Substructural Typestates
(Technical Appendix)

Filipe Militão
Carnegie Mellon University &
Universidade Nova de Lisboa
filipe.militao@cs.cmu.edu

Jonathan Aldrich
Carnegie Mellon University
jonathan.aldrich@cs.cmu.edu

Luís Caires
Universidade Nova de Lisboa
luis.caires@di.fct.unl.pt

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A. Abbreviations

We define a few convenient abbreviations that were used in the examples and can be encoded into the core language without the need for additional constructs.

- Let "compressed" expressions are encoded as:

  $$!e \ \triangleq\ \mathsf{let}\ x = e\ \mathsf{in}\ !x\ \mathsf{end}$$

  and similar to the remaining constructs (such as assignment, open, etc).

- **Sequence:** $e_0; e_1 \triangleq \mathsf{let}\ x = e_0\ \mathsf{in}\ e_1\ \mathsf{end}$
  Where x is not free in e_1 and e_0. Note that e_0 must have a pure type since the value will be discarded. However, it can have capabilities stacked on top of it since those get automatically threaded to e_1.

- **Unlabeled pairs (and their generalization, tuples)**, which can be encoded as function and application.
  Our core language only includes *choice* labeled products, so that the programmer must pick one (and only one) of a set of available fields — all of which produce the same effect in the Δ environment. The alternative would be to require *all* fields to be used, so that it is a linear labeled pair (where the order matters).

  We now show how unlabeled pairs can be encoded in the language, and leave out the generalization for arbitrary (but fixed length) tuples since it is straightforward.

  $$\frac{\Gamma;\Delta_0 \vdash e_0 : A_0 \dashv \Delta_1 \qquad \Gamma;\Delta_1 \vdash e_1 : A_1 \dashv \Delta_2}{\Gamma;\Delta_0 \vdash \{e_0, e_1\} : [A_0, A_1] \dashv \Delta_2}$$

  $$\frac{\begin{array}{c}\Gamma;\Delta_0 \vdash e_0 : [A_0, A_1] \dashv \Delta_1\\ \Gamma;\Delta_1, x_0 : A_0, x_1 : A_1 \vdash e_1 : A_2 \dashv \Delta_2\end{array}}{\Gamma;\Delta_0 \vdash \mathsf{let}\ [x_0, x_1] = e_0\ \mathsf{in}\ e_1\ \mathsf{end} : A_2 \dashv \Delta_2}\ (x_0, x_1 \text{ fresh in conclusion})$$

  Can be encoded as:

  $$\begin{array}{rl} \{e_0, e_1\} \triangleq & \mathsf{let}\ x_0 = e_0\ \mathsf{in} \\ & \ \mathsf{let}\ x_1 = e_1\ \mathsf{in} \\ & \quad < R > \mathsf{fun}(\mathtt{f} : A_0 \multimap A_1 \multimap R).(\ \mathtt{f}\ x_0\ x_1\ ) \\ & \ \mathsf{end} \\ & \mathsf{end} \end{array}$$

  $$\begin{array}{rl} \mathsf{let}\ [x_0, x_1] = e_p\ \mathsf{in}\ e_f\ \mathsf{end} \triangleq & \mathsf{let}\ p\ =\ e_p\ \mathsf{in} \\ & \ p[A_f]\ (\ \mathsf{fun}(x_0 : A_0).\mathsf{fun}(x_1 : A_1).e_f\ ) \\ & \mathsf{end} \end{array}$$

  where $e_0 : A_0$, $e_1 : A_1$ and $e_f : A_f$.

- **Recursion:** We use the traditional *call-by-value Y-combinator* encoded in our core language to provide recursion without using additional typing rules or reductions.

  Note that using a special construct, such as "$\mathsf{rec}\ x.e$" would require changing the *subtitution lemma* since if rec is a value, then no further reductions can occur, and if it is not a value, then the *substitution lemma* must account for expressions, not just values.

```
1 let fix = <A><B>fun( f : !( ( A ⊸ B ) ⊸ ( A ⊸ B ) ) ).
2   let r = fun( x : rec X.!( X ⊸ ( A ⊸ B ) ) ).
3       f ( fun( v : A ).(x(x))(v) ) in
4     r r
5   end : !∀A.∀B.( !(( A ⊸ B ) ⊸ ( A ⊸ B )) ⊸ ( A ⊸ B ) )
```

Noting that x in line 3 has types:

$$\begin{array}{ll} !(\, \mathbf{rec}\ X.!(\, X \multimap (\, A \multimap B\,)\,) \multimap (\, A \multimap B\,)\,) & \text{(function)} \\ \mathbf{rec}\ X.!(\, X \multimap (\, A \multimap B\,)\,) & \text{(argument)} \end{array}$$

Making the argument of `f` in that line to be $A \multimap B$, and `r` to be of type:

$$\mathbf{rec}\ X.!(\, X \multimap (\, A \multimap B\,)\,) \multimap (\, A \multimap B\,)$$

which, applied to itself, yields the type of the function without the recursive argument visible ($A \multimap B$).

Therefore, to use recursion, we must create a function that takes the recursive function as argument as shown in the literature.

In the examples, we make use of the construct "rec $x.e$" to define a recursive function (with body e), without having to use the expanded notation, and that automatically threads all location variables through its argument(s).

- **Shorter delete rule**.
  In the examples, we use a shorter (and more limited) delete typing rule to avoid having to carry existential types around.

  $$\text{delete}_{\text{examples}}\ x \triangleq \quad \text{open}\ < t_y, y >= (\,\text{delete}\ < t_x, x >\ )\ \text{in}\ y\ \text{end}$$

  where $x : \mathbf{ref}\ t_x$ and $y : A$ where t_y does not occur in A (and therefore does not need to be packed to leave that scope).

  Similar functionality could be achieved with the following typing rule (that is used in the prototype):

  $$\frac{\Gamma; \Delta_0 \vdash e : \mathbf{ref}\ p \dashv \Delta_1, \mathbf{rw}\ p\ A}{\Gamma; \Delta_0 \vdash \text{delete}\ e : A \dashv \Delta_1}\ \text{(t:Delete-Prototype)}$$

- **Girards' encoding of existential types**.
  However, this abbreviation is not used since it makes the use of existential types slightly more complex and less clear. Nonetheless, we leave it here as an observation on how it could be achieved.

  An existential type can be encoded into an universal type by consider the packed type to be hidden inside an universally quantified function that is not directly usable to client:

  $$\exists X.A \triangleq \forall \mathbf{R}.(\, \forall X.(\, A \multimap \mathbf{R}\,) \multimap \mathbf{R}\,)$$

  where $\mathbf{R}$ is the result of the expression that *uses* the packed existential and where X cannot occur in $\mathbf{R}$.

  Pack if we have:

  $$\langle A_0, e \rangle : \exists X.A_1$$

  then it can be encoded as:

  $$\langle \mathbf{R} \rangle (\, \text{fun}(\, x : \forall X.(\, A_1 \multimap \mathbf{R}\,)\,).(x[A_0](e))\,)$$

  so that it is a polymorphic function on $\mathbf{R}$, i.e. the result of *opening* the packed existential.

  Open if we have:

  $$\text{open}\ \langle X, x \rangle = e_0\ \text{in}\ e_1\ \text{end} : A_1$$

  where $e_0 : \exists X.A_0$, then it can be encoded as:

  $$e_0[A_1](\ \langle X \rangle (\, \text{fun}(\, x : A_0\,).e_1\,)\,)$$

  provided the resulting types A_0, A_1 are known.

It works in identical ways to abstract *locations*:

$$\exists t.A \triangleq \forall \mathbf{R}.(\, \forall t.(\, A \multimap \mathbf{R}\,) \multimap \mathbf{R}\,)$$

but the result must always be a type (not a location) since that is the type of the expression that is used on the *open*/*pack* constructs.

B. Proofs

B.1 Well-Formed Types and Environments

Our well-formed definition ensures that types are properly formed (i.e. type formation), be it in the environments or just in a regular type. Therefore, each type must have all the location variables it depends on declared in the corresponding Γ environment so that all *location variables* must be known in the same scope as the capability that refers a certain *location variable*. An analogous condition must hold for *type variables*.

Definition 1 (Well-Formed)**.** We have the following cases (defined by induction on the structure of the type/environment):

- $\boxed{\Gamma\ \mathbf{wf}}$ **(Gamma)**

$$\frac{}{\cdot\ \mathbf{wf}} \qquad \frac{\Gamma\ \mathbf{wf}}{\Gamma, p : \mathbf{loc}\ \mathbf{wf}} \qquad \frac{\Gamma\ \mathbf{wf}}{\Gamma, X : \mathbf{type}\ \mathbf{wf}} \qquad \frac{\Gamma\ \mathbf{wf} \qquad \Gamma \vdash A\ \mathbf{type}}{\Gamma, x : A\ \mathbf{wf}}$$

- $\boxed{\Gamma \vdash \Delta\ \mathbf{wf}}$ **(Delta)**

$$\frac{}{\Gamma \vdash \cdot\ \mathbf{wf}} \qquad \frac{\Gamma \vdash \Delta\ \mathbf{wf} \qquad \Gamma \vdash A\ \mathbf{type}}{\Gamma \vdash \Delta, x : A\ \mathbf{wf}}$$

$$\frac{\Gamma \vdash \Delta\ \mathbf{wf} \qquad \Gamma \vdash A\ \mathbf{type}}{\Gamma \vdash \Delta, A\ \mathbf{wf}}$$

- $\boxed{\Gamma \vdash A\ \mathbf{type}}$ **(Type)**

$$\frac{\Gamma \vdash A\ \mathbf{type}}{\Gamma \vdash !A\ \mathbf{type}} \qquad \frac{\overline{\Gamma \vdash A_i\ \mathbf{type}}}{\Gamma \vdash [\overline{\mathtt{f} : A}]\ \mathbf{type}}$$

$$\frac{\Gamma \vdash A_0\ \mathbf{type} \qquad \Gamma \vdash A_1\ \mathbf{type}}{\Gamma \vdash (A_0 \multimap A_1)\ \mathbf{type}} \qquad \frac{p : \mathbf{loc} \in \Gamma \qquad \Gamma \vdash A\ \mathbf{type}}{\Gamma \vdash (\mathbf{rw}\ p\ A)\ \mathbf{type}}$$

$$\frac{}{\Gamma, p : \mathbf{loc} \vdash (\mathbf{ref}\ p)\ \mathbf{type}} \qquad \frac{}{\Gamma, X\ \mathbf{type} \vdash X\ \mathbf{type}}$$

$$\frac{\Gamma \vdash A_0\ \mathbf{type} \qquad \Gamma \vdash A_1\ \mathbf{type}}{\Gamma \vdash (A_0 :: A_1)\ \mathbf{type}} \qquad \frac{\Gamma \vdash A_0\ \mathbf{type} \qquad \Gamma \vdash A_1\ \mathbf{type}}{\Gamma \vdash (A_0 * A_1)\ \mathbf{type}}$$

$$\frac{\Gamma, t : \mathbf{loc} \vdash A\ \mathbf{type}}{\Gamma \vdash \forall t.A\ \mathbf{type}} \qquad \frac{\Gamma, t : \mathbf{loc} \vdash A\ \mathbf{type}}{\Gamma \vdash \exists t.A\ \mathbf{type}}$$

$$\frac{\Gamma, X\ \mathbf{type} \vdash A\ \mathbf{type}}{\Gamma \vdash \forall X.A\ \mathbf{type}} \qquad \frac{\Gamma, X\ \mathbf{type} \vdash A\ \mathbf{type}}{\Gamma \vdash \exists X.A\ \mathbf{type}}$$

$$\frac{}{\Gamma \vdash \textsf{\textbf{none}}\ \mathbf{type}} \qquad \frac{\Gamma \vdash A_0\ \mathbf{type} \qquad \Gamma \vdash A_1\ \mathbf{type}}{\Gamma \vdash A_0 \oplus A_1\ \mathbf{type}}$$

$$\frac{\Gamma, X\ \mathbf{type} \vdash A\ \mathbf{type}}{\Gamma \vdash \mathbf{rec}\ X.A\ \mathbf{type}} \qquad \frac{\overline{\Gamma \vdash A_i\ \mathbf{type}}}{\Gamma \vdash \sum_i \mathtt{l}_i\#A_i\ \mathbf{type}}$$

Note that well-formed conditions are not explicitly mentioned and are assumed to be present whenever they are relevant.

B.2 Subtyping Inversion Lemma

Lemma 1 (Subtyping Inversion Lemma)**.** We have the following cases for *types* (A) and for the *linear typing environment* (Δ):

- (Type) If $A <: A'$ then one of the following holds:
  1. $A' = A$.
  2. if $A = !A_0$ then either:
     (a) $A' = A_0$, or;
     (b) $A' = !A_1$ and $A_0 <: A_1$, or;
     (c) $A' = ![]$.
  3. if $A = A_0 \multimap A_1$ then $A' = A_2 \multimap A_3$ and $A_1 <: A_3$ and $A_2 <: A_0$.
  4. if $A = A_0 :: A_2$ then $A' = A_1 :: A_3$ and $A_0 <: A_1$ and $A_2 <: A_3$.
  5. if $A = [\overline{\mathtt{f} : A}]$ then either:
     (a) $A = [\overline{\mathtt{f} : A}, \mathtt{f}_i : A_i]$ and $A' = [\overline{\mathtt{f} : A}]$ and $i > 0$.
     (b) $A = [\overline{\mathtt{f} : A}, \mathtt{f}_i : A_0]$ and $A' = [\overline{\mathtt{f} : A}, \mathtt{f}_i : A_1]$ and $A_0 <: A_1$.
     (c) $A = [\overline{\mathtt{f} : !A}]$ and $A' =![\overline{\mathtt{f} : !A}]$.
  6. if $A = \mathbf{rw}\ p\ A_0$ then $A' = \mathbf{rw}\ p\ A_1$ and $A_0 <: A_1$.
  7. if $A = \exists t.A_0$ then $A' = \exists t.A_1$ and $A_0 <: A_1$.
  8. if $A = \forall t.A_0$ then $A' = \forall t.A_1$ and $A_0 <: A_1$.
  9. if $A = \exists X.A_0$ then $A' = \exists X.A_1$ and $A_0 <: A_1$.
  10. if $A = \forall X.A_0$ then $A' = \forall X.A_1$ and $A_0 <: A_1$.
  11. if $A = \mathbf{ref}\ p$ then $A' = !(\mathbf{ref}\ p)$.
  12. if $A = A_0 * A_1$ then either:
      (a) $A' = A_1 * A_0$, or;
      (b) $A' = A_0 * A_2$ and $A_1 <: A_2$.
      (c) if $A_0 = (A_0' * A_0'')$ then $A' = A_0' * (A_0'' * A_1)$.
  13. if $A = \sum_i \mathtt{l}_i\#A_i$ then $A' = \mathtt{l}'\#A' + \sum_i \mathtt{l}_i\#A_i$.
  14. if $A = A_0\{X/\mathbf{rec}\ X.A_0\}$ then $A' = \mathbf{rec}\ X.A_0$.
  15. if $A = \mathbf{rec}\ X.A_0$ the either:
      (a) $A' = \mathbf{rec}\ X.A_1$ and $A_0 <: A_1$, or;
      (b) $A' = A_1\{X/\mathbf{rec}\ X.A_1\}$.
- (Delta) If $\Delta <: \Delta'$ then one of the following holds:
  1. $\Delta = \Delta'$.
  2. if $\Delta = \Delta_0, x : A_0$ then $\Delta' = \Delta_1, x : A_1$ and $\Delta_0 <: \Delta_1$ and $A_0 <: A_1$.
  3. if $\Delta = \Delta_0, A_0$ then either:
     (a) $\Delta' = \Delta_1, A_1$ and $\Delta_0 <: \Delta_1$ and $A_0 <: A_1$.
     (b) $\Delta' = \Delta_0, A_0 \oplus A_1$.
  4. if $\Delta = \Delta_0, A_0, A_1$ then either:
     (a) $\Delta' = \Delta_0, A_0 * A_1$, or;
     (b) case (3) with A_0, or;
     (c) case (3) with A_1.
  5. if $\Delta = \Delta_0, A_0 * A_1$ then $\Delta' = \Delta_0, A_0, A_1$.
  6. if $\Delta = \Delta_0, \textsf{\textbf{none}}$ then $\Delta' = \Delta_0$.
  7. $\Delta' = \Delta, \textsf{\textbf{none}}$.
  8. if $\Delta = \Delta_0, A_0 \oplus A_1$ then $\Delta_0, A_0 <: \Delta'$ and $\Delta_0, A_1 <: \Delta'$.

Proof. We only very informally sketch the proof, without going into detail on each case since they are straightforward to show.

1. (Type) By induction on the derivation of $A <: A'$.
   Case (st:Symmetry) Case 1 of the definition.
   Case (st:ToLinear) Case 2 (a) of the definition.
   Case (st:Pure) Case 2 (b) of the definition.
   Case (st:Top) Case 2 (c) of the definition.
   Case (st:Ref) Case 11 of the definition.
   Case (st:Function) Case 3 of the definition.
   Case (st:Loc-Exists) Case 7 of the definition.
   Case (st:Loc-Forall) Case 8 of the definition.
   Case (st:Type-Exists) Case 9 of the definition.
   Case (st:Type-Forall) Case 10 of the definition.
   Case (st:Record) Case 5 (b) of the definition.
   Case (st:Discard) Case 5 (a) of the definition.
   Case (st:PurifyRec) Case 5 (c) of the definition.
   Case (st:Stack) Case 4 of the definition.
   Case (st:Cap) Case 6 of the definition.
   Case (st:Com) Case 12 (a) of the definition.
   Case (st:Cong) Case 12 (b) of the definition.

Case (st:Assoc) Case 12 (c) of the definition.
Case (st:Sum) Case 13 of the definition.
Case (st:Fold) Case 14 of the definition.
Case (st:Unfold) Case 15 (a) of the definition.
Case (st:Rec) Case 15 (b) of the definition.

2. (Delta) By induction on the derivation of $\Delta <: \Delta'$.
Case (sd:Symmetry) - Case 1 of the definition.
Case (sd:Var) - Case 2 of the definition.
Case (sd:Type) - Case 3 (a), 4 (b) and 4 (c) of the definition.
Case (sd:Star), right - Case 4 of the definition.
Case (sd:Star), left - Case 5 of the definition.
Case (sd:None) - Cases 7 (for <:, right) and 6 (for :>, left) of the definition.
Case (sd:Alternative-R) - Case 3 (b) of the definition.
Case (sd:Alternative-L) - Case 8 of the definition.

□

B.3 Store Typing

We use the notation $\widehat{\Gamma}$ to mean that Γ is closed in the sense of only containing (ρ : **loc**) elements and nothing else. Therefore, it only lists the known location *constants*. Similarly, we use $\widehat{\Delta}$ to mean that Δ is closed, so that it only includes capabilities (of the form: **rw** $\rho\ A$ — note the location constant ρ). There is no inconsistency with the notation of A since if such type can only depend on closed environments (in order to be well-formed), then it too must be closed or it would not be well-formed.

Definition 2 (Store Typing)**.**

$$\text{(str:Empty)}\ \frac{}{\cdot;\cdot\vdash\cdot}\qquad \text{(str:Loc)}\ \frac{\widehat{\Gamma};\widehat{\Delta}\vdash H}{\widehat{\Gamma},\rho:\mathbf{loc};\widehat{\Delta}\vdash H}\qquad \text{(str:Star)}\ \frac{\widehat{\Gamma};\widehat{\Delta},A_0,A_1\vdash H}{\widehat{\Gamma};\widehat{\Delta},A_0 * A_1\vdash H}$$

$$\text{(str:None)}\ \frac{\widehat{\Gamma};\widehat{\Delta}\vdash H}{\widehat{\Gamma};\widehat{\Delta},\textsf{none}\vdash H}\qquad \text{(str:Alternative)}\ \frac{\widehat{\Gamma};\widehat{\Delta},A_0\vdash H}{\widehat{\Gamma};\widehat{\Delta},A_0\oplus A_1\vdash H}$$

$$\text{(str:Binding)}\ \frac{\widehat{\Gamma};\widehat{\Delta},\widehat{\Delta_v}\vdash H\qquad \widehat{\Gamma};\widehat{\Delta_v}\vdash v:A\dashv\cdot}{\widehat{\Gamma};\widehat{\Delta},\mathbf{rw}\ \rho\ A\vdash H,\rho\hookrightarrow v}$$

Note that, since the added capability on (str:Binding) must still be well-formed, such implies that $\widehat{\Gamma}$ must contain ρ. For the same reason, ρ must also *not* appear in $\widehat{\Delta}$ or H. On (str:Alternative), we only need one rule because such type is assumed to be commutative.

Lemma 2 (Store Typing Inversion Lemma)**.** If

$$\widehat{\Gamma};\widehat{\Delta}\vdash H$$

then one of the following holds:

1. $\widehat{\Gamma}=\cdot$ and $\widehat{\Delta}=\cdot$ and $H=\cdot$.
2. if $\widehat{\Gamma}=\widehat{\Gamma'},\rho:\mathbf{loc}$ then $\widehat{\Gamma'};\widehat{\Delta}\vdash H$.
3. if $\widehat{\Delta}=\widehat{\Delta'},A_0 * A_1$ then $\widehat{\Gamma};\widehat{\Delta'},A_0,A_1\vdash H$.
4. if $\widehat{\Delta}=\widehat{\Delta'},\mathbf{rw}\ \rho\ A$ and $H=H',\rho\hookrightarrow v$ then $\widehat{\Gamma};\widehat{\Delta'},\widehat{\Delta_v}\vdash H'$ and $\widehat{\Gamma};\widehat{\Delta_v}\vdash v:A\dashv\cdot$.
5. if $\widehat{\Delta}=\widehat{\Delta'},\textsf{none}$ then $\widehat{\Gamma};\widehat{\Delta'}\vdash H$.
6. if $\widehat{\Delta}=\widehat{\Delta'},A_0\oplus A_1$ then either:
   - $\widehat{\Gamma};\widehat{\Delta},A_0\vdash H$, or;
   - $\widehat{\Gamma};\widehat{\Delta},A_1\vdash H$.

   (note that $\oplus$ is commutative)

Proof. Straightforward induction on the derivation of $\widehat{\Gamma};\widehat{\Delta}\vdash H$. □

Lemma 3 (Subtyping Store Typing)**.** If $\widehat{\Gamma};\widehat{\Delta} \vdash H$ and $\widehat{\Delta} <: \widehat{\Delta'}$ then $\widehat{\Gamma};\widehat{\Delta'} \vdash H$.

Proof. By induction on the derivation of $\widehat{\Gamma};\widehat{\Delta} \vdash H$.

Case (str:Empty) We have:

$\cdot;\cdot \vdash \cdot$ (1)
$\cdot <: \widehat{\Delta'}$ (2)
by hypothesis
By (Subtyping Inversion Lemma) on (2), we have that either:
- [1] $\widehat{\Delta'} = \cdot$ (1.1)
Thus, we conclude by (1).
- [7] $\widehat{\Delta'} = \cdot, \textsf{\textbf{none}}$ (2.1)
$\cdot;\cdot, \textsf{\textbf{none}} \vdash \cdot$ (2.2)
by (str:None) on (1).
Thus, we conclude.

Case (str:Loc) We have:

$\widehat{\Gamma}, \rho : \textbf{loc}; \widehat{\Delta} \vdash H$ (1)
$\widehat{\Delta} <: \widehat{\Delta'}$ (2)
by hypothesis.
$\widehat{\Gamma};\widehat{\Delta} \vdash H$ (3)
by inversion on (str:Loc) with (1).
$\widehat{\Gamma};\widehat{\Delta'} \vdash H$ (4)
by induction hypothesis with (3) and (2).
$\widehat{\Gamma}, \rho : \textbf{loc}; \widehat{\Delta'} \vdash H$ (5)
by (str:Loc) with ρ and (4).
Thus, we conclude.

