

Assignment 1: Grammars and Induction

15-312: Foundations of Programming Languages
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Out: Thursday, August 29, 2002
Due: Thursday, September 5, 2002 (1:30 pm)

50 points total

Welcome to 15-312! This assignment focuses on context-free grammars and inductive proofs. It is due September 5th at the start of lecture. You are encouraged, but not required, to typeset your answers; if you write by hand, write legibly. If I can't read it, I can't give credit for it.

Please make sure you understand the policy on collaboration; refer to

<http://www.cs.cmu.edu/~fp/courses/312/assignments.html>

1 Grammars (35 points)

Consider the grammar G (which should look familiar) over the alphabet $\Sigma = \{\text{int}, \text{list}, \rightarrow, (,)\}$, with nonterminals tycon and type :

$$\begin{aligned}\text{tycon} &::= \text{int} \mid \text{tycon list} \mid (\text{type}) \\ \text{type} &::= \text{tycon} \mid \text{type} \rightarrow \text{type}\end{aligned}$$

Question 1.1 (5 points).

The first production for tycon can be written in rule notation as

$$\overline{\text{int tycon}}$$

Write grammar G in rule notation.

Grammar G is flawed: it does not pin down the associativity of \rightarrow . For example, there are two different derivations of the string $\text{int} \rightarrow \text{int} \rightarrow \text{int}$. We can fix the ambiguity by changing the second production of type from

$$\text{type} ::= \text{type} \rightarrow \text{type}$$

to

$$\text{type} ::= \text{tycon} \rightarrow \text{type}$$

Making this change results in the grammar G' below. To avoid confusion, we rename tycon to tycon' and type to type' .

$$\begin{aligned}\text{tycon}' &::= \text{int} \mid \text{tycon}' \text{ list} \mid (\text{type}') \\ \text{type}' &::= \text{tycon}' \mid \text{tycon}' \rightarrow \text{type}'\end{aligned}$$

Question 1.2 (5 points).

Write grammar G' in rule notation.

However, we haven't proved that this new grammar G' really is equivalent to G , that is, that the languages of type and type' are the same: $L(\text{type}) = L(\text{type}')$. This can be proved in two steps: first prove $L(\text{type}') \subseteq L(\text{type})$, then prove $L(\text{type}) \subseteq L(\text{type}')$.

Question 1.3 (10 points).

Prove $L(\text{type}') \subseteq L(\text{type})$ by proving

If $s \text{ type}'$ then $s \text{ type}$

by induction. If you need to generalize the induction hypothesis, be sure to clearly state your generalized induction hypothesis. If you need any lemmas, state them explicitly and prove them.

Question 1.4 (15 points).

Prove $L(\text{type}) \subseteq L(\text{type}')$ by proving

If $s \text{ type}$ then $s \text{ type}'$

As in the previous question, clearly state any generalized induction hypothesis and prove any lemmas you need.

2 Propositional logic (15 points)

In this question we will look at a subset of *Propositional Logic*. Our universe of terms consists of an infinite number of arity 0 operators P_0, P_1, \dots, P_n (“propositional variables”), and the binary operator \Rightarrow (“implication”). We define the sets `prop` and `thm` over this universe:

$$\begin{array}{c} \frac{}{P_i \text{ prop}} \text{Var} \qquad \frac{A \text{ prop} \quad B \text{ prop}}{A \Rightarrow (B \Rightarrow A) \text{ thm}} K \\[10pt] \frac{A \text{ prop} \quad B \text{ prop}}{A \Rightarrow B \text{ prop}} \text{Imp} \qquad \frac{A \text{ prop} \quad B \text{ prop} \quad C \text{ prop}}{(A \Rightarrow (B \Rightarrow C)) \Rightarrow (A \Rightarrow B) \Rightarrow (A \Rightarrow C) \text{ thm}} S \\[10pt] \frac{A \Rightarrow B \text{ thm} \quad A \text{ thm}}{B \text{ thm}} \text{App} \end{array}$$

Truth Value. If we have assignments (to true or false) for all of the propositional variables in a proposition, its truth value (either true or false) can be computed recursively using the following familiar truth table for \Rightarrow :

Proposition	Truth Value
false \Rightarrow false	true
false \Rightarrow true	true
true \Rightarrow false	false
true \Rightarrow true	true

Tautology. A proposition is a tautology iff for every assignment of truth values (true, false) to the propositional variables P_0, \dots, P_n , the truth value of the proposition is true.

Question 2.1 (15 points).

Prove, using rule induction, that if $A \text{ thm}$ then A is a tautology.

Question 2.2 (EXTRA CREDIT).

Find a proposition A that is a tautology, but not a theorem (that is, the judgment $A \text{ thm}$ cannot be derived). You do not need to prove that it is not a theorem!