B}}{\Gamma, u : B \vdash \mathsf{iffer}(M, v) : A}
}{\Gamma \vdash \mathsf{iffi}(u.\mathsf{iffel}(M, u), v.\mathsf{iffer}(M, v)) : A \equiv B}
$$