Case (str:Binding) We have:

$\widehat{\Gamma};\widehat{\Delta}, \textbf{rw}\,\rho\,A \vdash H, \rho \hookrightarrow v$ (1)
$\widehat{\Delta}, \textbf{rw}\,\rho\,A <: \widehat{\Delta'}$ (2)
by hypothesis.
$\widehat{\Gamma};\widehat{\Delta}, \widehat{\Delta_v} \vdash H$ (3)
$\widehat{\Gamma};\widehat{\Delta_v} \vdash v : A \dashv \cdot$ (4)
by inversion on (str:Binding) with (1).
By (Subtyping Inversion Lemma) on (2), we have that either:
- [1] $\widehat{\Delta'} = \widehat{\Delta}, \textbf{rw}\,\rho\,A$ (1.1)
by sub-case hypothesis.
Thus, we conclude by (1).
- [3(a)] $\widehat{\Delta'} = \widehat{\Delta_0}, A_0$ (2.1)
$\widehat{\Delta} <: \widehat{\Delta_0}$ (2.2)
$\textbf{rw}\,\rho\,A <: A_0$ (2.3)
by sub-case hypothesis.
$A_0 = \textbf{rw}\,\rho\,A_1$ (2.4)
$A <: A_1$ (2.5)
by (Subtyping Inversion Lemma) with (2.3).
(note the symmetric case is immediate, so we omit it).
$\widehat{\Gamma};\widehat{\Delta_v} \vdash v : A_1 \dashv \cdot$ (2.6)
by (t:Subsumption) on (4) with (2.5).
$\widehat{\Gamma};\widehat{\Delta_0}, \widehat{\Delta_v} \vdash H$ (2.7)
by induction hypothesis on (3) and (2.2) noting that Δ_v is unchanged.
$\widehat{\Gamma};\widehat{\Delta_0}, \textbf{rw}\,\rho\,A_1 \vdash H, \rho \hookrightarrow v$ (2.8)
by (str:Binding) with (2.6) and (2.7) with ρ.
$\widehat{\Gamma};\widehat{\Delta'} \vdash H, \rho \hookrightarrow v$ (2.9)
by rewriting (2.8) with (2.1) and (2.4).
Thus, we conclude.
- [3(b)] $\widehat{\Delta'} = \widehat{\Delta}, (\textbf{rw}\,\rho\,A) \oplus A_1$ (3.1)
by sub-case hypothesis.
$\widehat{\Gamma};\widehat{\Delta}, (\textbf{rw}\,\rho\,A) \oplus A_1 \vdash H, \rho \hookrightarrow v$ (3.2)
by (str:Alternative) on (1).
Thus, we conclude.
- [7] $\widehat{\Delta'} = \widehat{\Delta}, \textbf{rw}\,\rho\,A, \textsf{\textbf{none}}$ (4.1)
by sub-case hypothesis.
$\widehat{\Gamma};\widehat{\Delta}, \textbf{rw}\,\rho\,A, \textsf{\textbf{none}} \vdash H, \rho \hookrightarrow v$ (4.2)
by (str:None) on (1).
Thus, we conclude.

Case (str:Star) We have:

$\widehat{\Gamma};\widehat{\Delta}, A_0 * A_1 \vdash H$ (1)
$\widehat{\Delta}, A_0 * A_1 <: \widehat{\Delta'}$ (2)
by hypothesis.
$\widehat{\Gamma};\widehat{\Delta}, A_0, A_1 \vdash H$ (3)
by inversion on (str:Star) on (1).
by (Subtyping Inversion Lemma) on (2) we have that either:
- [1] $\widehat{\Delta'} = \widehat{\Delta}, A_0 * A_1$ (1.1)
Thus, we conclude by (1).
- [3(a)] $\widehat{\Delta'} = \widehat{\Delta''}, A$ and
$\widehat{\Delta} <: \widehat{\Delta''}$ (2.1)
$A_0 * A_1 <: A$ (2.2)
By (Subtyping Inversion Lemma) on (2.2) we have that either:
◇ [12(a)] $A = A_1 * A_0$
$\widehat{\Delta'} = \widehat{\Delta''}, A_1 * A_0$ (2.3)
by rewriting hypothesis.
$\widehat{\Delta''}, A_1 * A_0 <: \widehat{\Delta''}, A_1, A_0$ (2.4)
by (sd:Star) on (2.3).
$\widehat{\Gamma};\widehat{\Delta''}, A_0, A_1 \vdash H$ (2.5)
by induction hypothesis on (3) with (2.1).
$\widehat{\Gamma};\widehat{\Delta''}, A_1, A_0 \vdash H$ (2.6)
since Δ is a set, re-ordering is allowed.
Thus, we conclude by (2.6).
◇ [12(b)] $A = A_0 * A_2$ and $A_1 <: A_2$
$\widehat{\Delta'} = \widehat{\Delta}, A_0 * A_2$ (3.1)
by rewriting hypothesis.
$\widehat{\Delta}, A_0 * A_2 <: \widehat{\Delta}, A_0, A_2$ (3.2)
by (sd:Star) on (3.1).
$\widehat{\Gamma};\widehat{\Delta}, A_0, A_2 \vdash H$ (3.3)
by induction hypothesis on (3) with $A_1 <: A_2$.
Thus, we conclude.
◇ [12(c)] if $A_0 = A_0' * A_0''$ then $A = A_0' * (A_0'' * A_1)$
$\widehat{\Delta'} = \widehat{\Delta}, (A_0' * A_0'') * A_1$ (4.1)
$\widehat{\Gamma};\widehat{\Delta}, (A_0' * A_0''), A_1 \vdash H$ (4.2)
by rewriting hypothesis.
$\widehat{\Gamma};\widehat{\Delta}, A_1, (A_0' * A_0'') \vdash H$ (4.3)
since Δ is a set, re-ordering is allowed on (4.2).
$\widehat{\Gamma};\widehat{\Delta}, A_1, A_0', A_0'' \vdash H$ (4.4)
by (Store Typing Inversion Lemma) on (4.3).
$\widehat{\Gamma};\widehat{\Delta}, A_0', A_0'', A_1 \vdash H$ (4.5)
since Δ is a set, re-ordering is allowed on (4.4).
$\widehat{\Gamma};\widehat{\Delta}, A_0', (A_0'' * A_1) \vdash H$ (4.6)
by (str:Star) on (4.5).
$\widehat{\Gamma};\widehat{\Delta}, A_0' * (A_0'' * A_1) \vdash H$ (4.7)
by (str:Star) on (4.6).
Thus, we conclude.
- [3(b)] $\widehat{\Delta'} = \widehat{\Delta}, (A_0 * A_1) \oplus A_2$.
Thus, we conclude by (str:Alternative) on (1).
- [5] $\widehat{\Delta'} = \widehat{\Delta}, A_0, A_1$.
Thus, we conclude by (3).
- [7] $\widehat{\Delta'} = \widehat{\Delta}, \textsf{\textbf{none}}$.
Thus, we conclude by (str:None) on (1).

Case (str:None) We have:

$\widehat{\Gamma};\widehat{\Delta}, \textsf{\textbf{none}} \vdash H$ (1)
$\widehat{\Delta}, \textsf{\textbf{none}} <: \widehat{\Delta'}$ (2)
by hypothesis.
$\widehat{\Gamma};\widehat{\Delta} \vdash H$ (3)
by inversion on (str:Star) on (1).
By (Subtyping Inversion Lemma) on (2), we have that either:
- [1] $\widehat{\Delta'} = \widehat{\Delta}, \textsf{\textbf{none}}$
Thus, we conclude by (1).
- [6] $\widehat{\Delta'} = \widehat{\Delta}$
Thus, we conclude by (3).

Case (str:Alternative) We have:

$\widehat{\Gamma};\widehat{\Delta}, A_0 \oplus A_1 \vdash H$ (1)
$\widehat{\Delta}, A_0 \oplus A_1 <: \widehat{\Delta'}$ (2)
by hypothesis.
By (Subtyping Inversion Lemma) on (2), we have that either:
- [1] $\widehat{\Delta'} = \widehat{\Delta}, A_0 \oplus A_1$ (1.1)
Thus, we conclude by (1).
- [3(a)] $\widehat{\Delta'} = \widehat{\Delta_0}, A$ (2.1)
$\widehat{\Delta} <: \widehat{\Delta_0}$ (2.2)
$A_0 \oplus A_1 <: A$ (2.3)
by sub-case hypothesis.
$A = A_0 \oplus A_1$ (2.4)
by (Subtyping Inversion Lemma) on (2.3).
By inversion on (1) we have that either:
◇ $\widehat{\Gamma};\widehat{\Delta}, A_0 \vdash H$ (2.5)
$\widehat{\Delta}, A_0 <: \widehat{\Delta_0}, A_0$ (2.6)
by (sd:Type) on (2.2) and (st:Symmetry) with A_0.

$\widehat{\Gamma}; \widehat{\Delta_0}, A_0 \vdash H$ (2.7)

by induction hypothesis on (2.5) and (2.6).

$\widehat{\Gamma}; \widehat{\Delta_0}, A_0 \oplus A_1 \vdash H$ (2.8)

by (str:Alternative) on (2.7).

Thus, we conclude.

⋄ $\widehat{\Gamma}; \widehat{\Delta}, A_1 \vdash H$ (2.9)

Analogous to the previous case, noting that $\oplus$ is commutative.

• [3(*b*)] $\widehat{\Delta'} = \widehat{\Delta}, (A_0 \oplus A_1) \oplus A_2$ (3.1)

Thus, we conclude by (str:Alternative) on (1) with A_2.

• [7] $\widehat{\Delta'} = \widehat{\Delta}, A_0 \oplus A_1, \textbf{none}$ (4.1)

Thus, we conclude by (str:None) on (1).

• [8] $\widehat{\Delta}, A_0 <: \widehat{\Delta'}$ (5.1)

$\widehat{\Delta}, A_1 <: \widehat{\Delta'}$ (5.2)

By inversion on (1) we have that either:

⋄ $\widehat{\Gamma}; \widehat{\Delta}, A_0 \vdash H$ (5.3)

$\widehat{\Gamma}; \widehat{\Delta'} \vdash H$ (5.4)

by induction hypothesis on (5.1) and sub-case hypothesis.

⋄ $\widehat{\Gamma}; \widehat{\Delta}, A_1 \vdash H$ (5.5)

Analogous to the previous case, using (5.2).

□

B.4 Values Inversion Lemma

Lemma 4 (Values Inversion Lemma)**.** If v is a value such that:

$$\widehat{\Gamma}; \widehat{\Delta} \vdash v : A_0 \dashv \cdot$$

then one of the following holds:

1. if $A_0 = []$ then:
$$\widehat{\Delta} = \cdot \qquad \widehat{\Gamma}; \cdot \vdash v : [] \dashv \cdot$$
2. if $A_0 = !A_1$ then:
$$\widehat{\Delta} = \cdot \qquad \widehat{\Gamma}; \cdot \vdash v : A_1 \dashv \cdot$$
3. if $A_0 = A_1 :: A_2$ then:
$$\widehat{\Gamma}; \widehat{\Delta} \vdash v : A_1 \dashv A_2$$
4. if $A_0 = \textbf{ref}\,\rho$ then:
$$v = \rho \qquad \rho : \textbf{loc} \in \Gamma \qquad \widehat{\Delta} = \cdot$$
5. if $A_0 = A \multimap A'$ then:
$$A <: A'' \qquad v = \mathsf{fun}(x : A'').e \qquad \widehat{\Gamma}; \widehat{\Delta}, x : A'' \vdash e : A' \dashv \cdot$$
6. if $A_0 = \forall t.A$ then:
$$v = \langle t \rangle\, e \qquad \widehat{\Gamma}, t : \textbf{loc}; \widehat{\Delta} \vdash e : A \dashv \cdot$$
7. if $A_0 = \exists t.A$ then:
$$v = \langle p, v' \rangle \qquad \widehat{\Gamma}; \widehat{\Delta} \vdash v' : A\{p/t\} \dashv \cdot$$
8. if $A_0 = [\overline{\mathtt{f} : A}]$ then:
$$v = \{\overline{\mathtt{f} = v'}\} \qquad \overline{\widehat{\Gamma}; \widehat{\Delta} \vdash v'_i : \; A_i \dashv \cdot}$$
(Note that, although the record value can have more fields than those that are listed in the type, only the fields that are in the type will appear in the inversion.)
9. if $A_0 = \forall X.A$ then:
$$v = \langle X \rangle\, e \qquad \widehat{\Gamma}, X : \textbf{type}; \widehat{\Delta} \vdash e : A \dashv \cdot$$
10. if $A_0 = \exists X.A$ then:
$$v = \langle A', v' \rangle \qquad \widehat{\Gamma}; \widehat{\Delta} \vdash v' : A\{A'/X\} \dashv \cdot$$
11. if $A_0 = \sum_i \mathtt{l}_i\#A_i$ then:
$$v = \mathtt{l}_i\#v_i \qquad \widehat{\Gamma}; \widehat{\Delta} \vdash v_i : A_i \dashv \cdot$$
for some i.
12. if $A_0 = \textbf{rec}\ X.A$ then
$$\widehat{\Gamma}; \widehat{\Delta} \vdash v : A\{\textbf{rec}\ X.A/X\} \dashv \cdot$$
13. if $\Delta = \Delta', A_1 \oplus A_2$ then
$$\widehat{\Gamma}; \widehat{\Delta'}, A_1 \vdash v : A_0 \dashv \cdot \qquad \widehat{\Gamma}; \widehat{\Delta'}, A_2 \vdash v : A_0 \dashv \cdot$$

Proof. By induction on the derivation of $\widehat{\Gamma}; \widehat{\Delta} \vdash v : A_0 \dashv \cdot$.

Case (t:Ref) - We have:

$\widehat{\Gamma}, \rho : \textbf{loc}; \cdot \vdash \rho : \textbf{ref}\,\rho \dashv \cdot$ (1)

by hypothesis.

Thus, we conclude by case 4 of the definition.

Case (t:Pure) - We have:

$\widehat{\Gamma}; \cdot \vdash v : !A_1 \dashv \cdot$ (1)

by hypothesis.

$\widehat{\Gamma}; \cdot \vdash v : A_1 \dashv \cdot$ (2)

by inversion on (t:Pure).

Thus, we conclude by case 2 of the definition.

Case (t:Unit) - We have:

$\widehat{\Gamma}; \cdot \vdash v : [] \dashv \cdot$ (1)
by hypothesis.
Thus, we conclude by case 1 of the definition.

Case (t:Pure-Read), (t:Linear-Read), (t:Pure-Elim), (t:New) - Not applicable.
Case (t:Delete), (t:Assign), (t:Dereference-Linear), (t:Dereference-Pure) - Not applicable.
Case (t:Record) - We have:

$\widehat{\Gamma}; \widehat{\Delta} \vdash \{\overline{\mathtt{f} = v}\} : [\overline{\mathtt{f} : A}] \dashv \cdot$ (1)
by hypothesis.
$\overline{\widehat{\Gamma}; \widehat{\Delta} \vdash v_i : A_i \dashv \cdot}$ (2)
by inversion on (t:Record).
Thus, we conclude by case 8 of the definition.

Case (t:Selection), (t:Application) - Not applicable.
Case (t:Function) - We have:

$\widehat{\Gamma}; \widehat{\Delta} \vdash \mathsf{fun}(x : A_0).e : A_0 \multimap A_1 \dashv \cdot$ (1)
by hypothesis.
$\widehat{\Gamma}; \widehat{\Delta}, x : A_0 \vdash e : A_1 \dashv \cdot$ (2)
by inversion on (t:Function).
$A_0 <: A_0$ (3)
by (st:Symmetry) with A_0.
Thus, we conclude by case 5 of the definition.

Case (t:Cap-Elim) - Not applicable.
Case (t:Cap-Stack) - We have:

$\widehat{\Gamma}; \widehat{\Delta} \vdash v : A_0 :: A_1 \dashv \cdot$ (1)
by hypothesis.
$\widehat{\Gamma}; \widehat{\Delta} \vdash v : A_0 \dashv A_1$ (2)
by inversion on (t:Cap-Stack).
Thus, we conclude by case 3 of the definition.

Case (t:Cap-Unstack), (t:Application) - Not applicable.
Case (t:Forall-Loc) We have:

$\widehat{\Gamma}; \widehat{\Delta} \vdash \langle t \rangle\, e : \forall t.A \dashv \cdot$ (1)
by hypothesis.
$\widehat{\Gamma, t : \mathbf{loc}}; \widehat{\Delta} \vdash e : A \dashv \cdot$ (2)
by inversion on (t:Forall-Loc) with (1).
Thus, we conclude by case 6 of the definition.

Case (t:Loc-App) Not applicable.
Case (t:Loc-Pack) We have:

$\widehat{\Gamma}; \widehat{\Delta} \vdash \langle p, v \rangle : \exists t.A \dashv \cdot$ (1)
by hypothesis.
$\widehat{\Gamma}; \widehat{\Delta} \vdash v : A\{p/t\} \dashv \cdot$ (2)
by inversion on (t:Loc-Pack) with (1).
Thus, we conclude by case 7 of the definition.

Case (t:Loc-Open) Not applicable.
Case (t:Forall-Type) We have:

$\widehat{\Gamma}; \widehat{\Delta} \vdash \langle X \rangle\, e : \forall X.A \dashv \cdot$ (1)
by hypothesis.
$\widehat{\Gamma, X : \mathbf{type}}; \widehat{\Delta} \vdash e : A \dashv \cdot$ (2)
by inversion on (t:Forall-Loc) with (1).
Thus, we conclude by case 9 of the definition.

Case (t:Type-App) Not applicable.
Case (t:Type-Pack) We have:

$\widehat{\Gamma}; \widehat{\Delta} \vdash \langle A_0, v \rangle : \exists X.A_1 \dashv \cdot$ (1)
by hypothesis.
$\widehat{\Gamma}; \widehat{\Delta} \vdash v : A_1\{A_0/X\} \dashv \cdot$ (2)
by inversion on (t:Type-Pack) with (1).
Thus, we conclude by case 10 of the definition.

Case (t:Type-Open) Not applicable.
Case (t:Tag) We have:

$\widehat{\Gamma}; \widehat{\Delta} \vdash \mathtt{l}\#v : \mathtt{l}\#A \dashv \cdot$ (1)
by hypothesis.
$\widehat{\Gamma}; \widehat{\Delta} \vdash v : A \dashv \cdot$ (2)
by inversion on (t:Tag).
Thus, we conclude by case 11 of the definition.

Case (t:Case) Not applicable.
Case (t:Alternative-Left) We have:

$\widehat{\Gamma}; \widehat{\Delta}, A_0 \oplus A_1 \vdash v : A_2 \dashv \cdot$ (1)
by hypothesis.
$\widehat{\Gamma}; \widehat{\Delta}, A_0 \vdash v : A_2 \dashv \cdot$ (2)
$\widehat{\Gamma}; \widehat{\Delta}, A_1 \vdash v : A_2 \dashv \cdot$ (3)
by inversion on (t:Alternative-Left).
Thus, we conclude by case 13 of the definition.

Case (t:Frame) Not applicable, Δ environment on right is empty, otherwise direct application of induction hypothesis.
Case (t:Subsumption) We have:

$\widehat{\Gamma}; \widehat{\Delta} \vdash v : A_1 \dashv \cdot$ (1)
by hypothesis.
$\widehat{\Delta} <: \widehat{\Delta'}$ (2)
$\widehat{\Gamma}; \widehat{\Delta'} \vdash v : A_0 \dashv \cdot$ (3)
$A_0 <: A_1$ (4)
$\cdot <: \cdot$ (5)
by inversion on (t:Subsumption).
By induction hypothesis on (3) we have that one of the following holds:

1. if $A_0 = []$ then:
   $\widehat{\Delta'} = \cdot$ (1.1)
   $\widehat{\Gamma}; \cdot \vdash v : [] \dashv \cdot$ (1.2)
   $[] <: A_1$ (1.3)
   by case 1 of the hypothesis and rewriting (4).
   Then, by (Subtyping Inversion Lemma) on (1.3) we have that either:
   • [1] $A_1 = []$ (1.4)
   and we conclude as case 1 of the definition.
   • $[5(c)]$ $A_1 = ![]$ (1.5)
   and we conclude as case 2 of the definition.

2. if $A_0 = !A$ then:
   $\widehat{\Delta'} = \cdot$ (2.1)
   $\widehat{\Gamma}; \cdot \vdash v : A \dashv \cdot$ (2.2)
   $!A <: A_1$ (2.3)
   by case 2 of the hypothesis and rewriting (4).
   by (Subtyping Inversion Lemma) on (2.3) we have that either:
   • [1] $A_1 = !A$
   Thus, we conclude by case 2 of the definition through (2.2).
   • $[2(a)]$ $A_1 = A$
   Thus, we conclude by induction hypothesis on (2.2).
   • $[2(b)]$ $A_1 = !A'$ and $A <: A'$
   $\widehat{\Gamma}; \cdot \vdash v : A' \dashv \cdot$ (2.4)
   by (t:Subsumption) on (2.2) with $A <: A'$.
   Thus, we conclude by case 2 of the definition with (2.4).
   • $[2(c)]$ $A_1 = ![]$
   $\widehat{\Gamma}; \cdot \vdash v : [] \dashv \cdot$ (2.5)
   by (t:Unit) on v.
   Thus, we conclude by case 2 of the definition.

3. if $A_0 = A \multimap A'$ then:
   $v = \mathsf{fun}(x : A).e$ (3.1)
   $\widehat{\Gamma}; \widehat{\Delta'}, x : A \vdash e : A' \dashv \cdot$ (3.2)
   $A \multimap A' <: A_1$ (3.3)
   by case 5 of the hypothesis and rewriting (4).
   by (Subtyping Inversion Lemma) on (3.3) we have that:
   (note: we omit the case $A_1 = A_0$, since it is immediate)
   $A_1 = A'' \multimap A'''$ (3.4)
   $A' <: A'''$ (3.5)
   $A'' <: A$ (3.6)
   $\widehat{\Gamma}; \widehat{\Delta'}, x : A \vdash e : A''' \dashv \cdot$ (3.7)
   by (t:Subsumption) on (3.2) and (3.5)
   $\widehat{\Gamma}; \widehat{\Delta}, x : A \vdash e : A''' \dashv \cdot$ (3.8)
   by (t:Subsumption) on (3.7) and (sd:Var) with (2).
   (a defocus-guarantee can never be introduced by subtyping, thus $\widehat{\Delta}$)
   Thus, with (3.8), (3.6) and (3.1) we conclude by case 5 of the definition.

4. if $A_0 = A :: A'$ then:
   $\widehat{\Gamma}; \widehat{\Delta'} \vdash v : A \dashv A'$ (4.1)
   $A :: A' <: A_1$ (4.2)
   by case 3 of the hypothesis and rewriting (4).
   by (Subtyping Inversion Lemma) on (4.2) we have that:
   (note: we omit the case $A_1 = A_0$, since it is immediate)
   $A_1 = A'' :: A'''$ (4.3)
   $A <: A''$ (4.4)
   $A' <: A'''$ (4.5)
   $\widehat{\Gamma}; \widehat{\Delta} \vdash v : A'' \dashv A'''$ (4.6)
   by (t:Subsumption) on (4.1) with (4.4) and (4.5).
   Thus, we conclude by case 3 of the definition.

5. if $A_0 = [\overline{\mathtt{f} : A}]$ then:
   $v = \{\overline{\mathtt{f} = v'}\}$ (5.1)
   $\overline{\widehat{\Gamma}; \widehat{\Delta'} \vdash v'_i :\ A_i \dashv \cdot}$ (5.2)
   $[\overline{\mathtt{f} : A}] <: A_1$ (5.5)
   by case 8 of the hypothesis and rewriting (4).
   by (Subtyping Inversion Lemma) on (5.5) we have that either:
   (note: we omit the case $A_1 = A_0$, since it is immediate)
   • $[5(b)]$ $A_0 = [\overline{\mathtt{f} : A}\ ,\ \mathtt{f}_i : A']$ and
   $A_1 = [\overline{\mathtt{f} : A}\ ,\ \mathtt{f}_i : A'']$ (5.6)
   $A' <: A''$ (5.7)
   Thus, by (t:Subsumption) on (5.2) and (5.7) we conclude by case 8 of the definition.
   • $[5(a)]$ $A_0 = [\overline{\mathtt{f} : A}, \mathtt{f}_i : A]$ and
   $A_1 = [\overline{\mathtt{f} : A}]$ and $i > 0$.
   Thus, by (t:Record) with (5.1) and ignoring the dropped field, we conclude by case 8 of the definition. Note that all fields have the same effect and by $i > 0$ we ensure that subtyping leaves at least one field to do such effect.
   • $[5(c)]$ $A_0 = [\overline{\mathtt{f} : !A}]$ and
   $A_1 = ![\overline{\mathtt{f} : !A}]$ (5.8)
   $\overline{\widehat{\Gamma}; \widehat{\Delta'} \vdash v'_i : !A_i \dashv \cdot}$ (5.9)
   by rewriting (5.2) with (5.8).
   $\overline{\widehat{\Gamma}; \cdot \vdash v'_i : !A_i \dashv \cdot}$ (5.10)
   by induction hypothesis on (5.9), note the ! type.
   $\widehat{\Gamma}; \cdot \vdash \{\overline{\mathtt{f} = v'}\} : [\overline{\mathtt{f} : !A}] \dashv \cdot$ (5.11)
   by (t:Record) on (5.9).
   Thus, we conclude by case 2 of the definition.

6. if $A_0 = \exists t.A$ then:
   $v = \langle p, v' \rangle$ (6.1)
   $\widehat{\Gamma}; \widehat{\Delta'} \vdash v' : A\{p/t\} \dashv \cdot$ (6.2)
   $\exists t.A <: A_1$ (6.3)
   by case 7 of the hypothesis and rewriting (4).
   by (Subtyping Inversion Lemma) on (6.3) we have that:
   (note: we omit the case $A_1 = A_0$, since it is immediate)
   $A_1 = \exists t.A'$ (6.4)
   $A <: A'$ (6.5)
   $\widehat{\Gamma}; \widehat{\Delta} \vdash v' : A'\{p/t\} \dashv \cdot$ (6.6)
   by (t:Subsumption) on (6.2) and (6.5).
   Thus, we conclude by case 7 of the definition.

7. if $A_0 = \forall t.A$ then:
   $v = \langle t \rangle\, e$ (7.1)
   $\widehat{\Gamma, t : \mathbf{loc}}; \widehat{\Delta'} \vdash e : A \dashv \cdot$ (7.2)
   $\forall t.A <: A_1$ (7.3)
   by case 6 of the hypothesis and rewriting (4).
   by (Subtyping Inversion Lemma) on (7.3) we have that:
   (note: we omit the case $A_1 = A_0$, since it is immediate)
   $A_1 = \forall t.A'$ (7.4)
   $A <: A'$ (7.5)
   $\widehat{\Gamma, t : \mathbf{loc}}; \widehat{\Delta} \vdash e : A' \dashv \cdot$ (7.2)
   by (t:Subsumption) on (7.2) and (7.5).
   (note that a defocus-guarantee cannot be introduced by subtyping)
   Thus, we conclude by case 6 of the definition.

8. if $A_0 = \mathbf{ref}\, \rho$ then:
   $v = \rho$ (8.1)
   $\rho : \mathbf{loc}\ \in\ \widehat{\Gamma}$ (8.2)
   $\widehat{\Delta} = \cdot$ (8.3)
   $\mathbf{ref}\, \rho <: A_1$ (8.4)
   by case 4 of the hypothesis and rewriting (4).
   (note: we omit the case $A_1 = A_0$, since it is immediate)
   by (Subtyping Inversion Lemma) on (8.4) we have:
   • $[11]$ $A1 = !(\mathbf{ref}\, p)$
   Thus, we conclude by case 2 of the definition.

9. if $A_0 = \exists X.A$, analogous to $\exists t.A$.

10. if $A_0 = \forall X.A$, analogous to $\forall t.A$.

11. if $A_0 = \sum_i \mathtt{l}_i \# A'_i$ then:
    $v = \mathtt{l}_i \# v_i$ (11.1)
    $\widehat{\Gamma}; \widehat{\Delta'} \vdash v_i : A'_i \dashv \cdot$ (11.2)
    for some i.
    $\sum_i \mathtt{l}_i \# A'_i <: A_1$ (11.3)
    (note: we omit the case $A_1 = \sum_i \mathtt{l}_i \# A'_i$, since it is immediate)
    by (Subtyping Inversion Lemma) on (8.4) we have that:
    $A_1 = \mathtt{l}' \# A' + \sum_i \mathtt{l}_i \# A'_i$ (11.4)
    Thus, by (11.2) we conclude by case 11 of the definition.

12. if $A_0 = \mathbf{rec}\ X.A$ then:
    $\widehat{\Gamma}; \widehat{\Delta'} \vdash v : A\{\mathbf{rec}\ X.A/X\} \dashv \cdot$ (12.1)
    $\mathbf{rec}\ X.A <: A_1$ (12.2)
    by case 12 of the hypothesis and rewriting (4).
    (note: we omit the case $A_1 = A_0$, since it is immediate)
    by (Subtyping Inversion Lemma) on (12.2) we have that either:
    • $[15(a)]$ $A1 = \mathbf{rec}\ X.A$ and $A <: A'$
    $\widehat{\Gamma}; \widehat{\Delta} \vdash v : A'\{\mathbf{rec}\ X.A'/X\} \dashv \cdot$ (12.3)
    by (t:Subsumption) on (12.1).
    Thus, we conclude by case 12 of the definition.
    • $[15(b)]$ $A1 = A\{X/\mathbf{rec}\ X.A\}$
    Thus, we conclude by induction hypothesis on (12.1) combined with (t:Subsumption) on each case.

13. if $\Delta = \Delta', A_2 \oplus A_3$ then:
    $\widehat{\Gamma}; \widehat{\Delta'}, A_2 \vdash v : A_0 \dashv \cdot$ (13.1)
    $\widehat{\Gamma}; \widehat{\Delta'}, A_3 \vdash v : A_0 \dashv \cdot$ (13.2)
    $A_0 <: A_1$ (13.3)
    By induction hypothesis on each case and then (t:Subsumption).

Case (t:Let) Not a value.

□

B.5 Substitution

For clarity, substitution is defined on constructs that allow expressions even though our grammar (in some places) only allows values since such difference has no impact in the following definitions and is generally more readable.

1. **Variable Substitution**, (vs:*)

   We define the usual capture-avoiding (i.e. up to renaming of bounded variables) substitution rules:

$$\boxed{e_0\{v/x\} = e_1}$$

$$
\begin{array}{lrcll}
\text{(vs:1)} & \rho\{v/x\} & = & \rho & \\
\text{(vs:2)} & x\{v/x\} & = & v & \\
\text{(vs:3)} & x_0\{v/x_1\} & = & x_0 & (x_0 \neq x_1) \\
\text{(vs:4)} & (\mathsf{fun}(x_0 : A).e_0)\{v/x_1\} & = & \mathsf{fun}(x_0 : A).e_0\{v/x_1\} & (x_0 \neq x_1) \\
\text{(vs:5)} & \{\overline{\mathtt{f} = e}\}\{v/x\} & = & \{\overline{\mathtt{f} = e\{v/x\}}\} & \\
\text{(vs:6)} & (e.\mathtt{f})\{v/x\} & = & e\{v/x\}.\mathtt{f} & \\
\text{(vs:7)} & (e_0\ e_1)\{v/x\} & = & e_0\{v/x\}\ e_1\{v/x\} & \\
\text{(vs:8)} & (\mathsf{new}\ e)\{v/x\} & = & \mathsf{new}\ e\{v/x\} & \\
\text{(vs:9)} & (\mathsf{delete}\ e)\{v/x\} & = & \mathsf{delete}\ e\{v/x\} & \\
\text{(vs:10)} & (!e)\{v/x\} & = & !e\{v/x\} & \\
\text{(vs:11)} & (e_0 := e_1)\{v/x\} & = & e_0\{v/x\} := e_1\{v/x\} & \\
\text{(vs:12)} & \langle p, e\rangle\,\{v/x\} & = & \langle p, e\{v/x\}\rangle & \\
\text{(vs:13)} & e[p]\{v/x\} & = & e\{v/x\}[p] & \\
\text{(vs:14)} & (\langle t\rangle\, e)\{v/x\} & = & \langle t\rangle\, e\{v/x\} & \\
\text{(vs:15)} & (\mathsf{open}\ \langle t, x_0\rangle = e_0\ \mathsf{in}\ e_1\ \mathsf{end})\{v/x_1\} & = & \mathsf{open}\ \langle t, x_0\rangle = e_0\{v/x_1\}\ \mathsf{in}\ e_1\{v/x_1\}\ \mathsf{end} & (x_0 \neq x_1) \\
\text{(vs:16)} & \langle A, e\rangle\,\{v/x\} & = & \langle A, e\{v/x\}\rangle & \\
\text{(vs:17)} & e[A]\{v/x\} & = & e\{v/x\}[A] & \\
\text{(vs:18)} & (\langle X\rangle\, e)\{v/x\} & = & \langle X\rangle\, e\{v/x\} & \\
\text{(vs:19)} & (\mathsf{open}\ \langle X, x_0\rangle = e_0\ \mathsf{in}\ e_1\ \mathsf{end})\{v/x_1\} & = & \mathsf{open}\ \langle X, x_0\rangle = e_0\{v/x_1\}\ \mathsf{in}\ e_1\{v/x_1\}\ \mathsf{end} & (x_0 \neq x_1) \\
\text{(vs:20)} & (\mathtt{l}\#e)\{v/x\} & = & \mathtt{l}\#e\{v/x\} & \\
\text{(vs:21)} & (\mathsf{case}\ e\ \mathsf{of}\ \overline{\mathtt{l}_i\#x_i \rightarrow e_i}\ \mathsf{end})\{v/x\} & = & \mathsf{case}\ e\{v/x\}\ \mathsf{of}\ \overline{\mathtt{l}_i\#x_i \rightarrow e_i\{v/x\}}\ \mathsf{end} & (x_i \neq x) \\
\text{(vs:22)} & (\mathsf{let}\ x_0 = e_0\ \mathsf{in}\ e_1\ \mathsf{end})\{v/x_1\} & = & \mathsf{let}\ x_0 = e_0\{v/x_1\}\ \mathsf{in}\ e_1\{v/x_1\}\ \mathsf{end} & (x_0 \neq x_1)
\end{array}
$$

2. **Location Variable Substitution**, (ls:*)
   Similarly, we define location substitution (but here up to renaming of bounded *location* variables) as:

$$\boxed{e_0\{p/t\} = e_1}$$

$$
\begin{array}{lrcll}
(\textsc{ls}{:}1.1) & \rho\{p/t\} & = & \rho & \\
(\textsc{ls}{:}1.2) & x\{p/t\} & = & x & \\
(\textsc{ls}{:}1.3) & (\mathsf{fun}(x : A).e)\{p/t\} & = & \mathsf{fun}(x : A\{p/t\}).e\{p/t\} & \\
(\textsc{ls}{:}1.4) & \{\overline{\mathtt{f} = e}\}\{p/t\} & = & \{\overline{\mathtt{f} = e\{p/t\}}\} & \\
(\textsc{ls}{:}1.5) & (e.\mathtt{f})\{p/t\} & = & e\{p/t\}.\mathtt{f} & \\
(\textsc{ls}{:}1.6) & (e_0\ e_1)\{p/t\} & = & e_0\{p/t\}\ e_1\{p/t\} & \\
(\textsc{ls}{:}1.7) & (\mathsf{new}\ e)\{p/t\} & = & \mathsf{new}\ e\{p/t\} & \\
(\textsc{ls}{:}1.8) & (\mathsf{delete}\ e)\{p/t\} & = & \mathsf{delete}\ e\{p/t\} & \\
(\textsc{ls}{:}1.9) & (!e)\{p/t\} & = & !e\{p/t\} & \\
(\textsc{ls}{:}1.10) & (e_0 := e_1)\{p/t\} & = & e_0\{p/t\} := e_1\{p/t\} & \\
(\textsc{ls}{:}1.11) & \langle p_0, e\rangle\,\{p_1/t\} & = & \langle p_0\{p_1/t\}, e\{p_1/t\}\rangle & \\
(\textsc{ls}{:}1.12) & e[p_0]\{p_1/t\} & = & e\{p_1/t\}[p_0\{p_1/t\}] & \\
(\textsc{ls}{:}1.13) & (\langle t_0\rangle\, e)\{p/t_1\} & = & \langle t_1\rangle\, e\{p/t_1\} & (t_0 \neq t_1) \\
(\textsc{ls}{:}1.14) & (\mathsf{open}\ \langle t_0, x\rangle = e_0\ \mathsf{in}\ e_1\ \mathsf{end})\{p/t_1\} & = & \mathsf{open}\ \langle t_0, x\rangle = e_0\{p/t_1\}\ \mathsf{in}\ e_1\{p/t_1\}\ \mathsf{end} & (t_0 \neq t_1) \\
(\textsc{ls}{:}1.15) & \langle A, e\rangle\,\{p/t\} & = & \langle A\{p/t\}, e\{p/t\}\rangle & \\
(\textsc{ls}{:}1.16) & e[A]\{p/t\} & = & e\{p/t\}[A\{p/t\}] & \\
(\textsc{ls}{:}1.17) & (\langle X\rangle\, e)\{p/t\} & = & \langle X\rangle\, e\{p/t\} & \\
(\textsc{ls}{:}1.18) & (\mathsf{open}\ \langle X, x\rangle = e_0\ \mathsf{in}\ e_1\ \mathsf{end})\{p/t\} & = & \mathsf{open}\ \langle X, x\rangle = e_0\{p/t\}\ \mathsf{in}\ e_1\{p/t\}\ \mathsf{end} & \\
(\textsc{ls}{:}1.19) & (\mathtt{l}\#e)\{p/t\} & = & \mathtt{l}\#e\{p/t\} & \\
(\textsc{ls}{:}1.20) & (\mathsf{case}\ e\ \mathsf{of}\ \overline{\mathtt{l}_i\#x_i \to e_i}\ \mathsf{end})\{p/t\} & = & \mathsf{case}\ e\{p/t\}\ \mathsf{of}\ \overline{\mathtt{l}_i\#x_i \to e_i\{p/t\}}\ \mathsf{end} & \\
(\textsc{ls}{:}1.21) & (\mathsf{let}\ x = e_0\ \mathsf{in}\ e_1\ \mathsf{end})\{p/t\} & = & \mathsf{let}\ x_0 = e_0\{p/t\}\ \mathsf{in}\ e_1\{p/t\}\ \mathsf{end} &
\end{array}
$$

$$\boxed{A_0\{p/t\} = A_1}$$

$$
\begin{array}{lrcll}
(\textsc{ls}{:}2.1) & \rho\{p/t\} & = & \rho & \\
(\textsc{ls}{:}2.2) & t\{p/t\} & = & p & \\
(\textsc{ls}{:}2.3) & t_0\{p/t_1\} & = & t_0 & (t_0 \neq t_1) \\
(\textsc{ls}{:}2.4) & (!A)\{p/t\} & = & !A\{p/t\} & \\
(\textsc{ls}{:}2.5) & (A_0 \multimap A_1)\{p/t\} & = & A_0\{p/t\} \multimap A_1\{p/t\} & \\
(\textsc{ls}{:}2.6) & (A_0 :: A_1)\{p/t\} & = & A_0\{p/t\} :: A_1\{p/t\} & \\
(\textsc{ls}{:}2.7) & [\overline{\mathtt{f} : A}]\{p/t\} & = & [\overline{\mathtt{f} : A\{p/t\}}] & \\
(\textsc{ls}{:}2.8) & (\forall t_0.A)\{p/t_1\} & = & \forall t_0.A\{p/t_1\} & (t_0 \neq t_1) \\
(\textsc{ls}{:}2.9) & (\exists t_0.A)\{p/t_1\} & = & \exists t_0.A\{p/t_1\} & (t_0 \neq t_1) \\
(\textsc{ls}{:}2.10) & (\mathbf{ref}\ p_0)\{p_1/t\} & = & \mathbf{ref}\ p_0\{p_1/t\} & \\
(\textsc{ls}{:}2.12) & (\mathbf{rw}\ p_0\ A)\{p_1/t\} & = & \mathbf{rw}\ p_0\{p_1/t\}\ A\{p_1/t\} & \\
(\textsc{ls}{:}2.13) & (A_0 * A_1)\{p/t\} & = & A_0\{p/t\} * A_1\{p/t\} & \\
(\textsc{ls}{:}2.14) & (\forall X.A)\{p/t\} & = & \forall X.A\{p/t\} & \\
(\textsc{ls}{:}2.15) & (\exists X.A)\{p/t\} & = & \exists X.A\{p/t\} & \\
(\textsc{ls}{:}2.16) & X\{p/t\} & = & X & \\
(\textsc{ls}{:}2.17) & (\mathbf{rec}\ X.A)\{p/t\} & = & \mathbf{rec}\ X.A\{p/t\} & \\
(\textsc{ls}{:}2.18) & (\textstyle\sum_i \mathtt{l}_i\#A_i)\{p/t\} & = & \textstyle\sum_i \mathtt{l}_i\#A_i\{p/t\} & \\
(\textsc{ls}{:}2.19) & (A_0 \oplus A_1)\{p/t\} & = & A_0\{p/t\} \oplus A_1\{p/t\} & \\
(\textsc{ls}{:}2.20) & \boldsymbol{\mathsf{none}}\{p/t\} & = & \boldsymbol{\mathsf{none}} &
\end{array}
$$

$$\boxed{\Gamma_0\{p/t\} = \Gamma_1}$$

$$
\begin{array}{lrcll}
(\textsc{ls}{:}3.1) & \cdot\{p/t\} & = & \cdot & \\
(\textsc{ls}{:}3.2) & (\Gamma, x : A)\{p/t\} & = & \Gamma\{p/t\}, x : A\{p/t\} & \\
(\textsc{ls}{:}3.3) & (\Gamma, t_0 : \mathbf{loc})\{p/t_1\} & = & \Gamma\{p/t_1\}, t_0 : \mathbf{loc} & (t_0 \neq t_1) \\
(\textsc{ls}{:}3.4) & (\Gamma, X : \mathbf{type})\{p/t\} & = & \Gamma\{p/t\}, X : \mathbf{type} &
\end{array}
$$

$$\boxed{\Delta_0\{p/t\} = \Delta_1}$$

$$
\begin{array}{lrcl}
(\textsc{ls}{:}4.1) & \cdot\{p/t\} & = & \cdot \\
(\textsc{ls}{:}4.2) & (\Delta, x : A)\{p/t\} & = & \Delta\{p/t\}, x : A\{p/t\} \\
(\textsc{ls}{:}4.3) & (\Delta, A)\{p/t\} & = & \Delta\{p/t\}, A\{p/t\}
\end{array}
$$

3. **Type Variable Substitution**, (TS:*)
   Finally, we define type substitution (up to renaming of bounded *type* variables) as:

$$\boxed{e_0\{A/X\} = e_1}$$

$$
\begin{array}{lrcll}
\text{(TS:1.1)} & \rho\{A/X\} & = & \rho & \\
\text{(TS:1.2)} & x\{A/X\} & = & x & \\
\text{(TS:1.3)} & (\mathsf{fun}(x : A_0).e)\{A_1/X\} & = & \mathsf{fun}(x : A_0\{A_1/X\}).e\{A_1/X\} & \\
\text{(TS:1.4)} & \{\overline{\mathtt{f} = e}\}\{A/X\} & = & \{\overline{\mathtt{f} = e\{A/X\}}\} & \\
\text{(TS:1.5)} & (e.\mathtt{f})\{A/X\} & = & e\{A/X\}.\mathtt{f} & \\
\text{(TS:1.6)} & (e_0\ e_1)\{A/X\} & = & e_0\{A/X\}\ e_1\{A/X\} & \\
\text{(TS:1.7)} & (\mathsf{new}\ e)\{A/X\} & = & \mathsf{new}\ e\{A/X\} & \\
\text{(TS:1.8)} & (\mathsf{delete}\ e)\{A/X\} & = & \mathsf{delete}\ e\{A/X\} & \\
\text{(TS:1.9)} & (!e)\{A/X\} & = & !e\{A/X\} & \\
\text{(TS:1.10)} & (e_0 := e_1)\{A/X\} & = & e_0\{A/X\} := e_1\{A/X\} & \\
\text{(TS:1.11)} & \langle p, e\rangle\,\{A/X\} & = & \langle p, e\{A/X\}\rangle & \\
\text{(TS:1.12)} & e[p]\{A/X\} & = & e\{A/X\}[p] & \\
\text{(TS:1.13)} & (\langle t\rangle\, e)\{A/X\} & = & \langle t\rangle\, e\{A/X\} & \\
\text{(TS:1.14)} & (\mathsf{open}\ \langle t, x\rangle = e_0\ \mathsf{in}\ e_1\ \mathsf{end})\{A/X\} & = & \mathsf{open}\ \langle t, x\rangle = e_0\{A/X\}\ \mathsf{in}\ e_1\{A/X\}\ \mathsf{end} & \\
\text{(TS:1.15)} & \langle A_0, e\rangle\,\{A_1/X\} & = & \langle A_0\{A_1/X\}, e\{A_1/X\}\rangle & \\
\text{(TS:1.16)} & e[A_0]\{A_1/X\} & = & e\{A_1/X\}[A_0\{A_1/X\}] & \\
\text{(TS:1.17)} & (\langle X_0\rangle\, e)\{A/X_1\} & = & \langle X_0\rangle\, e\{A/X_1\} & (X_0 \neq X_1) \\
\text{(TS:1.18)} & (\mathsf{open}\ \langle X_0, x\rangle = e_0\ \mathsf{in}\ e_1\ \mathsf{end})\{A/X_1\} & = & \mathsf{open}\ \langle X_0, x\rangle = e_0\{A/X_1\}\ \mathsf{in}\ e_1\{A/X_1\}\ \mathsf{end} & (X_0 \neq X_1) \\
\text{(TS:1.19)} & (\mathtt{l}\#e)\{A/X\} & = & \mathtt{l}\#e\{A/X\} & \\
\text{(TS:1.20)} & (\mathsf{case}\ e\ \mathsf{of}\ \overline{\mathtt{l}_i\#x_i \rightarrow e_i}\ \mathsf{end})\{A/X\} & = & \mathsf{case}\ e\{A/X\}\ \mathsf{of}\ \overline{\mathtt{l}_i\#x_i \rightarrow e_i\{A/X\}}\ \mathsf{end} & \\
\text{(TS:1.21)} & (\mathsf{let}\ x = e_0\ \mathsf{in}\ e_1\ \mathsf{end})\{A/X\} & = & \mathsf{let}\ x_0 = e_0\{A/X\}\ \mathsf{in}\ e_1\{A/X\}\ \mathsf{end} &
\end{array}
$$

$$\boxed{A_0\{A_1/X\} = A_2}$$

$$
\begin{array}{lrcll}
\text{(TS:2.1)} & \rho\{A/X\} & = & \rho & \\
\text{(TS:2.2)} & t\{A/X\} & = & p & \\
\text{(TS:2.3)} & X\{A/X\} & = & A & \\
\text{(TS:2.4)} & X_0\{A/X_1\} & = & X_0 & (X_0 \neq X_1) \\
\text{(TS:2.5)} & (!A_0)\{A_1/X\} & = & !A_0\{A_1/X\} & \\
\text{(TS:2.6)} & (A_0 \multimap A_1)\{A_2/X\} & = & A_0\{A_2/X\} \multimap A_1\{A_2/X\} & \\
\text{(TS:2.7)} & (A_0 :: A_1)\{A_2/X\} & = & A_0\{A_2/X\} :: A_1\{A_2/X\} & \\
\text{(TS:2.8)} & [\overline{\mathtt{f} : A}]\{A_0/X\} & = & [\overline{\mathtt{f} : A\{A_0/X\}}] & \\
\text{(TS:2.9)} & (\forall t.A_0)\{A_1/X\} & = & \forall t.A_0\{A_1/X\} & \\
\text{(TS:2.10)} & (\exists t.A_0)\{A_1/X\} & = & \exists t.A_0\{A_1/X\} & \\
\text{(TS:2.11)} & (\mathbf{ref}\ p)\{A/X\} & = & \mathbf{ref}\ p & \\
\text{(TS:2.13)} & (\mathbf{rw}\ p\ A_0)\{A_1/X\} & = & \mathbf{rw}\ p\ A_0\{A_1/X\} & \\
\text{(TS:2.14)} & (A_0 * A_1)\{A_2/X\} & = & A_0\{A_2/X\} * A_1\{A_2/X\} & \\
\text{(TS:2.15)} & (\forall X_0.A_0)\{A_1/X_1\} & = & \forall X_0.A_0\{A_1/X_1\} & (X_0 \neq X_1) \\
\text{(TS:2.16)} & (\exists X_0.A_0)\{A_1/X_1\} & = & \exists X_0.A_0\{A_1/X_1\} & (X_0 \neq X_1) \\
\text{(TS:2.17)} & (\mathbf{rec}\ X_0.A_0)\{A_1/X_1\} & = & \mathbf{rec}\ X_0.A_0\{A_1/X_1\} & (X_0 \neq X_1) \\
\text{(TS:2.18)} & (\sum_i \mathtt{l}_i\#A_i)\{A/X\} & = & \sum_i \mathtt{l}_i\#A_i\{A/X\} & \\
\text{(TS:2.19)} & (A_0 \oplus A_1)\{A/X\} & = & A_0\{A/X\} \oplus A_1\{A/X\} & \\
\text{(TS:2.20)} & \mathbf{none}\{A/X\} & = & \mathbf{none} &
\end{array}
$$

$$\boxed{\Gamma_0\{A/X\} = \Gamma_1}$$

$$
\begin{array}{lrll}
\text{(TS:3.1)} & \cdot\{A/X\} = & \cdot & \\
\text{(TS:3.2)} & (\Gamma, x : A_0)\{A_1/X\} = & \Gamma\{A_1/X\}, x : A_0\{A_1/X\} & \\
\text{(TS:3.3)} & (\Gamma, t : \mathbf{loc})\{A/X\} = & \Gamma\{A/X\}, t : \mathbf{loc} & \\
\text{(TS:3.4)} & (\Gamma, X_0 : \mathbf{type})\{A/X_1\} = & \Gamma\{A/X_1\}, X_0 : \mathbf{type} & (X_0 \neq X_1)
\end{array}
$$

$$\boxed{\Delta_0\{A/X\} = \Delta_1}$$

$$
\begin{array}{lrl}
\text{(TS:4.1)} & \cdot\{A/X\} = & \cdot \\
\text{(TS:4.2)} & (\Delta, x : A_0)\{A_1/X\} = & \Delta\{A_1/X\}, x : A_0\{A_1/X\} \\
\text{(TS:4.3)} & (\Delta, A_0)\{A_1/X\} = & \Delta\{A_1/X\}, A_0\{A_1/X\}
\end{array}
$$

B.6 Free Variables Lemma

Lemma 5 (Free Variables Lemma)**.** If $\Gamma; \Delta_0, x : A_0 \vdash e : A_1 \dashv \Delta_1$ and $x \in \mathtt{fv}(e)$ then $x \notin \Delta_1$.
$\mathtt{fv}(e) \triangleq$ "set of all free variables inside the expression e"

Proof. We proceed by induction on the derivation of $\Gamma; \Delta_0, x : A_0 \vdash e : A_1 \dashv \Delta_1$.

Case (t:Ref), (t:Pure), (t:Unit), (t:Pure-Read) - Δ is empty.
Case (t:Linear-Read) - We have:

$\Gamma; x : A \vdash x : A \dashv \cdot$ (1)
$x \in \mathtt{fv}(x)$ (2)
by hypothesis.

Therefore, we immediately conclude $x \notin \cdot$.

Case (t:Pure-Elim) - We have:

$\Gamma; \Delta_0, x : !A_0 \vdash e : A_1 \dashv \Delta_1$ (1)
$x \in \mathtt{fv}(e)$ (2)
by hypothesis.
$\Gamma, x : A_0; \Delta_0 \vdash e : A_1 \dashv \Delta_1$ (3)
by inversion on (t:Pure-Elim).
$x \notin \Delta_1$ (4)
because x is in the linear environment (and cannot appear duplicated in Δ's).

Therefore, we conclude.

(Note: the case when x is not the one use in the (t:Pure-Elim) rule is a direct application of the induction hypothesis.)

Case (t:New) - We have:

$\Gamma; \Delta_0, x : A_0 \vdash \textsf{new}\ v : \exists t.(\mathbf{ref}\ t :: \mathbf{rw}\ t\ A) \dashv \Delta_1$ (1)
$x \in \mathtt{fv}(\textsf{new}\ v)$ (2)
by hypothesis.
$\Gamma; \Delta_0, x : A_0 \vdash v : A \dashv \Delta_1$ (3)
by inversion on (t:New) with (1).
$x \in \mathtt{fv}(v)$ (4)
[$\mathtt{fv}(\textsf{new}\ v) = \mathtt{fv}(v)$]
by definition of $\mathtt{fv}$ and (2).
$x \notin \Delta_1$ (5)
by induction hypothesis on (3) and (4).

Therefore, we conclude.

Case (t:Delete) - We have:

$\Gamma; \Delta_0, x : A_0 \vdash \textsf{delete}\ v : \exists t.A \dashv \Delta_1$ (1)
$x \in \mathtt{fv}(\textsf{delete}\ v)$ (2)
by hypothesis.
$\Gamma; \Delta_0, x : A_0 \vdash v : \exists t.(\mathbf{ref}\ t :: \mathbf{rw}\ t\ A) \dashv \Delta_1$ (3)
by inversion on (t:Delete) with (1).
$x \in \mathtt{fv}(v)$ (4)
[$\mathtt{fv}(\textsf{delete}\ v) = \mathtt{fv}(v)$]
by definition of $\mathtt{fv}$ and (2).
$x \notin \Delta_1$ (5)
by induction hypothesis on (3) and (4).

Therefore, we conclude.

Case (t:Assign) - We have:

$\Gamma; \Delta_0, x : A \vdash v_0 := v_1 : A_1 \dashv \Delta_2, \mathbf{rw}\ p\ A_0$ (1)
$x \in \mathtt{fv}(v_0 := v_1)$ (2)
by hypothesis.
$\Gamma; \Delta_0, x : A \vdash v_1 : A_0 \dashv \Delta_1$ (3)
$\Gamma; \Delta_1 \vdash v_0 : \mathbf{ref}\ p \dashv \Delta_2, \mathbf{rw}\ p\ A_1$ (4)
by inversion on (t:Assign) with (1).
[$\mathtt{fv}(v_0 := v_1) = \mathtt{fv}(v_0) \cup \mathtt{fv}(v_1)$]

Therefore, we have the following possibilities:

1. $x \in \mathtt{fv}(v_0) \wedge x \notin \mathtt{fv}(v_1)$
   $(x : A) \in \Delta_1$ (1.1)
   by $x \notin \mathtt{fv}(v_1)$.
   $x \notin \Delta_2, \mathbf{rw}\ p\ A_1$ (1.2)
   by induction hypothesis on (4) with (1.1).
   $x \notin \Delta_2, \mathbf{rw}\ p\ A_0$ (1.3)
   since the capability trivially obeys the restriction (since x is not a type).
   Thus, we conclude.
2. $x \in \mathtt{fv}(v_1) \wedge x \notin \mathtt{fv}(v_0)$
   $x \notin \Delta_1$ (2.1)
   by induction hypothesis on (3) and case assumption.
   $x \notin \Delta_2, \mathbf{rw}\ p\ A_1$ (2.2)
   by (2.1) and (4).
   $x \notin \Delta_2, \mathbf{rw}\ p\ A_0$ (2.3)
   since the capability trivially obeys the restriction on (2.2).
   Thus, we conclude.
3. $x \in \mathtt{fv}(v_0) \wedge x \in \mathtt{fv}(v_1)$
   $x \notin \Delta_1$ (3.1)
   by induction hypothesis on (3) and case assumption.
   We reach a contradiction since v_0 is well-typed by (4) but $x \in \mathtt{fv}(v_1)$ contradicts (3.1). Thus, such case is impossible to occur in a well-typed expression.
   Thus, we conclude.

Case (t:Dereference-Linear) - We have:

$\Gamma; \Delta_0, x : A_0 \vdash !v : A \dashv \Delta_1, \mathbf{rw}\ p\ []$ (1)
$x \in \mathtt{fv}(!v)$ (2)
by hypothesis.
$\Gamma; \Delta_0, x : A_0 \vdash v : \mathbf{ref}\ p \dashv \Delta_1, \mathbf{rw}\ p\ A$ (3)
by inversion on (t:Dereference-Linear).
[$\mathtt{fv}(!v) = \mathtt{fv}(v)$]
$x \in \mathtt{fv}(v)$ (4)
by definition of $\mathtt{fv}$ and (2).
$x \notin \Delta_1, \mathbf{rw}\ p\ A$ (5)
by induction hypothesis on (3) and (4).
$x \notin \Delta_1, \mathbf{rw}\ p\ []$ (6)
by (5) and since x cannot be in $\mathbf{rw}\ p\ []$.

Thus, we conclude.

Case (t:Dereference-Pure) - We have:

$\Gamma; \Delta_0, x : A_0 \vdash !v : !A_1 \dashv \Delta_1, \mathbf{rw}\ p\ !A_1$ (1)
$x \in \mathtt{fv}(!v)$ (2)
by hypothesis.
$\Gamma; \Delta_0, x : A_0 \vdash v : \mathbf{ref}\ p \dashv \Delta_1, \mathbf{rw}\ p\ !A_1$ (3)
by inversion on (t:Dereference-Pure).
[$\mathtt{fv}(!e) = \mathtt{fv}(v)$]
$x \in \mathtt{fv}(v)$ (4)
by definition of $\mathtt{fv}$ and (2).
$x \notin \Delta_1, \mathbf{rw}\ p\ !A_1$ (5)
by induction hypothesis on (3) and (4).

Thus, we conclude.

Case (t:Record) - We have:

$\Gamma; \Delta, x : A_0 \vdash \{\overline{\mathtt{f} = v}\} : [\overline{\mathtt{f} : A}] \dashv \cdot$ (1)
$x \in \mathtt{fv}(\{\overline{\mathtt{f} = v}\})$ (2)
by hypothesis.

Therefore, we immediately conclude $x \notin \cdot$.

Case (t:Selection) - We have:

$\Gamma; \Delta_0, x : A_0 \vdash v.\mathtt{f}_i : A_i \dashv \Delta_1$ (1)
$x \in \mathtt{fv}(v.\mathtt{f})$ (2)
by hypothesis.
$\Gamma; \Delta_0, x : A_0 \vdash v : [\overline{\mathtt{f} : A}] \dashv \Delta_1$ (3)
by inversion on (t:Selection).
[$\mathtt{fv}(v.\mathtt{f}) = \mathtt{fv}(v)$]
$x \in \mathtt{fv}(v)$ (4)
by definition of $\mathtt{fv}$ and (2).
$x \notin \Delta_1$ (5)
by induction hypothesis on (3) and (4).

Thus, we conclude.

Case (t:Application) - We have:

$\Gamma; \Delta_0, x : A \vdash v_0\ v_1 : A_1 \dashv \Delta_2$ (1)
$x \in \mathtt{fv}(v_0\ v_1)$ (2)
[$\mathtt{fv}(v_0\ v_1) = \mathtt{fv}(v_0) \cup \mathtt{fv}(v_1)$]
by hypothesis.
$\Gamma; \Delta_0 \vdash v_1 : A_0 \dashv \Delta_1$ (3)
$\Gamma; \Delta_1 \vdash v_0 : A_0 \multimap A_1 \dashv \Delta_2$ (4)
by inversion on (t:Application) with (1).

Therefore, we have the following possibilities:

1. $x \in \mathtt{fv}(v_0) \wedge x \notin \mathtt{fv}(v_1)$
   $\Gamma; \Delta_0 \vdash v_1 : A_0 \dashv \Delta'_1$ (1.1)
   $\Delta_1 = \Delta'_1, x : A$ (1.2)
   by $x \notin \mathtt{fv}(v_1)$.
   $\Gamma; \Delta'_1, x : A \vdash v_0 : A_0 \multimap A_1 \dashv \Delta_2$ (1.3)
   by rewriting (4) with (1.2).
   $x \notin \Delta_2$ (1.4)

Thus, we conclude.

2. $x \in \mathtt{fv}(v_0) \wedge x \in \mathtt{fv}(v_1)$

   $x \notin \Delta_1$ (2.1)

   by induction hypothesis on (3) and case assumption.

   We reach a contradiction since v_0 is well-typed by (4) but $x \in \mathtt{fv}(v_1)$ contradicts (2.1). Thus, such case is impossible to occur in a well-typed expression. Therefore, we conclude.

3. $x \in \mathtt{fv}(v_1) \wedge x \notin \mathtt{fv}(v_0)$

   $x \notin \Delta_1$ (3.1)

   by induction hypothesis on (3) and case assumption.

   $x \notin \Delta_2$ (3.2)

   by (3.1) and (4).

   Thus, we conclude.

Case (t:Function) - We have:

$\Gamma; \Delta, x : A_0 \vdash \mathsf{fun}(x_0 : A_2).e : A_2 \multimap A_1 \dashv \cdot$ (1)

$x \in \mathtt{fv}(\mathsf{fun}(x_0 : A_2).e)$ (2)

by hypothesis.

$x \notin \cdot$ (3)

since it is the empty environment.

Thus, we conclude.

Case (t:Forall-Loc) - We have:

$\Gamma; \Delta, x : A_0 \vdash \langle t \rangle\, e : \forall t.A \dashv \cdot$ (1)

$x \in \mathtt{fv}(\langle t \rangle\, e)$ (2)

by hypothesis.

$x \notin \cdot$ (3)

since it is the empty environment.

Thus, we conclude.

Case (t:Loc-App) - We have:

$\Gamma; \Delta, x : A_0 \vdash v[p] : A\{p/t\} \dashv \Delta_1$ (1)

$x \in \mathtt{fv}(v[p])$ (2)

by hypothesis.

$p : \mathbf{loc} \in \Gamma$ (3)

$\Gamma; \Delta, x : A_0 \vdash v : \ \forall t.A \dashv \Delta_1$ (4)

by inversion on (t:Loc-App) on (1).

[$\mathtt{fv}(v[p]) = \mathtt{fv}(v)$]

$x \in \mathtt{fv}(v)$ (5)

by definition of $\mathtt{fv}$ and (2).

$x \notin \Delta_1$ (6)

by induction hypothesis on (5) and (4).

Thus, we conclude.

Case (t:Loc-Open) - We have:

$\Gamma; \Delta_0, x : A \vdash \mathsf{open}\, \langle t, x_0 \rangle = v_0 \ \mathsf{in}\ e_1\ \mathsf{end} : A_1 \dashv \Delta_2$ (1)

$x \in \mathtt{fv}(\mathsf{open}\, \langle t, x_0 \rangle = v_0 \ \mathsf{in}\ e_1\ \mathsf{end})$ (2)

[$\mathtt{fv}(\mathsf{open}\, \langle t, x_0 \rangle = v_0 \ \mathsf{in}\ e_1\ \mathsf{end}) = \mathtt{fv}(v_0) \cup \mathtt{fv}(e_1)$]

by hypothesis.

$\Gamma; \Delta_0, x : A \vdash v_0 : \exists t.A_0 \dashv \Delta_1$ (3)

$\Gamma, t : \mathbf{loc}; \Delta_1, x_0 : A_0 \vdash e_1 : A_1 \dashv \Delta_2$ (4)

by inversion on (t:Loc-Open) with (1).

Therefore, we have the following possibilities:

1. $x \in \mathtt{fv}(e_1) \wedge x \notin \mathtt{fv}(v_0)$

   $(x : A) \in \Delta_1$ (1.1)

   by $x \notin \mathtt{fv}(v_0)$.

   $x \notin \Delta_2$ (1.2)

   by induction hypothesis on (4) with (1.1).

   Thus, we conclude.

2. $x \in \mathtt{fv}(v_0) \wedge x \in \mathtt{fv}(e_1)$

   $x \notin \Delta_1$ (2.1)

   by induction hypothesis on (3) and case assumption.

   We reach a contradiction since v_0 is well-typed by (4) but $x \in \mathtt{fv}(e_1)$ contradicts (2.1). Thus, such case is impossible to occur in a well-typed expression.

3. $x \in \mathtt{fv}(v_0) \wedge x \notin \mathtt{fv}(e_1)$

   $x \notin \Delta_1$ (3.1)

   by induction hypothesis on (3) and case assumption.

   $x \notin \Delta_2$ (3.2)

   by (3.1) and (4).

   Thus, we conclude.

Case (t:Loc-Pack) - We have:

$\Gamma; \Delta, x : A_0 \vdash \langle p, v \rangle : \exists t.A \dashv \Delta_1$ (1)

$x \in \mathtt{fv}(\langle p, v \rangle)$ (2)

by hypothesis.

$\Gamma; \Delta, x : A_0 \vdash v : A\{p/t\} \dashv \Delta_1$ (3)

by inversion on (t:Loc-Pack) on (1).

[$\mathtt{fv}(\langle p, v \rangle) = \mathtt{fv}(v)$]

$x \in \mathtt{fv}(v)$ (4)

by definition of $\mathtt{fv}$ and (2).

$x \notin \Delta_1$ (5)

by induction hypothesis on (4) and (3).

Thus, we conclude.

Case (t:Forall-Type) - We have:

$\Gamma; \Delta, x : A_0 \vdash \langle X \rangle\, e : \forall X.A \dashv \cdot$ (1)

$x \in \mathtt{fv}(\langle X \rangle\, e)$ (2)

by hypothesis.

$x \notin \cdot$ (3)

since it is the empty environment.

Thus, we conclude.

Case (t:Type-App) - We have:

$\Gamma; \Delta, x : A_0 \vdash v[A_1] : A_2\{A_1/X\} \dashv \Delta_1$ (1)

$x \in \mathtt{fv}(v[A_1])$ (2)

by hypothesis.

$\Gamma \vdash A_1\ \mathbf{type}$ (3)

$\Gamma; \Delta, x : A_0 \vdash v : \forall X.A_2 \dashv \Delta_1$ (4)

by inversion on (t:Type-App) on (1).

[$\mathtt{fv}(v[A_1]) = \mathtt{fv}(v)$]

$x \in \mathtt{fv}(v)$ (5)

by definition of $\mathtt{fv}$ and (2).

$x \notin \Delta_1$ (6)

by induction hypothesis on (5) and (4).

Thus, we conclude.

Case (t:Type-Pack) - We have:

$\Gamma; \Delta, x : A_0 \vdash \langle A_1, v \rangle : \exists X.A_2 \dashv \Delta_1$ (1)

$x \in \mathtt{fv}(\langle A_1, v \rangle)$ (2)

by hypothesis.

$\Gamma; \Delta, x : A_0 \vdash v : A_2\{A_1/X\} \dashv \Delta_1$ (3)

by inversion on (t:Type-Pack) on (1).

[$\mathtt{fv}(\langle A_1, v \rangle) = \mathtt{fv}(v)$]

$x \in \mathtt{fv}(v)$ (4)

by definition of $\mathtt{fv}$ and (2).

$x \notin \Delta_1$ (5)

by induction hypothesis on (4) and (3).

Thus, we conclude.

Case (t:Type-Open) - Analogous to (t:Loc-Open).

Case (t:Cap-Elim) - We have:

$\Gamma; \Delta_0, x : A_1 :: A_2 \vdash e : A_0 \dashv \Delta_1$ (1)

$x \in \mathtt{fv}(e)$ (2)

by hypothesis.

$\Gamma; \Delta_0, x : A_1, A_2 \vdash e : A_0 \dashv \Delta_1$ (3)

by inversion on (t:Cap-Elim) on (1).

$x \notin \Delta_1$ (4)

by induction hypothesis on (2) and (3).

Thus, we conclude.

Case (t:Cap-Stack) - We have:

$\Gamma; \Delta_0, x : A_0 \vdash e : A_1 :: A_2 \dashv \Delta_1$ (1)

$x \in \mathtt{fv}(e)$ (2)

by hypothesis.

$\Gamma; \Delta_0 \vdash e : A_1 \dashv \Delta_1, A_2$ (3)

by inversion on (t:Cap-Stack) on (1).

$x \notin \Delta_1, A_2$ (4)

by induction hypothesis on (3) and (2).

$x \notin \Delta_1$ (5)

by (4).

Thus, we conclude.

Case (t:Cap-Unstack) - We have:

$\Gamma; \Delta_0, x : A_0 \vdash e : A_1 \dashv \Delta_1, A_2$ (1)

$x \in \mathtt{fv}(e)$ (2)

by hypothesis.

$\Gamma; \Delta_0, x : A_0 \vdash e : A_1 :: A_2 \dashv \Delta_1$ (3)

by inversion on (t:Cap-Unstack) with (1).

$x \notin \Delta$ (4)

by induction hypothesis with (3) and (2).

Thus, we conclude.

Case (T:Frame) - We have:

$\Gamma; (\Delta_0, x : A_0), \Delta_2 \vdash e : A \dashv \Delta_1, \Delta_2$ (1)
$x \in \mathtt{fv}(e)$ (2)
by hypothesis.
$\Gamma; \Delta_0, x : A_0 \vdash e : A \dashv \Delta_1$ (3)
by inversion on (T:Frame) with (1), note by (2) x must be in environment.
$x \notin \Delta_1$ (4)
by induction hypothesis.
$x \notin (\Delta_1, \Delta_2)$ (5)
since by (1) x cannot be in Δ_2.
Thus, we conclude.

Case (T:Subsumption) - We have:

$\Gamma; \Delta_0, x : A \vdash e : A_1 \dashv \Delta_1$ (1)
$x \in \mathtt{fv}(e)$ (2)
by hypothesis.
$\Delta_0, x : A <: \Delta'_0, x : A'$ (3)
$\Gamma; \Delta'_0 \vdash e : A_0 \dashv \Delta'_1$ (4)
$A_0 <: A_1$ (5)
$\Delta'_1 <: \Delta_1$ (6)
by inversion on (T:Subsumption) with (1).
$x \notin \Delta'_1$ (7)
by induction hypothesis on (2) and (4).
$x \notin \Delta_1$ (8)
by (6) and (7) noting the members of Δ_1 and Δ'_1 are the same.
Thus, we conclude.

Case (T:Tag) - We have:

$\Gamma; \Delta_0, x : A_0 \vdash \mathtt{l}\#v : A_1 \dashv \Delta_1$ (1)
$x \in \mathtt{fv}(\mathtt{l}\#v)$ (2)
by hypothesis.
$\Gamma; \Delta_0, x : A_0 \vdash v : A_1 \dashv \Delta_1$ (3)
by inversion on (T:Tag) with (1).
[$\mathtt{fv}(\mathtt{l}\#v) = \mathtt{fv}(v)$]
$x \in \mathtt{fv}(e)$ (4)
by definition of fv and (2).
$x \notin \Delta_1$ (5)
by induction hypothesis on (3) and (4).
Thus, we conclude.

Case (T:Case) - We have:

$\Gamma; \Delta_0, x : A' \vdash \mathsf{case}\ v\ \mathsf{of}\ \overline{\mathtt{l}_j\#x_j \rightarrow e_j}\ \mathsf{end} : A \dashv \Delta_1$ (1)
$x \in \mathtt{fv}(\mathsf{case}\ v\ \mathsf{of}\ \overline{\mathtt{l}_j\#x_j \rightarrow e_j}\ \mathsf{end})$ (2)
[$\mathtt{fv}(\mathsf{case}\ v\ \mathsf{of}\ \overline{\mathtt{l}_j\#x_j \rightarrow e_j}\ \mathsf{end}) = \mathtt{fv}(v) \cup \overline{\mathtt{fv}(e_i)}$], for some $i \leq j$
by hypothesis.
$\Gamma; \Delta_0, x : A' \vdash v : \sum_i \mathtt{l}_i\#A_i \dashv \Delta'$ (3)
$\overline{\Gamma; \Delta', x_i : A_i \vdash e_i : A \dashv \Delta_1}$ (4)
$i \leq j$ (5)
by inversion on (T:Case) with (1).

Therefore, we have the following possibilities:

1. $x \in \mathtt{fv}(v) \wedge x \notin \overline{\mathtt{fv}(e_i)}$
   $x \notin \Delta'$ (1.1)
   by induction hypothesis on (3) and case assumption.
   $x \notin \Delta_1$ (1.2)
   by (1.1) and (4).
   Thus, we conclude.
2. $x \notin \mathtt{fv}(v) \wedge x \in \overline{\mathtt{fv}(e_i)}$
   $(x : A') \in \Delta'$ (2.1)
   by $x \notin \mathtt{fv}(e)$.
   $x \notin \Delta_1$ (2.2)
   by induction hypothesis on (4) and (2.1).
   Thus, we conclude.
3. $x \in \mathtt{fv}(v) \wedge x \in \overline{\mathtt{fv}(e_i)}$
   $x \notin \Delta_1$ (3.1)
   by induction hypothesis on (3) and sub-case hypothesis.

   We reach a contradiction since v is well-typed by (4) but $x \in \overline{\mathtt{fv}(e_i)}$ contradicts (3.1). Thus, such case is impossible to occur in a well-typed expression.

Case (T:Alternative-Left) - We have:

$\Gamma; \Delta_0, x : A_0, A_1 \oplus A_2 \vdash e : A_3 \dashv \Delta_1$ (1)
$x \in \mathtt{fv}(e)$ (2)
by hypothesis.
$\Gamma; \Delta_0, x : A_0, A_1 \vdash e : A_3 \dashv \Delta_1$ (3)
$\Gamma; \Delta_0, x : A_0, A_2 \vdash e : A_3 \dashv \Delta_1$ (4)
by inversion on (T:Alternative-Left) with (1).
$x \notin \Delta_1$ (5)
by induction hypothesis with (2) and (3).
Thus, we conclude.

Case (T:Let) - We have:

$\Gamma; \Delta_0, x : A \vdash \mathsf{let}\ x_0 = e_0\ \mathsf{in}\ e_1\ \mathsf{end} : A_1 \dashv \Delta_2$ (1)
$x \in \mathtt{fv}(\mathsf{let}\ x_0 = e_0\ \mathsf{in}\ e_1\ \mathsf{end})$ (2)
[$\mathtt{fv}(\mathsf{let}\ x_0 = e_0\ \mathsf{in}\ e_1\ \mathsf{end}) = \mathtt{fv}(e_0) \cup \mathtt{fv}(e_1)$]
by hypothesis.
$\Gamma; \Delta_0, x : A \vdash e_0 : A_0 \dashv \Delta_1$ (3)
$\Gamma; \Delta_1, x_0 : A_0 \vdash e_1 : A_1 \dashv \Delta_2$ (4)
by inversion on (T:Let) with (1).

Therefore, we have the following possibilities:

1. $x \in \mathtt{fv}(e_1) \wedge x \notin \mathtt{fv}(e_0)$
   $(x : A) \in \Delta_1$ (1.1)
   by $x \notin \mathtt{fv}(e_0)$.
   $x \notin \Delta_2$ (1.2)
   by induction hypothesis on (4) with (1.1).
   Thus, we conclude.
2. $x \in \mathtt{fv}(e_0) \wedge x \in \mathtt{fv}(e_1)$
   $x \notin \Delta_1$ (2.1)
   by induction hypothesis on (3) and case assumption.

   We reach a contradiction since e_0 is well-typed by (4) but $x \in \mathtt{fv}(e_1)$ contradicts (2.1). Thus, such case is impossible to occur in a well-typed expression.
3. $x \in \mathtt{fv}(e_0) \wedge x \notin \mathtt{fv}(e_1)$
   $x \notin \Delta_1$ (3.1)
   by induction hypothesis on (3) and case assumption.
   $x \notin \Delta_2$ (3.2)
   by (3.1) and (4).
   Thus, we conclude.

□

B.7 Well-Form Lemmas

Lemma 6 (Well-Formed Type Substitution)**.** We have:

- For *location variables*:
  1. If
     $$\Gamma, t : \mathbf{loc}\ \ \mathbf{wf} \qquad \rho : \mathbf{loc} \in \Gamma$$
     then $\Gamma\{\rho/t\}\ \ \mathbf{wf}$.
  2. If
     $$\Gamma, t : \mathbf{loc} \vdash \Delta\ \ \mathbf{wf} \qquad \rho : \mathbf{loc} \in \Gamma$$
     then $\Gamma\{\rho/t\} \vdash \Delta\{\rho/t\}\ \ \mathbf{wf}$.
  3. If
     $$\Gamma, t : \mathbf{loc} \vdash A\ \ \mathbf{type} \qquad \rho : \mathbf{loc} \in \Gamma$$
     then $\Gamma\{\rho/t\} \vdash A\{\rho/t\}\ \ \mathbf{type}$.
- For *type variables*:
  1. If
     $$\Gamma, X\ \mathbf{type}\ \ \mathbf{wf} \qquad \Gamma \vdash A\ \mathbf{type}$$
     then $\Gamma\{A/X\}\ \ \mathbf{wf}$.
  2. If
     $$\Gamma, X\ \mathbf{type} \vdash \Delta\ \ \mathbf{wf} \qquad \Gamma \vdash A\ \mathbf{type}$$
     then $\Gamma\{A/X\} \vdash \Delta\{A/X\}\ \ \mathbf{wf}$.
  3. If
     $$\Gamma, X\ \mathbf{type} \vdash A\ \ \mathbf{type} \qquad \Gamma \vdash A'\ \mathbf{type}$$
     then $\Gamma\{A'/X\} \vdash A\{A'/X\}\ \ \mathbf{type}$.

Proof. Straightforward by induction on the structure of Γ, Δ and types. □

Lemma 7 (Well-Formed Subtyping)**.** We have two cases:

1. (Type) If $\Gamma \vdash A\ \ \mathbf{type}$ and $A <: A'$ then $\Gamma \vdash A'\ \ \mathbf{type}$.
2. (Delta) If $\Gamma \vdash \Delta\ \ \mathbf{wf}$ and $\Delta <: \Delta'$ then $\Gamma \vdash \Delta'\ \ \mathbf{wf}$.

Proof. Straightforward by induction on the definition of <: for types and Δ, respectively. □

B.8 Substitution Lemma

Lemma 8 (Substitution Lemma)**.** We have the following substitution properties for both *expression typing* and *type formation*:

1. (Linear) If
   $$\Gamma; \Delta_0 \vdash v : A_0 \dashv \Delta_1 \qquad \Gamma; \Delta_1, x : A_0 \vdash e : A_1 \dashv \Delta_2$$
   then
   $$\Gamma; \Delta_0 \vdash e\{v/x\} : A_1 \dashv \Delta_2$$
2. (Pure) If
   $$\Gamma; \cdot \vdash v : !A_0 \dashv \cdot \qquad \Gamma, x : A_0; \Delta_0 \vdash e : A_1 \dashv \Delta_1$$
   then
   $$\Gamma; \Delta_0 \vdash e\{v/x\} : A_1 \dashv \Delta_1$$
   (note that due to the required pure types, the Δ environments to check v must be empty)
3. (Location Variable) If
   $$\Gamma, t : \mathbf{loc}; \Delta_0 \vdash e : A \dashv \Delta_1 \qquad \rho : \mathbf{loc} \in \Gamma$$
   then
   $$\Gamma\{\rho/t\}; \Delta_0\{\rho/t\} \vdash e\{\rho/t\} : A\{\rho/t\} \dashv \Delta_1\{\rho/t\}$$
   Note that, since t may appear free in all typing environments, the expression and in its type, we must substitute into all those elements.
4. (Type Variable) If
   $$\Gamma, X\ \mathbf{type}; \Delta_0 \vdash e : A_0 \dashv \Delta_1 \qquad \Gamma \vdash A_1\ \mathbf{type}$$
   then
   $$\Gamma\{A_1/X\}; \Delta_0\{A_1/X\} \vdash e\{A_1/X\} : A_0\{A_1/X\} \dashv \Delta_1\{A_1/X\}$$
   (replaces X in all places it may occur free)

Proof. We split the proof on each of the lemma's sub-parts:

1. (Linear)

   Proof. We proceed by induction on the typing derivation of $\Gamma; \Delta_1, x : A_0 \vdash e : A_1 \dashv \Delta_2$.

   Case (t:Ref), (t:Pure), (t:Unit), (t:Pure-Read) - Not applicable since these rules require an empty Δ environment.

   Case (t:Linear-Read) - We have:

   $\Gamma; \Delta \vdash v : A \dashv \cdot$ (1)

   $\Gamma; x : A \vdash x : A \dashv \cdot$ (2)

   by hypothesis.

   (note v's ending environment must be $\cdot$ to apply (t:Linear-Read)).

   $\Gamma; \Delta \vdash x\{v/x\} : A \dashv \cdot$ (3)

   by (vs:2) with (1) and x.

   Thus, we conclude.

   Case (t:Pure-Elim) - We have:

   $\Gamma; \Delta_0 \vdash v : A_0 \dashv \Delta_1$ (1)

   $\Gamma; \Delta_1, x_1 : !A_2, x_0 : A_0 \vdash e : A_1 \dashv \Delta_2$ (2)

   by hypothesis.

   $\Gamma, x_1 : A_2; \Delta_1, x_0 : A_0 \vdash e : A_1 \dashv \Delta_2$ (3)

   by inversion on (t:Pure-Elim) with (2).

   $\Gamma, x_1 : A_2; \Delta_1 \vdash e\{v/x_0\} : A_1 \dashv \Delta_2$ (4)

   by induction hypothesis on (3) with (1).

   $\Gamma; \Delta_1, x_1 : !A_2 \vdash e\{v/x_0\} : A_1 \dashv \Delta_2$ (5)

   by (t:Pure-Elim) with (4).

   Thus, we conclude.

   Case (t:New) - We have:

   $\Gamma; \Delta_0 \vdash v : A_0 \dashv \Delta_1$ (1)

   $\Gamma; \Delta_1, x : A_0 \vdash \mathsf{new}\ v_0 : \exists t.(\mathbf{ref}\ t :: \mathbf{rw}\ t\ A_1) \dashv \Delta_2$ (2)

   by hypothesis.

   $\Gamma; \Delta_1, x : A_0 \vdash v_0 : A_1 \dashv \Delta_2$ (3)

   by inversion on (t:New) with (2).

   $\Gamma; \Delta_0 \vdash v_0\{v/x\} : A_1 \dashv \Delta_2$ (4)

   by induction hypothesis with (1) and (3).

   $\Gamma; \Delta_0 \vdash \mathsf{new}\ v_0\{v/x\} : \exists t.(\mathbf{ref}\ t :: \mathbf{rw}\ t\ A_1) \dashv \Delta_2$ (5)

by (t:New) with (4).

$\Gamma; \Delta_0 \vdash (\mathsf{new}\ v_0)\{v/x\} : \exists t.(\mathbf{ref}\ t :: \mathbf{rw}\ t\ A_1) \dashv \Delta_2$ (6)

by (vs:8) with (5).

Thus, we conclude.

Case (t:Delete) - We have:

$\Gamma; \Delta_0 \vdash v : A_0 \dashv \Delta_1$ (1)

$\Gamma; \Delta_1, x : A_0 \vdash \mathsf{delete}\ v_0 : \exists t.A_1 \dashv \Delta_2$ (2)

by hypothesis.

$\Gamma; \Delta_1, x : A_0 \vdash v_0 : \exists t.(\mathbf{ref}\ t :: \mathbf{rw}\ t\ A_1) \dashv \Delta_2$ (3)

by inversion on (t:Delete) with (2).

$\Gamma; \Delta_0 \vdash v_0\{v/x\} : \exists t.(\mathbf{ref}\ t :: \mathbf{rw}\ t\ A_1) \dashv \Delta_2$ (4)

by induction hypothesis with (1) and (3).

$\Gamma; \Delta_0 \vdash \mathsf{delete}\ v_0\{v/x\} : \exists t.A_1 \dashv \Delta_2$ (5)

by (t:Delete) with (4).

$\Gamma; \Delta_0 \vdash (\mathsf{delete}\ v_0)\{v/x\} : \exists t.A_1 \dashv \Delta_2$ (6)

by (vs:9) with (5).

Thus, we conclude.

Case (t:Assign) - We have:

$\Gamma; \Delta_0 \vdash v : A_0 \dashv \Delta_1$ (1)

$\Gamma; \Delta_1, x : A_0 \vdash v_0 := v_1 : A_1 \dashv \Delta_2, \mathbf{rw}\ p\ A_2$ (2)

by hypothesis.

$\Gamma; \Delta_1, x : A_0 \vdash v_1 : A_2 \dashv \Delta'$ (3)

$\Gamma; \Delta' \vdash v_0 : \mathbf{ref}\ p \dashv \Delta_2, \mathbf{rw}\ p\ A_1$ (4)

by inversion on (t:Assign) with (2).

We have that either:

(a) $x \in \mathtt{fv}(v_1)$

$x \notin \Delta'$ (1.1)

by (Free Variables Lemma) on (3).

$\Gamma; \Delta' \vdash v_0\{v/x\} : \mathbf{ref}\ p \dashv \Delta_2, \mathbf{rw}\ p\ A_1$ (1.2)

since x cannot occur in e_0 by (1.1).

$\Gamma; \Delta_1 \vdash v_1\{v/x\} : A_2 \dashv \Delta'$ (1.3)

by induction hypothesis on (1) and (3).

$\Gamma; \Delta_1 \vdash v_0\{v/x\} := v_1\{v/x\} : A_1 \dashv \Delta_2, \mathbf{rw}\ p\ A_2$ (1.4)

by (t:Assign) on (1.2) and (1.3).

$\Gamma; \Delta_1 \vdash (v_0 := v_1)\{v/x\} : A_1 \dashv \Delta_2, \mathbf{rw}\ p\ A_2$ (1.5)

by (vs:11) on (1.4).

Thus, we conclude.

(b) $x \notin \mathtt{fv}(v_1)$

$(x : A_0) \in \Delta'$ (2.1)

by (9) and $x \notin \mathtt{fv}(v_1)$.

$\Gamma; \Delta'' \vdash v_0\{v/x\} : \mathbf{ref}\ p \dashv \Delta_2, \mathbf{rw}\ p\ A_1$ (2.2)

by induction hypothesis (since it is applied to x wherever is in the environment) and where Δ'' is the same as Δ' without x.

$\Gamma; \Delta_1 \vdash v_1\{v/x\} : A_2 \dashv \Delta''$ (2.3)

since x cannot occur in e_1 by $x \notin \mathtt{fv}(e_1)$.

$\Gamma; \Delta_1 \vdash v_0\{v/x\} := v_1\{v/x\} : A_1 \dashv \Delta_2, \mathbf{rw}\ p\ A_2$ (2.4)

by (t:Assign) using (2.4) and (2.5).

$\Gamma; \Delta_1 \vdash (v_0 := v_1)\{v/x\} : A_1 \dashv \Delta_2, \mathbf{rw}\ p\ A_2$ (2.5)

by (vs:11) on (2.6).

Thus, we conclude.

Case (t:Dereference-Linear) - We have:

$\Gamma; \Delta_0 \vdash v : A_0 \dashv \Delta_1$ (1)

$\Gamma; \Delta_1, x : A_0 \vdash !v_0 : A_1 \dashv \Delta_2, \mathbf{rw}\ p\ []$ (2)

by hypothesis.

$\Gamma; \Delta_1, x : A_0 \vdash v_0 : \mathbf{ref}\ p \dashv \Delta_2, \mathbf{rw}\ p\ A_1$ (3)

by inversion on (t:Dereference-Linear) on (2).

$\Gamma; \Delta_1 \vdash v_0\{v/x\} : \mathbf{ref}\ p \dashv \Delta_2, \mathbf{rw}\ p\ A_1$ (4)

by induction hypothesis with (1) and (3).

$\Gamma; \Delta_1 \vdash !v_0\{v/x\} : A_1 \dashv \Delta_2, \mathbf{rw}\ p\ []$ (5)

by (t:Dereference-Linear) on (4).

$\Gamma; \Delta_1 \vdash (!v_0)\{v/x\} : A_1 \dashv \Delta_2, \mathbf{rw}\ p\ []$ (6)

by (vs:10) on (5).

Thus, we conclude.

Case (t:Dereference-Pure) - Analogous to (t:Dereference-Linear).

Case (t:Record) - We have:

$\Gamma; \Delta_0 \vdash v : A_0 \dashv \Delta_1$ (1)

$\Gamma; \Delta_1, x : A_0 \vdash \{\overline{\mathtt{f} = v'}\} : [\overline{\mathtt{f} : A}] \dashv \cdot$ (2)

by hypothesis.

$\overline{\Gamma; \Delta_1, x : A_0 \vdash v'_i : A_i \dashv \cdot}$ (3)

by inversion with (t:Record) on (2).

$\overline{\Gamma; \Delta_1 \vdash v'_i\{v/x\} : A_i \dashv \cdot}$ (4)

by induction hypothesis with (1) and (3).

$\Gamma; \Delta_1 \vdash \{\overline{\mathtt{f} = v'\{v/x\}}\} : [\overline{\mathtt{f} : A}] \dashv \cdot$ (5)

by (t:Record) on (4).

$\Gamma; \Delta_1 \vdash (\{\overline{\mathtt{f} = v'}\})\{v/x\} : [\overline{\mathtt{f} : A}] \dashv \cdot$ (6)

by (vs:5) on (5).

Thus, we conclude.

Case (t:Selection) - We have:

$\Gamma; \Delta_0 \vdash v : A_0 \dashv \Delta_1$ (1)

$\Gamma; \Delta_1, x : A_0 \vdash v_0.\mathtt{f} : A_1 \dashv \Delta_2$ (2)

by hypothesis.

$\Gamma; \Delta_1, x : A_0 \vdash v_0 : [\mathtt{f} : A_1] \dashv \Delta_2$ (3)

by inversion on (t:Selection) with (2).

$\Gamma; \Delta_1 \vdash v_0\{v/x\} : [\mathtt{f} : A_1] \dashv \Delta_2$ (4)

by induction hypothesis on (3) with (1).

$\Gamma; \Delta_1 \vdash v_0\{v/x\}.\mathtt{f} : [\mathtt{f} : A_1] \dashv \Delta_2$ (5)

by (t:Selection) on (4).

$\Gamma; \Delta_1 \vdash (v_0.\mathtt{f})\{v/x\} : [\mathtt{f} : A_1] \dashv \Delta_2$ (6)

by (vs:6) on (5).

Thus, we conclude.

Case (t:Application) - We have:

$\Gamma; \Delta_0 \vdash v : A_0 \dashv \Delta_1$ (1)

$\Gamma; \Delta_1, x_0 : A_0 \vdash v_0\ v_1 : A_1 \dashv \Delta_2$ (2)

by hypothesis.

$\Gamma; \Delta_0, x_0 : A_0 \vdash v_1 : A_2 \dashv \Delta'$ (3)

$\Gamma; \Delta' \vdash v_0 : A_2 \multimap A_1 \dashv \Delta_2$ (4)

by inversion on (t:Application) with (2).

We have that either:

(a) $x \in \mathtt{fv}(v_1)$

$x \notin \Delta'$ (1.1)

by (Free Variables Lemma) on (3).

$\Gamma; \Delta' \vdash v_0\{v/x\} : A_2 \multimap A_1 \dashv \Delta_2$ (1.2)

since x cannot occur in e_0 by (1.1).

$\Gamma; \Delta_0 \vdash v_1\{v/x\} : A_2 \dashv \Delta'$ (1.3)

by induction hypothesis with (1) and (3).

$\Gamma; \Delta_0 \vdash v_0\{v/x\}\ v_1\{v/x\} : A_1 \dashv \Delta_2$ (1.4)

by (t:Application) with (1.2) and (1.3).

$\Gamma; \Delta_0 \vdash (v_0\ v_1)\{v/x\} : A_1 \dashv \Delta_2$ (1.5)

by (vs:7) on (1.4).

Thus, we conclude.

(b) $x \notin \mathtt{fv}(v_1)$

$(x : A_0) \in \Delta'$ (2.1)

by $x \notin \mathtt{fv}(v_1)$.

$\Gamma; \Delta'' \vdash v_0\{v/x\} : A_2 \multimap A_1 \dashv \Delta_2$ (2.2)

by induction hypothesis where Δ'' is Δ' without x.

$\Gamma; \Delta_0 \vdash v_1\{v/x\} : A_2 \dashv \Delta''$ (2.3)

since x cannot occur in v_1 by $x \notin \mathtt{fv}(v_1)$ and (2.1).

$\Gamma; \Delta_0 \vdash v_0\{v/x\}\ v_1\{v/x\} : A_1 \dashv \Delta_2$ (2.4)

by (t:Application) on (2.2) and (2.3).

$\Gamma; \Delta_0 \vdash (v_0\ v_1)\{v/x\} : A_1 \dashv \Delta_2$ (2.5)

by (vs:7) on (2.4).

Thus, we conclude.

Case (t:Function) - We have:

$\Gamma; \Delta_0 \vdash v : A_0 \dashv \Delta_1$ (1)

$\Gamma; \Delta_1, x_0 : A_0 \vdash \mathsf{fun}(x_1 : A_1).e : A_1 \multimap A_2 \dashv \cdot$ (2)

by hypothesis.

$\Gamma; \Delta_1, x_1 : A_1, x_0 : A_0 \vdash e : A_2 \dashv \cdot$ (3)

$x_1 \neq x_0$ (4)

by def. of substitution up to rename of bounded variables.

$\Gamma; \Delta_1, x_1 : A_1 \vdash e\{v/x\} : A_2 \dashv \cdot$ (5)

by induction hypothesis with (1) and (3).

$\Gamma; \Delta_1 \vdash \mathsf{fun}(x_1 : A_1).e\{v/x\} : A_1 \multimap A_2 \dashv \cdot$ (6)

by (t:Function) with (5).

$\Gamma; \Delta_1 \vdash (\mathsf{fun}(x_1 : A_1).e)\{v/x\} : A_1 \multimap A_2 \dashv \cdot$ (7)

by (vs:4) on (6) and (4).

Thus, we conclude.

Case (t:Forall-Loc) - We have:

$\Gamma; \Delta_0 \vdash v : A_0 \dashv \Delta_1$ (1)

$\Gamma; \Delta_1, x : A_0 \vdash \langle t \rangle\, e : \forall t.A_1 \dashv \cdot$ (2)

by hypothesis.

$\Gamma, t : \mathbf{loc}; \Delta_1, x : A_0 \vdash e : A_1 \dashv \cdot$ (3)

by inversion on (t:Forall-Loc) with (2).

$\Gamma, t : \mathbf{loc}; \Delta_1 \vdash e\{v/x\} : A_1 \dashv \cdot$ (4)

by induction hypothesis with (1) and (3).

$\Gamma; \Delta_1 \vdash \langle t \rangle\, e\{v/x\} : \forall t.A_1 \dashv \cdot$ (5)

by (t:Forall-Loc) on (4).

$\Gamma; \Delta_1 \vdash (\langle t \rangle\, e)\{v/x\} : \forall t.A_1 \dashv \cdot$ (6)

by (vs:14) on (5).

Thus, we conclude.

Case (T:Loc-App) - We have:

$\Gamma; \Delta_0 \vdash v : A_0 \dashv \Delta_1$ (1)

$\Gamma; \Delta_1, x : A_0 \vdash v_0[p] : A_1\{p/t\} \dashv \Delta_2$ (2)

by hypothesis.

$p : \mathbf{loc} \in \Gamma$ (3)

$\Gamma; \Delta_1, x : A_0 \vdash v_0 : \forall t.A_1 \dashv \Delta_2$ (4)

by inversion on (T:Loc-App) with (2).

$\Gamma; \Delta_1 \vdash v_0\{v/x\} : \forall t.A_1 \dashv \Delta_2$ (5)

by induction hypothesis on (4) and (1).

$\Gamma; \Delta_1 \vdash v_0\{v/x\}[p] : A_1\{p/t\} \dashv \Delta_2$ (6)

by (T:Loc-App) on (5) and (3).

$\Gamma; \Delta_1 \vdash (v_0[p])\{v/x\} : A_1\{p/t\} \dashv \Delta_2$ (7)

by (vs:13) on (6).

Thus, we conclude.

Case (T:Loc-Pack) - We have:

$\Gamma; \Delta_0 \vdash v : A_0 \dashv \Delta_1$ (1)

$\Gamma; \Delta_1, x : A_0 \vdash \langle p, v_0\rangle : \exists t.A_1 \dashv \Delta_2$ (2)

by hypothesis.

$\Gamma; \Delta_1, x : A_0 \vdash v_0 : \; A_1\{p/t\} \dashv \Delta_2$ (3)

by inversion on (T:Loc-Pack) with (2).

$\Gamma; \Delta_1 \vdash v_0\{v/x\} : \; A_1\{p/t\} \dashv \Delta_2$ (4)

by induction hypothesis on (1) and (3).

$\Gamma; \Delta_1 \vdash \langle p, v_0\{v/x\}\rangle : \exists t.A_1 \dashv \Delta_2$ (5)

by (T:Loc-Pack) on (4).

$\Gamma; \Delta_1 \vdash (\langle p, v_0\rangle)\{v/x\} : \exists t.A_1 \dashv \Delta_2$ (6)

by (vs:12) on (5).

Thus, we conclude.

Case (T:Loc-Open) - We have:

$\Gamma; \Delta_0 \vdash v : A_0 \dashv \Delta_1$ (1)

$\Gamma; \Delta_1, x_0 : A_0 \vdash \mathsf{open}\,\langle t, x_1\rangle = v_0 \;\mathsf{in}\; e_1 \;\mathsf{end} : A_1 \dashv \Delta_2$ (2)

by hypothesis.

$\Gamma; \Delta_1, x_0 : A_0 \vdash v_0 : \exists t.A_2 \dashv \Delta'$ (3)

$\Gamma, t : \mathbf{loc}; \Delta', x_1 : A_2 \vdash e_1 : A_1 \dashv \Delta_2$ (4)

by inversion on (T:Loc-Open) with (2).

We have that either:

(a) $x_0 \in \mathtt{fv}(v_0)$

$x_0 \notin \Delta'$ (1.1)

by (Free Variables Lemma) on (3).

$x_0 \neq x_1$ (1.2)

by def. of substitution up to rename of bounded variables.

$\Gamma, t : \mathbf{loc}; \Delta', x_1 : A_2 \vdash e_1\{v/x_0\} : A_1 \dashv \Delta_2$ (1.3)

since x_0 cannot occur in e_1 and by (1.1) nor in Γ by (3).

$\Gamma; \Delta_1 \vdash v_0\{v/x_0\} : \exists t.A_2 \dashv \Delta'$ (1.4)

by induction hypothesis on (1) and (3).

$\Gamma; \Delta_1 \vdash \mathsf{open}\,\langle t, x_1\rangle = v_0\{v/x_0\} \;\mathsf{in}\; e_1\{v/x_0\} \;\mathsf{end} : A_1 \dashv \Delta_2$ (1.5)

by (T:Loc-Open) on (1.3) and (1.4).

$\Gamma; \Delta_1 \vdash (\mathsf{open}\,\langle t, x_1\rangle = v_0 \;\mathsf{in}\; e_1 \;\mathsf{end})\{v/x_0\} : A_1 \dashv \Delta_2$ (1.6)

by (vs:15) on (1.6) and (1.2).

Thus, we conclude.

(b) $x_0 \notin \mathtt{fv}(v_0)$

$(x_0 : A_0) \in \Delta'$ (2.1)

by $x_0 \notin \mathtt{fv}(v_0)$.

$x_0 \neq x_1$ (2.2)

by def. of substitution up to rename of bounded variables.

$\Gamma, t : \mathbf{loc}; \Delta'', x_1 : A_2 \vdash e_1\{v/x_0\} : A_1 \dashv \Delta_2$ (2.3)

by induction hypothesis with Δ'' equal to Δ' without x_0.

$\Gamma; \Delta_1 \vdash v_0\{v/x_0\} : \exists t.A_2 \dashv \Delta''$ (2.4)

since x_0 cannot occur in v_0 by $x_0 \notin \mathtt{fv}(v_0)$.

$\Gamma; \Delta_1 \vdash \mathsf{open}\,\langle t, x_1\rangle = v_0\{v/x_0\} \;\mathsf{in}\; e_1\{v/x_0\} \;\mathsf{end} : A_1 \dashv \Delta_2$ (2.5)

by (T:Loc-Open) on (2.3) and (2.4).

$\Gamma; \Delta_1 \vdash (\mathsf{open}\,\langle t, x_1\rangle = v_0 \;\mathsf{in}\; e_1 \;\mathsf{end})\{v/x_0\} : A_1 \dashv \Delta_2$ (2.6)

by (vs:15) on (2.2) and (2.5).

Thus, we conclude.

Case (T:Forall-Type) - Analogous to (T:Forall-Loc) with (vs:18).

Case (T:Type-App) - Analogous to (T:Loc-App) with (vs:17).

Case (T:Type-Pack) - Analogous to (T:Loc-Pack) with (vs:16).

Case (T:Type-Open) - Analogous to (T:Loc-Open) with (vs:19).

Case (T:Cap-Elim) - We have:

$\Gamma; \Delta_0 \vdash v : A_0 \dashv \Delta_1, x_1 : A_2 :: A_3$ (1)

$\Gamma; \Delta_1, x_1 : A_2 :: A_3, x_0 : A_0 \vdash e : A_1 \dashv \Delta_2$ (2)

by hypothesis.

$\Gamma; \Delta_1, x_1 : A_2, A_3, x_0 : A_0 \vdash e : A_1 \dashv \Delta_2$ (3)

by inversion on (T:Cap-Elim) with (2).

$\Gamma; \Delta_1, x_1 : A_2, A_3 \vdash e\{v/x_0\} : A_1 \dashv \Delta_2$ (4)

by induction hypothesis with (1) and (3).

$\Gamma; \Delta_1, x_1 : A_2 :: A_3 \vdash e\{v/x_0\} : A_1 \dashv \Delta_2$ (5)

by (T:Cap-Elim) with (4).

Thus, we conclude.

Case (T:Cap-Stack) - We have:

$\Gamma; \Delta_0 \vdash v : A_0 \dashv \Delta_1$ (1)

$\Gamma; \Delta_1, x : A_0 \vdash e : A_1 :: A_2 \dashv \Delta_2$ (2)

by hypothesis.

$\Gamma; \Delta_1, x : A_0 \vdash e : A_1 \dashv \Delta_2, A_2$ (3)

by inversion on (T:Cap-Stack) with (2).

$\Gamma; \Delta_1 \vdash e\{v/x\} : A_1 \dashv \Delta_2, A_2$ (4)

by induction hypothesis with (1) and (3).

$\Gamma; \Delta_1 \vdash e\{v/x\} : A_1 :: A_2 \dashv \Delta_2$ (5)

by (T:Cap-Stack) on (4).

Thus, we conclude.

Case (T:Cap-Unstack) - We have:

$\Gamma; \Delta_0 \vdash v : A_0 \dashv \Delta_1$ (1)

$\Gamma; \Delta_1, x : A_0 \vdash e : A_1 \dashv \Delta_2, A_2$ (2)

by hypothesis.

$\Gamma; \Delta_1, x : A_0 \vdash e : A_1 :: A_2 \dashv \Delta_2$ (3)

by inversion (T:Cap-Unstack) with (2).

$\Gamma; \Delta_1 \vdash e\{v/x\} : A_1 :: A_2 \dashv \Delta_2$ (4)

by induction hypothesis with (1) and (3).

$\Gamma; \Delta_1 \vdash e\{v/x\} : A_1 \dashv \Delta_2, A_2$ (5)

by (T:Cap-Unstack) with (4).

Thus, we conclude.

Case (T:Subsumption) - We have:

$\Gamma; \Delta_0 \vdash v : A_0 \dashv \Delta_1$ (1)

$\Gamma; \Delta_1, x : A_0 \vdash e : A_1 \dashv \Delta_2$ (2)

by hypothesis.

$\Delta_1, x : A_0 <: \Delta'_1, x : A'_0$ (3)

$\Gamma; \Delta'_1, x : A'_0 \vdash e : A_2 \dashv \Delta'_2$ (4)

$A_2 <: A_1$ (5)

$\Delta'_2 <: \Delta_2$ (6)

by inversion on (T:Subsumption) on (2).

$A_0 <: A'_0$ (7)

by (Subtyping Inversion Lemma) on (3) on x.

$\Gamma; \Delta_0 \vdash v : A'_0 \dashv \Delta'_1$ (8)

by (T:Subsumption) on (1) with (7).

$\Gamma; \Delta'_1 \vdash e\{v/x\} : A_2 \dashv \Delta'_2$ (9)

by induction hypothesis on (4) and (8).

$\Delta_1 <: \Delta'_1$ (10)

by (Subtyping Inversion Lemma) on (3).

$\Gamma; \Delta_1 \vdash e\{v/x\} : A_1 \dashv \Delta_2$ (11)

by (T:Subsumption) on (9) with (10), (5) and (6).

Thus, we conclude.

Case (T:Frame) - We have:

$\Gamma; \Delta_0 \vdash v : A_0 \dashv \Delta_1$ (1)

$\Gamma; (\Delta_1, x : A_0), \Delta_3 \vdash e : A_1 \dashv \Delta_2, \Delta_3$ (2)

by hypothesis.

$\Gamma; \Delta_1, x : A_0 \vdash e : A_1 \dashv \Delta_2$ (3)

by inversion on (T:Frame) with (2).

$\Gamma; \Delta_1 \vdash e\{v/x\} : A_1 \dashv \Delta_2$ (4)

by induction hypothesis with (1) and (3).

$\Gamma; \Delta_1, \Delta_3 \vdash e\{v/x\} : A_1 \dashv \Delta_2, \Delta_3$ (5)

by (T:Frame) on (4) with Δ_3.

Thus, we conclude.

Case (T:Tag) - We have:

$\Gamma; \Delta_0 \vdash v : A_0 \dashv \Delta_1$ (1)

$\Gamma; \Delta_1, x : A_0 \vdash \mathtt{l}\#v_0 : \mathtt{l}\#A_1 \dashv \Delta_2$ (2)

by hypothesis.

$\Gamma; \Delta_1, x : A_0 \vdash v_0 : A_1 \dashv \Delta_2$ (3)

by inversion (T:Tag) with (2).

$\Gamma; \Delta_1 \vdash v_0\{v/x\} : A_1 \dashv \Delta_2$ (4)

by induction hypothesis with (1) and (3).

$\Gamma; \Delta_1 \vdash \mathtt{l}\#v_0\{v/x\} : \mathtt{l}\#A_1 \dashv \Delta_2$ (5)

by (T:Tag) with (4).

$\Gamma; \Delta_1 \vdash (\mathtt{l}\#v_0)\{v/x\} : \mathtt{l}\#A_1 \dashv \Delta_2$ (6)

by (vs:20) on (5).

Thus, we conclude.

Case (T:Case) - We have:

$\Gamma; \Delta_0 \vdash v : A_0 \dashv \Delta_1$ (1)

$\Gamma ; \Delta_1 , x : A_0 \vdash \mathsf{case}\ v_0\ \mathsf{of}\ \overline{\mathtt{l}_j\#x_j \to e_j}\ \mathsf{end} : A \dashv \Delta_2$ (2)

by hypothesis.

$\Gamma ; \Delta_1 , x : A_0 \vdash v_0 : \sum_i \mathtt{l}_i\#A'_i \dashv \Delta'$ (3)

$\overline{\Gamma ; \Delta' , x_i : A'_i \vdash e_i : A \dashv \Delta_2}$ (4)

$i \leq j$ (5)

by inversion (t:Case) with (2).

We have that either:

(a) $x \in \mathtt{fv}(v_0)$

$x \notin \Delta'$ (1.1)

by (Free Variables Lemma) on (3).

$\overline{x \neq x_j}$ (1.2)

by def. of substitution up to rename of bounded variables.

$\overline{\Gamma ; \Delta' , x_i : A'_i \vdash e_i\{v/x\} : A \dashv \Delta_2}$ (1.3)

since x cannot occur in e_i and by (1.1) nor in Γ by (3).

$\Gamma ; \Delta_1 , x : A_0 \vdash v_0\{v/x\} : \sum_i \mathtt{l}_i\#A'_i \dashv \Delta'$ (1.4)

by induction hypothesis on (1) and (3).

$\Gamma ; \Delta_1 \vdash \mathsf{case}\ v_0\{v/x\}\ \mathsf{of}\ \overline{\mathtt{l}_j\#x_j \to e_j\{v/x\}}\ \mathsf{end} : A \dashv \Delta_2$ (1.5)

by (t:Case) on (5), (1.3) and (1.4).

$\Gamma ; \Delta_1 \vdash (\mathsf{case}\ v_0\ \mathsf{of}\ \overline{\mathtt{l}_j\#x_j \to e_j}\ \mathsf{end})\{v/x\} : A \dashv \Delta_2$ (1.6)

by (vs:21) on (1.6) and (1.2).

Thus, we conclude.

(b) $x \notin \mathtt{fv}(v_0)$

$(x : A_0) \in \Delta'$ (2.1)

by $x \notin \mathtt{fv}(e)$.

$\overline{x \neq x_j}$ (2.2)

by def. of substitution up to rename of bounded variables.

$\overline{\Gamma ; \Delta'' , x_i : A'_i \vdash e_i\{v/x\} : A \dashv \Delta_2}$ (2.3)

by induction hypothesis where Δ'' is same as Δ' without x.

$\Gamma ; \Delta_1 \vdash v_0\{v/x\} : \sum_i \mathtt{l}_i\#A'_i \dashv \Delta''$ (2.4)

since x cannot occur in e by $x \notin \mathtt{fv}(e)$.

$\Gamma ; \Delta_1 \vdash \mathsf{case}\ v_0\{v/x\}\ \mathsf{of}\ \overline{\mathtt{l}_j\#x_j \to e_j\{v/x\}}\ \mathsf{end} : A \dashv \Delta_2$ (2.5)

by (t:Case) on (5), (2.3) and (2.4).

$\Gamma ; \Delta_1 \vdash (\mathsf{case}\ v_0\ \mathsf{of}\ \overline{\mathtt{l}_j\#x_j \to e_j}\ \mathsf{end})\{v/x\} : A \dashv \Delta_2$ (2.6)

by (vs:21) on (2.1) and (2.5).

Thus, we conclude.

Case (t:Alternative-Left) - Immediate by applying the induction hypothesis on the inversion and then re-applying the rule.

Case (t:Let) - Analogous to previous cases.

□

2. (Pure)

Proof. We proceed by induction on the typing derivation of $\Gamma , x : A_0 ; \Delta_0 \vdash e : A_1 \dashv \Delta_1$.

Case (t:Ref) - We have:

$\Gamma , \rho : \mathbf{loc} ; \cdot \vdash v_0 : !A_0 \dashv \cdot$ (1)

$\Gamma , \rho : \mathbf{loc} , x : A_0 ; \cdot \vdash \rho : \mathbf{ref}\,\rho \dashv \cdot$ (2)

by hypothesis.

$\Gamma , \rho : \mathbf{loc} ; \cdot \vdash \rho : \mathbf{ref}\,\rho \dashv \cdot$ (3)

by $x \notin \mathtt{fv}(\rho)$ on (2).

$\Gamma , \rho : \mathbf{loc} ; \cdot \vdash \rho\{v/x\} : \mathbf{ref}\,\rho \dashv \cdot$ (4)

by (vs:1) on (3) using x and v.

Thus, we conclude.

Case (t:Pure) - We have:

$\Gamma ; \cdot \vdash v_0 : !A_0 \dashv \cdot$ (1)

$\Gamma , x_0 : A_0 ; \cdot \vdash v_1 : !A_1 \dashv \cdot$ (2)

by hypothesis.

$\Gamma , x_0 : A_0 ; \cdot \vdash v_1 : A_1 \dashv \cdot$ (3)

by inversion on (t:Pure) with (2).

$\Gamma ; x_0 : !A_0 \vdash v_1 : A_1 \dashv \cdot$ (4)

by (t:Pure-Elim) on (3) with x_0.

$\Gamma ; \cdot \vdash v_1\{v_0/x_0\} : A_1 \dashv \cdot$ (5)

by (Substitution Lemma - Linear) with (1) and (4).

$\Gamma ; \cdot \vdash v_1\{v_0/x_0\} : !A_1 \dashv \cdot$ (6)

by (t:Pure) on (5).

Thus, we conclude.

Case (t:Unit) - We have:

$\Gamma ; \cdot \vdash v_0 : !A_0 \dashv \cdot$ (1)

$\Gamma , x : A_0 ; \cdot \vdash v_1 : [] \dashv \cdot$ (2)

by hypothesis.

$\Gamma ; \cdot \vdash v_1\{v_0/x\} : [] \dashv \cdot$ (3)

substitution on x cannot change the type since [] is always valid by (t:Unit).
(and substitution cannot change a value to become an expression).

Thus, we conclude.

Case (t:Pure-Read) - We have:

$\Gamma ; \cdot \vdash v : !A_0 \dashv \cdot$ (1)

$\Gamma , x_0 : A_0 ; \cdot \vdash x_1 : !A_1 \dashv \cdot$ (2)

by hypothesis (matching environments and type with (t:Pure-Read)).

We have that either:

(a) $x_0 = x_1$

$\Gamma ; \cdot \vdash v : !A \dashv \cdot$ (1.1)

$\Gamma , x : A ; \cdot \vdash x : !A \dashv \cdot$ (1.2)

by restated hypothesis with $x = x_0 = x_1$.
and with $A = A_0 = A_1$.

$\Gamma ; \cdot \vdash x\{v/x\} : !A \dashv \cdot$ (1.3)

by (vs:2) on (1.1) using x and v.

Thus, we conclude.

(b) $x_0 \neq x_1$

$\Gamma ; \cdot \vdash x_1 : !A_1 \dashv \cdot$ (2.1)

by $x_0 \notin \mathtt{fv}(x_1)$ on (2).

$\Gamma ; \cdot \vdash x_1\{v/x_0\} : !A_1 \dashv \cdot$ (2.2)

by (vs:3) on (2.1) using x_0 and v.

Thus, we conclude.

Case (t:Linear-Read) - We have:

$\Gamma ; \cdot \vdash v : !A_0 \dashv \cdot$ (1)

$\Gamma , x_0 : A_0 ; x_1 : A_1 \vdash x_1 : A_1 \dashv \cdot$ (2)

by hypothesis.

$x_0 \neq x_1$ (3)

since Γ and Δ identifiers cannot collide.

$\Gamma ; x_1 : A_1 \vdash x_1\{v/x_0\} : A_1 \dashv \cdot$ (4)

by (vs:3) on (2) using x_0 and v.

Thus, we conclude.

Case (t:Pure-Elim) - We have:

$\Gamma ; \cdot \vdash v : !A_0 \dashv \cdot$ (1)

$\Gamma , x_0 : A_0 ; \Delta_0 , x_1 : !A_2 \vdash e : A_1 \dashv \Delta_1$ (2)

by hypothesis.

$\Gamma , x_0 : A_0 , x_1 : A_2 ; \Delta_0 \vdash e : A_1 \dashv \Delta_1$ (3)

by inversion on (t:Pure-Elim) with (2)

$\Gamma , x_1 : A_2 ; \Delta_0 \vdash e\{v/x_0\} : A_1 \dashv \Delta_1$ (4)

by induction hypothesis on (1) with (3).

$\Gamma ; \Delta_0 , x_1 : !A_2 \vdash e\{v/x_0\} : A_1 \dashv \Delta_1$ (5)

by (t:Pure-Elim) on (4).

Thus, we conclude.

Case (t:New) - We have:

$\Gamma ; \cdot \vdash v : !A_0 \dashv \cdot$ (1)

$\Gamma , x : A_0 ; \Delta_0 \vdash \mathsf{new}\ v_0 : \exists t.(\mathbf{ref}\ t :: \mathbf{rw}\ t\ A_1) \dashv \Delta_1$ (2)

by hypothesis.

$\Gamma , x : A_0 ; \Delta_0 \vdash v_0 : A_1 \dashv \Delta_1$ (3)

by inversion on (t:New) with (2).

$\Gamma ; \Delta_0 \vdash v_0\{v/x\} : A_1 \dashv \Delta_1$ (4)

by induction hypothesis with (3) and (1).

$\Gamma ; \Delta_0 \vdash \mathsf{new}\ v_0\{v/x\} : \exists t.(\mathbf{ref}\ t :: \mathbf{rw}\ t\ A_1) \dashv \Delta_1$ (5)

by (t:New) with (4).

$\Gamma ; \Delta_0 \vdash (\mathsf{new}\ v_0)\{v/x\} : \exists t.(\mathbf{ref}\ t :: \mathbf{rw}\ t\ A_1) \dashv \Delta_1$ (6)

by (vs:8) on (5).

Thus, we conclude.

Case (t:Delete) - We have:

$\Gamma ; \cdot \vdash v : !A_0 \dashv \cdot$ (1)

$\Gamma , x : A_0 ; \Delta_0 \vdash \mathsf{delete}\ v_0 : \exists t.A_1 \dashv \Delta_1$ (2)

by hypothesis.

$\Gamma , x : A_0 ; \Delta_0 \vdash v_0 : \exists t.(\mathbf{ref}\ t :: \mathbf{rw}\ t\ A_1) \dashv \Delta_1$ (3)

by inversion on (t:Delete) with (2).

$\Gamma ; \Delta_0 \vdash v_0\{v/x\} : \exists t.(\mathbf{ref}\ t :: \mathbf{rw}\ t\ A_1) \dashv \Delta_1$ (4)

by induction hypothesis with (3) and (1).

$\Gamma ; \Delta_0 \vdash \mathsf{delete}\ v_0\{v/x\} : \exists t.A_1 \dashv \Delta_1$ (5)

by (t:Delete) with (4).

$\Gamma ; \Delta_0 \vdash (\mathsf{delete}\ v_0)\{v/x\} : \exists t.A_1 \dashv \Delta_1$ (6)

by (vs:9) on (5).

Thus, we conclude.

Case (t:Assign) - We have:

$\Gamma ; \cdot \vdash v : !A_0 \dashv \cdot$ (1)

$\Gamma , x : A_0 ; \Delta_0 \vdash v_0 := v_1 : A_1 \dashv \Delta_2 , \mathbf{rw}\ p\ A_2$ (2)

by hypothesis.

$\Gamma , x : A_0 ; \Delta_0 \vdash v_1 : A_2 \dashv \Delta_1$ (3)

$\Gamma , x : A_0 ; \Delta_1 \vdash v_0 : \mathbf{ref}\ p \dashv \Delta_2 , \mathbf{rw}\ p\ A_1$ (4)

by inversion on (t:Assign) with (2).

$\Gamma ; \Delta_0 \vdash v_1\{v/x\} : A_2 \dashv \Delta_1$ (5)

by induction hypothesis on (3) with (1).

$\Gamma; \Delta_1 \vdash v_0\{v/x\} : \textbf{ref}\ p \dashv \Delta_2, \textbf{rw}\ p\ A_1$ (6)

by induction hypothesis on (4) with (1).

$\Gamma; \Delta_0 \vdash v_0\{v/x\} := v_1\{v/x\} : A_1 \dashv \Delta_2, \textbf{rw}\ p\ A_2$ (7)

by (t:Assign) with (5) and (6).

$\Gamma; \Delta_0 \vdash (v_0 := v_1)\{v/x\} : A_1 \dashv \Delta_2, \textbf{rw}\ p\ A_2$ (8)

by (vs:11) on (7).

Thus, we conclude.

Case (t:Dereference-Linear) - We have:

$\Gamma; \cdot \vdash v : !A_0 \dashv \cdot$ (1)

$\Gamma, x : A_0; \Delta_0 \vdash !v_0 : A_1 \dashv \Delta_1, \textbf{rw}\ p\ []$ (2)

by hypothesis.

$\Gamma, x : A_0; \Delta_0 \vdash v_0 : \textbf{ref}\ p \dashv \Delta_1, \textbf{rw}\ p\ A_1$ (3)

by inversion on (t:Dereference-Linear) with (2).

$\Gamma; \Delta_0 \vdash v_0\{v/x\} : \textbf{ref}\ p \dashv \Delta_1, \textbf{rw}\ p\ A_1$ (4)

by induction hypothesis on (3) with (1).

$\Gamma; \Delta_0 \vdash !v_0\{v/x\} : A_1 \dashv \Delta_1, \textbf{rw}\ p\ []$ (5)

by (t:Dereference-Linear) with (4).

$\Gamma; \Delta_0 \vdash (!v_0)\{v/x\} : A_1 \dashv \Delta_1, \textbf{rw}\ p\ []$ (6)

by (vs:10) on (5).

Thus, we conclude.

Case (t:Dereference-Pure) - Analogous to (t:Dereference-Linear).

Case (t:Record) - We have:

$\Gamma; \cdot \vdash v : !A' \dashv \cdot$ (1)

$\Gamma, x : A'; \Delta \vdash \{\overline{\mathtt{f} = v'}\} : [\overline{\mathtt{f} : A}] \dashv \cdot$ (2)

by hypothesis.

$\overline{\Gamma, x : A'; \Delta \vdash v'_i : A_i \dashv \cdot}$ (3)

by inversion on (t:Record) with (2).

$\overline{\Gamma; \Delta \vdash v'_i\{v/x\} : A_i \dashv \cdot}$ (4)

by induction hypothesis on (3) with (1).

$\Gamma; \Delta \vdash \{\overline{\mathtt{f} = v'\{v/x\}}\} : [\overline{\mathtt{f} : A}] \dashv \cdot$ (5)

by (t:Record) on (4).

$\Gamma; \Delta \vdash (\{\overline{\mathtt{f} = v'}\})\{v/x\} : [\overline{\mathtt{f} : A}] \dashv \cdot$ (6)

by (vs:5) on (5).

Thus, we conclude.

Case (t:Selection) - We have:

$\Gamma; \cdot \vdash v : !A' \dashv \cdot$ (1)

$\Gamma, x : A'; \Delta_0 \vdash v_0.\mathtt{f} : A \dashv \Delta_1$ (2)

by hypothesis.

$\Gamma, x : A'; \Delta_0 \vdash v_0 : [\mathtt{f} : A] \dashv \Delta_1$ (3)

by inversion on (t:Selection) with (2).

$\Gamma; \Delta_0 \vdash v_0\{v/x\} : [\mathtt{f} : A] \dashv \Delta_1$ (4)

by induction hypothesis with (1) and (3).

$\Gamma; \Delta_0 \vdash v_0\{v/x\}.\mathtt{f} : A \dashv \Delta_1$ (5)

by (t:Selection) with (4).

$\Gamma; \Delta_0 \vdash (v_0.\mathtt{f})\{v/x\} : A \dashv \Delta_1$ (6)

by (vs:6) on (5).

Thus, we conclude.

Case (t:Application) - We have:

$\Gamma; \cdot \vdash v : !A' \dashv \cdot$ (1)

$\Gamma, x : A'; \Delta_0 \vdash v_0\ v_1 : A_1 \dashv \Delta_2$ (2)

by hypothesis.

$\Gamma, x : A'; \Delta_0 \vdash v_1 : A_0 \dashv \Delta_1$ (3)

$\Gamma, x : A'; \Delta_1 \vdash v_0 : A_0 \multimap A_1 \dashv \Delta_2$ (4)

by inversion on (t:Application) with (2).

$\Gamma; \Delta_0 \vdash v_1\{v/x\} : A_0 \dashv \Delta_1$ (5)

by induction hypothesis with (1) on (3).

$\Gamma; \Delta_1 \vdash v_0\{v/x\} : A_0 \multimap A_1 \dashv \Delta_2$ (6)

by induction hypothesis with (1) on (4).

$\Gamma; \Delta_0 \vdash v_0\{v/x\}\ v_1\{v/x\} : A_1 \dashv \Delta_2$ (7)

by (t:Application) with (5) and (6).

$\Gamma; \Delta_0 \vdash (v_0\ v_1)\{v/x\} : A_1 \dashv \Delta_2$ (8)

by (vs:7) on (7).

Thus, we conclude.

Case (t:Function) - We have:

$\Gamma; \cdot \vdash v : !A' \dashv \cdot$ (1)

$\Gamma, x_0 : A'; \Delta \vdash \mathsf{fun}(x_1 : A_0).e : A_0 \multimap A_1 \dashv \cdot$ (2)

by hypothesis.

$\Gamma, x_0 : A'; \Delta, x_1 : A_0 \vdash e : A_1 \dashv \cdot$ (3)

by inversion on (t:Function) with (2).

$x_0 \neq x_1$ (4)

by def. of substitution up to rename of bounded variables.

$\Gamma; \Delta, x_1 : A_0 \vdash e\{v/x_0\} : A_1 \dashv \cdot$ (5)

by induction hypothesis with (3) and (1).

$\Gamma; \Delta \vdash \mathsf{fun}(x_1 : A_0).e\{v/x_0\} : A_0 \multimap A_1 \dashv \cdot$ (6)

by (t:Function) with (6).

$\Gamma; \Delta \vdash (\mathsf{fun}(x_1 : A_0).e)\{v/x_0\} : A_0 \multimap A_1 \dashv \cdot$ (7)

by (vs:4) on (6) and (4).

Thus, we conclude.

Case (t:Forall-Loc) - We have:

$\Gamma; \cdot \vdash v : !A' \dashv \cdot$ (1)

$\Gamma, x : A'; \Delta_0 \vdash \langle t \rangle\, e : \forall t.A \dashv \cdot$ (2)

by hypothesis.

$\Gamma, t : \textbf{loc}, x : A'; \Delta_0 \vdash e : A \dashv \cdot$ (3)

by inversion on (t:Forall-Loc) with (2).

$\Gamma, t : \textbf{loc}; \Delta_0 \vdash e\{v/x\} : A \dashv \cdot$ (4)

by induction hypothesis on (3) with (1).

$\Gamma; \Delta_0 \vdash \langle t \rangle\, e\{v/x\} : \forall t.A \dashv \cdot$ (5)

by (t:Forall-Loc) with (4).

$\Gamma; \Delta_0 \vdash (\langle t \rangle\, e)\{v/x\} : \forall t.A \dashv \cdot$ (6)

by (vs:14) on (5).

Thus, we conclude.

Case (t:Loc-App) - We have:

$\Gamma; \cdot \vdash v : !A' \dashv \cdot$ (1)

$\Gamma, x : A'; \Delta_0 \vdash v_0[p] : A\{p/t\} \dashv \Delta_1$ (2)

by hypothesis.

$p : \textbf{loc} \in \Gamma, x : A'$ (3)

$\Gamma, x : A'; \Delta_0 \vdash v_0 : \forall t.A \dashv \Delta_1$ (4)

by inversion on (t:Loc-App) with (2).

$\Gamma; \Delta_0 \vdash v_0\{v/x\} : \forall t.A \dashv \Delta_1$ (5)

by induction hypothesis with (1) and (4).

$\Gamma; \Delta_0 \vdash v_0\{v/x\}[p] : A\{p/t\} \dashv \Delta_1$ (6)

by (t:Loc-App) with (5) and (3).

$\Gamma; \Delta_0 \vdash (v_0[p])\{v/x\} : A\{p/t\} \dashv \Delta_1$ (7)

by (vs:13) on (6).

Thus, we conclude.

Case (t:Loc-Open) - We have:

$\Gamma; \cdot \vdash v : !A' \dashv \cdot$ (1)

$\Gamma, x : A'; \Delta_0 \vdash \mathsf{open}\ \langle t, x_1 \rangle = v_0\ \mathsf{in}\ e_1\ \mathsf{end} : A_1 \dashv \Delta_1$ (2)

by hypothesis.

$\Gamma, x : A'; \Delta_0 \vdash v_0 : \exists t.A_0 \dashv \Delta_1$ (3)

$\Gamma, t : \textbf{loc}, x : A'; \Delta_1, x_1 : A_0 \vdash e_1 : A_1 \dashv \Delta_2$ (4)

by inversion on (t:Loc-Open) with (2).

$x_0 \neq x_1$ (5)

by def. of substitution up to rename of bounded variables.

$\Gamma; \Delta_0 \vdash v_0\{v/x\} : \exists t.A_0 \dashv \Delta_1$ (6)

by induction hypothesis on (3) and (1).

$\Gamma, t : \textbf{loc}; \Delta_1, x_1 : A_0 \vdash e_1\{v/x\} : A_1 \dashv \Delta_2$ (7)

by induction hypothesis on (4) and (1).

$\Gamma; \Delta_0 \vdash \mathsf{open}\ \langle t, x_1 \rangle = v_0\{v/x\}\ \mathsf{in}\ e_1\{v/x\}\ \mathsf{end} : A_1 \dashv \Delta_1$ (8)

by (t:Loc-Open) with (6) and (7).

$\Gamma; \Delta_0 \vdash (\mathsf{open}\ \langle t, x_1 \rangle = v_0\ \mathsf{in}\ e_1\ \mathsf{end})\{v/x\} : A_1 \dashv \Delta_1$ (9)

by (vs:15) on (8) and (5).

Thus, we conclude.

Case (t:Loc-Pack) - We have:

$\Gamma; \cdot \vdash v : !A' \dashv \cdot$ (1)

$\Gamma, x : A'; \Delta_0 \vdash \langle p, v_0 \rangle : \exists t.A \dashv \Delta_1$ (2)

by hypothesis.

$\Gamma, x : A'; \Delta_0 \vdash v_0 : A\{p/t\} \dashv \Delta_1$ (3)

by inversion on (t:Loc-Pack) with (2).

$\Gamma; \Delta_0 \vdash v_0\{v/x\} : A\{p/t\} \dashv \Delta_1$ (4)

by induction hypothesis with (1) and (3).

$\Gamma; \Delta_0 \vdash \langle p, v_0\{v/x\} \rangle : \exists t.A \dashv \Delta_1$ (5)

by (t:Loc-Pack) with (4).

$\Gamma; \Delta_0 \vdash (\langle p, v_0 \rangle)\{v/x\} : \exists t.A \dashv \Delta_1$ (6)

by (vs:12) on (5).

Thus, we conclude.

Case (t:Forall-Type) - Analogous to (t:Forall-Loc) with (vs:18).

Case (t:Type-App) - Analogous to (t:Loc-App) with (vs:17).

Case (t:Type-Pack) - Analogous to (t:Loc-Pack) with (vs:16).

Case (t:Type-Open) - Analogous to (t:Loc-Open) with (vs:19).

Case (t:Cap-Elim) - We have:

$\Gamma; \cdot \vdash v : !A' \dashv \cdot$ (1)

$\Gamma, x : A'; \Delta_0, x_0 : A_0 :: A_2 \vdash e : A_1 \dashv \Delta_1$ (2)

by hypothesis.

$\Gamma, x : A'; \Delta_0, x_0 : A_0, A_2 \vdash e : A_1 \dashv \Delta_1$ (3)

by inversion on (t:Cap-Elim) with (2).

$\Gamma; \Delta_0, x_0 : A_0, A_2 \vdash e\{v/x\} : A_1 \dashv \Delta_1$ (4)

by induction hypothesis with (1) and (3).

$\Gamma; \Delta_0, x_0 : A_0 :: A_2 \vdash e\{v/x\} : A_1 \dashv \Delta_1$ (5)

by (t:Cap-Elim) with (4).

Thus, we conclude.

Case (t:Cap-Stack) - We have:

$\Gamma; \cdot \vdash v : !A' \dashv \cdot$ (1)

$\Gamma, x : A'; \Delta_0 \vdash e : A_0 :: A_1 \dashv \Delta_1$ (2)

by hypothesis.

$\Gamma, x : A'; \Delta_0 \vdash e : A_0 \dashv \Delta_1, A_1$ (3)

by inversion on (t:Cap-Stack) with (2).

$\Gamma; \Delta_0 \vdash e\{v/x\} : A_0 \dashv \Delta_1, A_1$ (4)

by induction hypothesis with (1) and (3).

$\Gamma; \Delta_0 \vdash e\{v/x\} : A_0 :: A_1 \dashv \Delta_1$ (5)

by (t:Cap-Stack) with (4).

Thus, we conclude.

Case (t:Cap-Unstack) - We have:

$\Gamma; \cdot \vdash v : !A' \dashv \cdot$ (1)

$\Gamma, x : A'; \Delta_0 \vdash e : A_0 \dashv \Delta_1, A_1$ (2)

by hypothesis.

$\Gamma, x : A'; \Delta_0 \vdash e : A_0 :: A_1 \dashv \Delta_1$ (3)

by inversion on (t:Cap-Unstack) with (2).

$\Gamma; \Delta_0 \vdash e\{v/x\} : A_0 :: A_1 \dashv \Delta_1$ (4)

by induction hypothesis with (1) and (3).

$\Gamma; \Delta_0 \vdash e\{v/x\} : A_0 \dashv \Delta_1, A_1$ (5)

by (t:Cap-Unstack) with (4).

Thus, we conclude.

Case (t:Frame) - We have:

$\Gamma; \cdot \vdash v : !A' \dashv \cdot$ (1)

$\Gamma, x : A'; \Delta_0, \Delta_2 \vdash e : A \dashv \Delta_1, \Delta_2$ (2)

by hypothesis.

$\Gamma, x : A'; \Delta_0 \vdash e : A \dashv \Delta_1$ (3)

by inversion on (t:Frame) with (2).

$\Gamma; \Delta_0 \vdash e\{v/x\} : A \dashv \Delta_1$ (4)

by induction hypothesis with (1) and (3).

$\Gamma; \Delta_0, \Delta_2 \vdash e\{v/x\} : A \dashv \Delta_1, \Delta_2$ (5)

by (t:Frame) with Δ_2.

Thus, we conclude.

Case (t:Subsumption) - We have:

$\Gamma; \cdot \vdash v : !A' \dashv \cdot$ (1)

$\Gamma, x : A'; \Delta_0 \vdash e : A_1 \dashv \Delta_1$ (2)

by hypothesis.

$\Delta_0 <: \Delta_0'$ (3)

$\Gamma, x : A'; \Delta_0' \vdash e : A_0 \dashv \Delta_1'$ (4)

$A_0 <: A_1$ (5)

$\Delta_1' <: \Delta_1$ (6)

by inversion on (t:Subsumption) with (2).

$\Gamma; \Delta_0' \vdash e\{v/x\} : A_0 \dashv \Delta_1'$ (7)

by induction hypothesis with (1) and (4).

$\Gamma; \Delta_0 \vdash e\{v/x\} : A_1 \dashv \Delta_1$ (8)

by (t:Subsumption) with (7), (3), (5) and (6).

Thus, we conclude.

Case (t:Tag) - We have:

$\Gamma; \cdot \vdash v : !A' \dashv \cdot$ (1)

$\Gamma, x : A'; \Delta_0 \vdash \mathtt{l}\#v_0 : \mathtt{l}\#A_1 \dashv \Delta_1$ (2)

by hypothesis.

$\Gamma, x : A'; \Delta_0 \vdash v_0 : A_1 \dashv \Delta_1$ (3)

by inversion (t:Tag) with (2).

$\Gamma; \Delta_0 \vdash v_0\{v/x\} : A_1 \dashv \Delta_1$ (4)

by induction hypothesis with (1) and (3).

$\Gamma; \Delta_0 \vdash \mathtt{l}\#v_0\{v/x\} : \mathtt{l}\#A_1 \dashv \Delta_1$ (5)

by (t:Tag) with (4).

$\Gamma; \Delta_0 \vdash (\mathtt{l}\#v_0)\{v/x\} : \mathtt{l}\#A_1 \dashv \Delta_1$ (6)

by (vs:20) on (5).

Thus, we conclude.

Case (t:Case) - We have:

$\Gamma; \cdot \vdash v : !A' \dashv \cdot$ (1)

$\Gamma, x : A'; \Delta_0 \vdash \mathsf{case}\ v_0\ \mathsf{of}\ \overline{\mathtt{l}_j\#x_j \rightarrow e_j}\ \mathsf{end} : A \dashv \Delta_1$ (2)

by hypothesis.

$\underline{\Gamma, x : A'; \Delta_1 \vdash v_0 : \sum_i \mathtt{l}_i\#A_i' \dashv \Delta'}$ (3)

$\overline{\Gamma, x : A'; \Delta', x_i : A_i' \vdash e_i : A \dashv \Delta_2}$ (4)

$i \leq j$ (5)

by inversion (t:Case) with (2).

$\overline{x \neq x_j}$ (6)

by def. of substitution up to rename of bounded variables.

$\Gamma; \Delta_1 \vdash v_0\{v/x\} : \sum_i \mathtt{l}_i\#A_i' \dashv \Delta'$ (7)

by induction hypothesis on (3) and (1).

$\overline{\Gamma; \Delta', x_i : A_i' \vdash e_i\{v/x\} : A \dashv \Delta_2}$ (8)

by induction hypothesis on (4) and (1).

$\Gamma; \Delta_1 \vdash \mathsf{case}\ v_0\{v/x\}\ \mathsf{of}\ \overline{\mathtt{l}_j\#x_j \rightarrow e_j\{v/x\}}\ \mathsf{end} : A \dashv \Delta_2$ (9)

by (t:Case) on (5), (7) and (8).

$\Gamma; \Delta_1 \vdash (\mathsf{case}\ v_0\ \mathsf{of}\ \overline{\mathtt{l}_j\#x_j \rightarrow e_j}\ \mathsf{end})\{v/x\} : A \dashv \Delta_2$ (10)

by (vs:21) on (9) and (6).

Thus, we conclude.

Case (t:Alternative-Left) - Immediate by applying the induction hypothesis on the inversion and then re-applying the rule.

Case (t:Let) - Analogous to other cases such as (t:Loc-Open).

□

3. (Location Variable)

Proof. We proceed by induction on the typing derivation of $\Gamma, t : \mathbf{loc}; \Delta_0 \vdash e : A \dashv \Delta_1$.

Case (t:Ref) - We have:

$\Gamma, \rho_0 : \mathbf{loc}, t : \mathbf{loc}; \cdot \vdash \rho_0 : \mathbf{ref}\,\rho_0 \dashv \cdot$ (1)

$\rho : \mathbf{loc} \in \Gamma$ (2)

by hypothesis.

$\Gamma, \rho_0 : \mathbf{loc}, t : \mathbf{loc}\ \mathbf{wf}$ (3)

by typing.

$(\Gamma, \rho_0 : \mathbf{loc})\{\rho/t\}\ \mathbf{wf}$ (4)

by (Well-Formed Type Substitution - Gamma) on (3), (2).

$\Gamma\{\rho/t\}, \rho_0\{\rho/t\} : \mathbf{loc}\ \ \mathbf{wf}$ (5)

by (ls:3.3) on (4)

$\Gamma\{\rho/t\}, \rho_0\{\rho/t\} : \mathbf{loc}; \cdot \vdash \rho_0\{\rho/t\} : \mathbf{ref}\,\rho_0\{\rho/t\} \dashv \cdot$ (6)

by (t:Ref) with (5).

$(\Gamma, \rho_0 : \mathbf{loc})\{\rho/t\}; \cdot \vdash \rho_0\{\rho/t\} : (\mathbf{ref}\,\rho_0)\{\rho/t\} \dashv \cdot$ (7)

by (ls:3.3), (ls:2.10) on (6)

Thus, we conclude.

Case (t:Pure) - We have:

$\Gamma, t : \mathbf{loc}; \cdot \vdash v : !A \dashv \cdot$ (1)

$\rho : \mathbf{loc} \in \Gamma$ (2)

by hypothesis.

$\Gamma, t : \mathbf{loc}; \cdot \vdash v : A \dashv \cdot$ (3)

by inversion on (t:Pure) with (1).

$\Gamma\{\rho/t\}; \cdot\{\rho/t\} \vdash v\{\rho/t\} : A\{\rho/t\} \dashv \cdot\{\rho/t\}$ (4)

by induction hypothesis with (2) and (3).

$\Gamma\{\rho/t\}; \cdot\{\rho/t\} \vdash v\{\rho/t\} : !A\{\rho/t\} \dashv \cdot\{\rho/t\}$ (5)

by (t:Pure) on (4).

$\Gamma\{\rho/t\}; \cdot\{\rho/t\} \vdash v\{\rho/t\} : (!A)\{\rho/t\} \dashv \cdot\{\rho/t\}$ (6)

by (ls:2.4) on (5)

Thus, we conclude.

Case (t:Unit) - We have:

$\Gamma, t : \mathbf{loc}; \cdot \vdash v : [] \dashv \cdot$ (1)

$\rho : \mathbf{loc} \in \Gamma$ (2)

by hypothesis.

$\Gamma, t : \mathbf{loc}\ \mathbf{wf}$ (3)

by typing.

$\Gamma\{\rho/t\}\ \mathbf{wf}$ (4)

by (Well-Formed Type Substitution - Gamma) on (3), (2).

$\Gamma\{\rho/t\}; \cdot \vdash v : [] \dashv \cdot$ (5)

by (t:Unit) with (4).

$\Gamma\{\rho/t\}; \cdot\{\rho/t\} \vdash v\{\rho/t\} : []\{\rho/t\} \dashv \cdot\{\rho/t\}$ (6)

by (ls2.7), (ls4.1) on (5) and noting that regardless if t occurs or not in v its type remains unchanged.

Thus, we conclude.

Case (t:Pure-Read) - We have:

$\Gamma, x : A, t : \mathbf{loc}; \cdot \vdash x : !A \dashv \cdot$ (1)

$\rho : \mathbf{loc} \in \Gamma$ (2)

by hypothesis.

$\Gamma, x : A, t : \mathbf{loc}\ \ \mathbf{wf}$ (3)

by typing.

$(\Gamma, x : A)\{\rho/t\}\ \ \mathbf{wf}$ (4)

by (Well-Formed Type Substitution) on (3), (2).

$\Gamma\{\rho/t\}, x : A\{\rho/t\}\ \ \mathbf{wf}$ (5)

by (ls:3.2) on (4)

$\Gamma\{\rho/t\}, x : A\{\rho/t\}; \cdot \vdash x : !A\{\rho/t\} \dashv \cdot$ (6)

by (t:Pure-Read) with (5).

$\Gamma\{\rho/t\}, x : A\{\rho/t\}; \cdot\{\rho/t\} \vdash x\{\rho/t\} : (!A)\{\rho/t\} \dashv \cdot\{\rho/t\}$ (7)

by (ls:3.1), (ls:2.4), (ls:1.2) on (6)

Thus, we conclude.

Case (t:Linear-Read) - We have:

$\Gamma, t : \mathbf{loc};, x : A \vdash x : A \dashv \cdot$ (1)

$\rho : \mathbf{loc} \in \Gamma$ (2)

by hypothesis.

$(\Gamma, t : \mathbf{loc})\ \ \mathbf{wf}$ (3)

by typing.

$\Gamma\{\rho/t\}\ \ \mathbf{wf}$ (4)

by (Well-Formed Type Substitution) with (3) and (2).

$\Gamma, t : \mathbf{loc} \vdash A\ \ \mathbf{type}$ (5)

by (Well-Formed Delta) on (1)

$\Gamma\{\rho/t\} \vdash A\{\rho/t\}\ \ \mathbf{type}$ (6)

by (Well-Formed Type Substitution) with (6) and (2).

$\Gamma\{\rho/t\}; x : A\{\rho/t\} \vdash x : A\{\rho/t\} \dashv \cdot$ (7)

by (t:Linear-Read) with (5).

$\Gamma\{\rho/t\}; (x : A)\{\rho/t\} \vdash x\{\rho/t\} : A\{\rho/t\} \dashv \cdot\{\rho/t\}$ (8)

by (ls:4.2), (ls:4.1), (ls:1.2) on (7).

Thus, we conclude.

Case (t:Pure-Elim) - We have:

$\Gamma, t : \mathbf{loc}; \Delta_0, x : !A_0 \vdash e : A_1 \dashv \Delta_1$ (1)

$\rho : \mathbf{loc} \in \Gamma$ (2)

by hypothesis.

$\Gamma, t : \mathbf{loc}, x : A_0; \Delta_0 \vdash e : A_1 \dashv \Delta_1$ (3)

by inversion on (t:Pure-Elim) with (1).

$(\Gamma, x : A_0)\{\rho/t\}; \Delta_0\{\rho/t\} \vdash e\{\rho/t\} : A_1\{\rho/t\} \dashv \Delta_1\{\rho/t\}$ (4)

by induction hypothesis on (3) and (2).

$\Gamma\{\rho/t\}, x : A_0\{\rho/t\}; \Delta_0\{\rho/t\} \vdash e\{\rho/t\} : A_1\{\rho/t\} \dashv \Delta_1\{\rho/t\}$ (5)

by (ls:3.2) on (4)

$\Gamma\{\rho/t\}; \Delta_0\{\rho/t\}, x : !A_0\{\rho/t\} \vdash e\{\rho/t\} : A_1\{\rho/t\} \dashv \Delta_1\{\rho/t\}$ (6)

by (t:Pure-Elim) on (5).

$\Gamma\{\rho/t\}; (\Delta_0, x : !A_0)\{\rho/t\} \vdash e\{\rho/t\} : A_1\{\rho/t\} \dashv \Delta_1\{\rho/t\}$ (7)

by (ls:4.2) on (6)

Thus, we conclude.

Case (t:New) - We have:

$\Gamma, t : \mathbf{loc}; \Delta_0 \vdash \textsf{new}\ v : \exists t_0.(\mathbf{ref}\ t_0 :: \mathbf{rw}\ t_0\ A) \dashv \Delta_1$ (1)

$\rho : \mathbf{loc} \in \Gamma$ (2)

by hypothesis.

$\Gamma, t : \mathbf{loc}; \Delta_0 \vdash v : A \dashv \Delta_1$ (3)

by inversion on (t:New) with (1).

$\Gamma\{\rho/t\}; \Delta_0\{\rho/t\} \vdash v\{\rho/t\} : A\{\rho/t\} \dashv \Delta_1\{\rho/t\}$ (4)

by induction hypothesis on (2) and (3).

$\Gamma\{\rho/t\}; \Delta_0\{\rho/t\} \vdash \textsf{new}\ v\{\rho/t\} : \exists t_0.(\mathbf{ref}\ t_0 :: \mathbf{rw}\ t_0\ A\{\rho/t\}) \dashv \Delta_1\{\rho/t\}$ (5)

by (t:New) with (4).

$t_0 \neq t$ (6)

by def. of substitution up to rename of bounded location variables.

$\Gamma\{\rho/t\}; \Delta_0\{\rho/t\} \vdash (\textsf{new}\ v)\{\rho/t\} : \exists t_0.(\mathbf{ref}\ t_0 :: \mathbf{rw}\ t_0\ A\{\rho/t\}) \dashv \Delta_1\{\rho/t\}$ (7)

by (ls:1.7) on (5).

$\Gamma\{\rho/t\}; \Delta_0\{\rho/t\} \vdash (\textsf{new}\ v)\{\rho/t\} : \exists t_0.(\mathbf{ref}\ t_0 :: (\mathbf{rw}\ t_0\ A)\{\rho/t\}) \dashv \Delta_1\{\rho/t\}$ (8)

by (ls:2.12) on (7).

$\Gamma\{\rho/t\}; \Delta_0\{\rho/t\} \vdash (\textsf{new}\ v)\{\rho/t\} : \exists t_0.((\mathbf{ref}\ t_0 :: \mathbf{rw}\ t_0\ A)\{\rho/t\}) \dashv \Delta_1\{\rho/t\}$ (9)

by (ls:2.6) on (8) and (6).

$\Gamma\{\rho/t\}; \Delta_0\{\rho/t\} \vdash (\textsf{new}\ v)\{\rho/t\} : (\exists t_0.(\mathbf{ref}\ t_0 :: \mathbf{rw}\ t_0\ A))\{\rho/t\} \dashv \Delta_1\{\rho/t\}$ (10)

by (ls:2.9) on (9) and (6).

Thus, we conclude.

Case (t:Delete) - We have:

$\Gamma, t : \mathbf{loc}; \Delta_0 \vdash \textsf{delete}\ v : \exists t_0.A \dashv \Delta_1$ (1)

$\rho : \mathbf{loc} \in \Gamma$ (2)

by hypothesis.

$\Gamma, t : \mathbf{loc}; \Delta_0 \vdash v : \exists t_0.(\mathbf{ref}\ t_0 :: \mathbf{rw}\ t_0\ A) \dashv \Delta_1$ (3)

by inversion on (t:Delete) with (1).

$\Gamma\{\rho/t\}; \Delta_0\{\rho/t\} \vdash v\{\rho/t\} : (\exists t_0.(\mathbf{ref}\ t_0 :: \mathbf{rw}\ t_0\ A))\{\rho/t\} \dashv \Delta_1\{\rho/t\}$ (4)

by induction hypothesis on (2) and (3).

$t_0 \neq t$ (5)

by def. of substitution up to rename of bounded location variables.

$\Gamma\{\rho/t\}; \Delta_0\{\rho/t\} \vdash v\{\rho/t\} : \exists t_0.((\mathbf{ref}\ t_0 :: \mathbf{rw}\ t_0\ A)\{\rho/t\}) \dashv \Delta_1\{\rho/t\}$ (6)

by (ls:2.9) on (4) and (5).

$\Gamma\{\rho/t\}; \Delta_0\{\rho/t\} \vdash v\{\rho/t\} : \exists t_0.((\mathbf{ref}\ t_0)\{\rho/t\} :: (\mathbf{rw}\ t_0\ A)\{\rho/t\}) \dashv \Delta_1\{\rho/t\}$ (7)

by (ls:2.12) on (6).

$\Gamma\{\rho/t\}; \Delta_0\{\rho/t\} \vdash v\{\rho/t\} : \exists t_0.(\mathbf{ref}\ t_0 :: \mathbf{rw}\ t_0\ A\{\rho/t\}) \dashv \Delta_1\{\rho/t\}$ (8)

by (ls:2.10), (ls:2.3), (ls:2.12) on (7).

$\Gamma\{\rho/t\}; \Delta_0\{\rho/t\} \vdash v\{\rho/t\} : \exists t_0.(A\{\rho/t\}) \dashv \Delta_1\{\rho/t\}$ (9)

by (t:Delete) on (8).

$\Gamma\{\rho/t\}; \Delta_0\{\rho/t\} \vdash v\{\rho/t\} : (\exists t_0.A)\{\rho/t\} \dashv \Delta_1\{\rho/t\}$ (10)

by (ls:2.9) on (5) and (9).

Thus, we conclude.

Case (t:Assign) - We have:

$\Gamma, t : \mathbf{loc}; \Delta_0 \vdash v_0 := v_1 : A_1 \dashv \Delta_2, \mathbf{rw}\ p\ A_0$ (1)

$\rho : \mathbf{loc} \in \Gamma$ (2)

by hypothesis.

$\Gamma, t : \mathbf{loc}; \Delta_0 \vdash v_1 : A_0 \dashv \Delta_1$ (3)

$\Gamma, t : \mathbf{loc}; \Delta_1 \vdash v_0 : \mathbf{ref}\ p \dashv \Delta_2, \mathbf{rw}\ p\ A_1$ (4)

by inversion on (t:Assign) with (1).

$\Gamma\{\rho/t\}; \Delta_0\{\rho/t\} \vdash v_1\{\rho/t\} : A_0\{\rho/t\} \dashv \Delta_1\{\rho/t\}$ (5)

by induction hypothesis on (3) with (2).

$\Gamma\{\rho/t\}; \Delta_1\{\rho/t\} \vdash v_0\{\rho/t\} : (\mathbf{ref}\ p)\{\rho/t\} \dashv (\Delta_2, \mathbf{rw}\ p\ A_1)\{\rho/t\}$ (6)

by induction hypothesis on (4) with (2).

$\Gamma\{\rho/t\}; \Delta_1\{\rho/t\} \vdash v_0\{\rho/t\} : \mathbf{ref}\ p\{\rho/t\} \dashv \Delta_2\{\rho/t\}, \mathbf{rw}\ p\{\rho/t\}\ A_1\{\rho/t\}$ (7)

by (ls:2.10), (ls:4.3), (ls:2.12) on (6).

$\Gamma\{\rho/t\}; \Delta_1\{\rho/t\} \vdash v_0\{\rho/t\} := v_1\{\rho/t\} : A_1\{\rho/t\}$
$\dashv \Delta_2\{\rho/t\}, \mathbf{rw}\ p\{\rho/t\}\ A_0\{\rho/t\}$ (8)

by (t:Assign) on (6) and (7).

$\Gamma\{\rho/t\}; \Delta_1\{\rho/t\} \vdash (v_0 := v_1)\{\rho/t\} : A_1\{\rho/t\} \dashv (\Delta_2, \mathbf{rw}\ p\ A_0)\{\rho/t\}$ (9)

by (ls:1.10), (ls:2.12), (ls:4.3) on (8).

Thus, we conclude.

Case (t:Dereference-Linear) - We have:

$\Gamma, t : \mathbf{loc}; \Delta_0 \vdash !v : A \dashv \Delta_1, \mathbf{rw}\ p\ []$ (1)

$\rho : \mathbf{loc} \in \Gamma$ (2)

by hypothesis.

$\Gamma, t : \mathbf{loc}; \Delta_0 \vdash v : \mathbf{ref}\ p \dashv \Delta_1, \mathbf{rw}\ p\ A$ (3)

by inversion on (t:Dereference-Linear) with (1).

$\Gamma\{\rho/t\}; \Delta_0\{\rho/t\} \vdash v\{\rho/t\} : (\mathbf{ref}\ p)\{\rho/t\} \dashv (\Delta_1, \mathbf{rw}\ p\ A)\{\rho/t\}$ (4)

by induction hypothesis with (2) and (3).

$\Gamma\{\rho/t\}; \Delta_0\{\rho/t\} \vdash v\{\rho/t\} : \mathbf{ref}\ p\{\rho/t\} \dashv \Delta_1\{\rho/t\}, \mathbf{rw}\ p\{\rho/t\}\ A\{\rho/t\}$ (5)

by (ls:4.3), (ls:2.12), (ls:2.10) on (4).

$\Gamma\{\rho/t\}; \Delta_0\{\rho/t\} \vdash !v\{\rho/t\} : A\{\rho/t\} \dashv \Delta_1\{\rho/t\}, \mathbf{rw}\ p\{\rho/t\}\ []$ (6)

by (t:Dereference-Linear) on (5).

$\Gamma\{\rho/t\}; \Delta_0\{\rho/t\} \vdash (!v)\{\rho/t\} : A\{\rho/t\} \dashv (\Delta_1, \mathbf{rw}\ p\ [])$ (7)

by (ls:1.9), (ls:4.3), (ls:2.12), (ls:2.3) on (6).

Thus, we conclude.

Case (t:Dereference-Pure) - Analogous to (t:Dereference-Linear).

Case (t:Record) - We have:

$\Gamma, t : \mathbf{loc}; \Delta \vdash \{\overline{\mathtt{f} = v}\} : [\overline{\mathtt{f} : A}] \dashv \cdot$ (1)

$\rho : \mathbf{loc} \in \Gamma$ (2)

by hypothesis.

$\overline{\Gamma, t : \mathbf{loc}; \Delta \vdash v_i : A_i \dashv \cdot}$ (3)

by inversion on (t:Record) with (1).

$\overline{\Gamma\{\rho/t\}; \Delta\{\rho/t\} \vdash e_i\{\rho/t\} : A_i\{\rho/t\} \dashv \cdot\{\rho/t\}}$ (4)

by induction hypothesis with (2) and (3).

$\Gamma\{\rho/t\}; \Delta\{\rho/t\} \vdash \{\overline{\mathtt{f} = v\{\rho/t\}}\} : [\overline{\mathtt{f} : A\{\rho/t\}}] \dashv \cdot\{\rho/t\}$ (5)

by (t:Record) with (4).

$\Gamma\{\rho/t\}; \Delta\{\rho/t\} \vdash (\{\overline{\mathtt{f} = v}\})\{\rho/t\} : ([\overline{\mathtt{f} : A}])\{\rho/t\} \dashv \cdot\{\rho/t\}$ (6)

by (ls:1.4), (ls:2.7) on (5).

Thus, we conclude.

Case (t:Selection) - We have:

$\Gamma, t : \mathbf{loc}; \Delta_0 \vdash v.\mathtt{f}_i : A_i \dashv \Delta_1$ (1)

$\rho : \mathbf{loc} \in \Gamma$ (2)

by hypothesis.

$\Gamma, t : \mathbf{loc}; \Delta_0 \vdash v : [\overline{\mathtt{f} : A}] \dashv \Delta_1$ (3)

by inversion on (t:Selection) with (1).

$\Gamma\{\rho/t\}; \Delta_0\{\rho/t\} \vdash v\{\rho/t\} : [\overline{\mathtt{f} : A}]\{\rho/t\} \dashv \Delta_1\{\rho/t\}$ (4)

by induction hypothesis on (1) and (3).

$\Gamma\{\rho/t\}; \Delta_0\{\rho/t\} \vdash v\{\rho/t\} : [\overline{\mathtt{f} : A\{\rho/t\}}] \dashv \Delta_1\{\rho/t\}$ (5)

by (ls:2.7) on (4).

$\Gamma\{\rho/t\}; \Delta_0\{\rho/t\} \vdash v\{\rho/t\}.\mathtt{f}_i : A_i\{\rho/t\} \dashv \Delta_1\{\rho/t\}$ (6)

by (t:Selection) on (5).

$\Gamma\{\rho/t\}; \Delta_0\{\rho/t\} \vdash (v.\mathtt{f}_i)\{\rho/t\} : A_i\{\rho/t\} \dashv \Delta_1\{\rho/t\}$ (7)

by (ls:1.5) on (6).

Thus, we conclude.

Case (t:Application) - We have:

$\Gamma, t : \mathbf{loc}; \Delta_0 \vdash v_0\ v_1 : A_1 \dashv \Delta_2$ (1)

$\rho : \mathbf{loc} \in \Gamma$ (2)

by hypothesis.

$\Gamma, t : \mathbf{loc}; \Delta_0 \vdash v_1 : A_0 \dashv \Delta_1$ (3)

$\Gamma, t : \mathbf{loc}; \Delta_1 \vdash v_0 : A_0 \multimap A_1 \dashv \Delta_2$ (4)

by inversion on (t:Application) with (1).

$\Gamma\{\rho/t\}; \Delta_0\{\rho/t\} \vdash v_1\{\rho/t\} : A_0\{\rho/t\} \dashv \Delta_1\{\rho/t\}$ (3)

by induction hypothesis on (2) and (3).

$\Gamma\{\rho/t\}; \Delta_1\{\rho/t\} \vdash v_0\{\rho/t\} : (A_0 \multimap A_1)\{\rho/t\} \dashv \Delta_2\{\rho/t\}$ (4)

by induction hypothesis on (2) and (4).

$\Gamma\{\rho/t\}; \Delta_1\{\rho/t\} \vdash v_0\{\rho/t\} : A_0\{\rho/t\} \multimap A_1\{\rho/t\} \dashv \Delta_2\{\rho/t\}$ (5)

by (ls:2.5) on (4).

$\Gamma\{\rho/t\}; \Delta_0\{\rho/t\} \vdash (v_0\ v_1)\{\rho/t\} : A_0\{\rho/t\} \dashv \Delta_2\{\rho/t\}$ (6)

by (t:Application) on (5) and (3), and (ls:1.6).

Thus, we conclude.

Case (T:Function) - We have:

$\Gamma, t : \mathbf{loc}; \Delta \vdash \mathsf{fun}(x : A_0).e : A_0 \multimap A_1 \dashv \cdot$ (1)

$\rho : \mathbf{loc} \in \Gamma$ (2)

by hypothesis.

$\Gamma, t : \mathbf{loc}; \Delta, x : A_0 \vdash e : A_1 \dashv \cdot$ (3)

by inversion on (T:Function) with (1).

$\Gamma\{\rho/t\}; (\Delta, x : A_0)\{\rho/t\} \vdash e\{\rho/t\} : A_1\{\rho/t\} \dashv \cdot\{\rho/t\}$ (4)

by induction hypothesis on (2) and (3).

$\Gamma\{\rho/t\}; \Delta\{\rho/t\}, x : A_0\{\rho/t\} \vdash e\{\rho/t\} : A_1\{\rho/t\} \dashv \cdot\{\rho/t\}$ (5)

by (LS:4.2) on (4).

$\Gamma\{\rho/t\}; \Delta\{\rho/t\} \vdash \mathsf{fun}(x : A_0\{\rho/t\}).e\{\rho/t\} : A_0\{\rho/t\} \multimap A_1\{\rho/t\} \dashv \cdot\{\rho/t\}$ (6)

by (T:Function) on (5).

$\Gamma\{\rho/t\}; \Delta\{\rho/t\} \vdash (\mathsf{fun}(x : A_0).e)\{\rho/t\} : (A_0 \multimap A_1)\{\rho/t\} \dashv \cdot\{\rho/t\}$ (7)

by (LS:1.3), (LS:2.5) on (6).

Thus, we conclude.

Case (T:Forall-Loc) - We have:

$\Gamma, t : \mathbf{loc}; \Delta_0 \vdash \langle t_0 \rangle\, e : \forall t_0.A \dashv \cdot$ (1)

$\rho : \mathbf{loc} \in \Gamma$ (2)

by hypothesis.

$\Gamma, t : \mathbf{loc}, t_0 : \mathbf{loc}; \Delta_0 \vdash e : A \dashv \cdot$ (3)

by inversion on (T:Forall-Loc) with (1).

$t_0 \neq t$ (4)

by def. of substitution up to rename of bounded location variables.

$(\Gamma, t_0 : \mathbf{loc})\{\rho/t\}; \Delta_0\{\rho/t\} \vdash e\{\rho/t\} : A\{\rho/t\} \dashv \cdot\{\rho/t\}$ (5)

by induction hypothesis with (2) and (3).

$\Gamma\{\rho/t\}, t_0 : \mathbf{loc}; \Delta_0\{\rho/t\} \vdash e\{\rho/t\} : A\{\rho/t\} \dashv \cdot\{\rho/t\}$ (6)

by (LS:3.3), (LS:2.3) with (4) on (5).

$\Gamma\{\rho/t\}; \Delta_0\{\rho/t\} \vdash \langle t_0 \rangle\, e\{\rho/t\} : \forall t_0.A\{\rho/t\} \dashv \cdot\{\rho/t\}$ (7)

by (T:Forall-Loc) on (6).

$\Gamma\{\rho/t\}; \Delta_0\{\rho/t\} \vdash (\langle t_0 \rangle\, e)\{\rho/t\} : (\forall t_0.A)\{\rho/t\} \dashv \cdot\{\rho/t\}$ (8)

by (LS:1.13), (LS:2.8) with (4) on (7).

Thus, we conclude.

Case (T:Loc-App) - We have:

$\Gamma, t : \mathbf{loc}; \Delta_0 \vdash v[p] : A\{p/t_0\} \dashv \Delta_1$ (1)

$\rho : \mathbf{loc} \in \Gamma$ (2)

by hypothesis.

$p : \mathbf{loc} \in \Gamma$ (3)

$\Gamma, t : \mathbf{loc}; \Delta_0 \vdash v : \forall t_0.A \dashv \Delta_1$ (4)

by inversion on (T:Loc-App) with (1).

$\Gamma\{\rho/t\}; \Delta_0\{\rho/t\} \vdash v\{\rho/t\} : (\forall t_0.A)\{\rho/t\} \dashv \Delta_1\{\rho/t\}$ (5)

by induction hypothesis with (2) and (4).

$p\{\rho/t\} : \mathbf{loc} \in \Gamma\{\rho/t\}$ (6)

by induction hypothesis with (2) and (3), and by (LS:3.3).

$t_0 \neq t$ (7)

by def. of substitution up to rename of bounded location variables.

$\Gamma\{\rho/t\}; \Delta_0\{\rho/t\} \vdash v\{\rho/t\} : \forall t_0.A\{\rho/t\} \dashv \Delta_1\{\rho/t\}$ (8)

by (LS:2.8), (7) on (5).

$\Gamma\{\rho/t\}; \Delta_0\{\rho/t\} \vdash v\{\rho/t\}[p\{\rho/t\}] : A\{\rho/t\}\{p/t_0\} \dashv \Delta_1\{\rho/t\}$ (9)

by (T:Loc-App) on (8) and (6).

$\Gamma\{\rho/t\}; \Delta_0\{\rho/t\} \vdash (v[p])\{\rho/t\} : A\{\rho/t\}\{p/t_0\} \dashv \Delta_1\{\rho/t\}$ (10)

by (LS:1.12) on (8).

Thus, we conclude.

Case (T:Loc-Pack) - We have:

$\Gamma, t : \mathbf{loc}; \Delta_0 \vdash \langle p, v \rangle : \exists t_0.A \dashv \Delta_1$ (1)

$\rho : \mathbf{loc} \in \Gamma$ (2)

by hypothesis.

$\Gamma, t : \mathbf{loc}; \Delta_0 \vdash v : A\{p/t_0\} \dashv \Delta_1$ (3)

by inversion on (T:Loc-Pack) with (1).

$\Gamma\{\rho/t\}; \Delta_0\{\rho/t\} \vdash v\{\rho/t\} :\ A\{p/t_0\}\{\rho/t\} \dashv \Delta_1\{\rho/t\}$ (4)

by induction hypothesis on (3) and (2).

$t_0 \neq t$ (5)

by def. of substitution up to rename of bounded location variables.

$\Gamma; \Delta_0\{\rho/t\} \vdash v\{\rho/t\} :\ A\{\rho/t\}\{p/t_0\} \dashv \Delta_1\{\rho/t\}$ (6)

by (4) and (5).

$\Gamma\{\rho/t\}; \Delta_0\{\rho/t\} \vdash \langle p\{\rho/t\}, v\{\rho/t\} \rangle : \exists t_0.A\{\rho/t\} \dashv \Delta_1\{\rho/t\}$ (7)

by (T:Loc-Pack) on (6) and because p must be in Γ.
(therefore, its substitution must also occurred by (LS:3.3)).

$\Gamma\{\rho/t\}; \Delta_0\{\rho/t\} \vdash (\langle p, v \rangle)\{\rho/t\} : (\exists t_0.A)\{\rho/t\} \dashv \Delta_1\{\rho/t\}$ (8)

by (LS:1.11), (LS:2.9) on (7), (5).

Thus, we conclude.

Case (T:Loc-Open) - We have:

$\Gamma, t : \mathbf{loc}; \Delta_0 \vdash \mathsf{open}\,\langle t_0, x \rangle = v_0 \;\mathsf{in}\; e_1 \;\mathsf{end} : A_1 \dashv \Delta_2$ (1)

$\rho : \mathbf{loc} \in \Gamma$ (2)

by hypothesis.

$\Gamma, t : \mathbf{loc}; \Delta_0 \vdash v_0 : \exists t_0.A_0 \dashv \Delta_1$ (3)

$\Gamma, t : \mathbf{loc}, t_0 : \mathbf{loc}; \Delta_1, x : A_0 \vdash e_1 : A_1 \dashv \Delta_2$ (4)

by inversion on (T:Loc-Open) with (1).

$\Gamma\{\rho/t\}; \Delta_0\{\rho/t\} \vdash v_0\{\rho/t\} : (\exists t_0.A_0)\{\rho/t\} \dashv \Delta_1\{\rho/t\}$ (5)

by induction hypothesis on (2) and (3).

$(\Gamma, t_0 : \mathbf{loc})\{\rho/t\}; (\Delta_1, x : A_0)\{\rho/t\} \vdash e_1\{\rho/t\} : A_1\{\rho/t\} \dashv \Delta_2\{\rho/t\}$ (6)

by induction hypothesis on (2) and (4).

$t_0 \neq t$ (7)

by def. of substitution up to rename of bounded location variables.

$\Gamma\{\rho/t\}, t_0 : \mathbf{loc}; \Delta_1\{\rho/t\}, x : A_0\{\rho/t\} \vdash e_1\{\rho/t\} : A_1\{\rho/t\} \dashv \Delta_2\{\rho/t\}$ (8)

by (LS:3.3), (LS:4.2) on (7), (6).

$\Gamma\{\rho/t\}; \Delta_0\{\rho/t\} \vdash v_0\{\rho/t\} : \exists t_0.A_0\{\rho/t\} \dashv \Delta_1\{\rho/t\}$ (9)

by (LS:2.10) on (5), (7).

$\Gamma\{\rho/t\}; \Delta_0\{\rho/t\} \vdash \mathsf{open}\,\langle t_0, x \rangle = v_0\{\rho/t\}$
$\mathsf{in}\; e_1\{\rho/t\} \;\mathsf{end} : A_1\{\rho/t\} \dashv \Delta_2\{\rho/t\}$ (10)

by (T:Loc-Open) on (8) and (9).

$\Gamma\{\rho/t\}; \Delta_0\{\rho/t\} \vdash (\mathsf{open}\,\langle t_0, x \rangle = v_0 \;\mathsf{in}\; e_1 \;\mathsf{end})\{\rho/t\} : A_1\{\rho/t\} \dashv \Delta_2\{\rho/t\}$ (11)

by (LS:1.14) on (10).

Thus, we conclude.

Case (T:Forall-Type) - Analogous to (T:Forall-Loc).
Case (T:Type-App) - Analogous to (T:Loc-App).
Case (T:Type-Pack) - Analogous to (T:Loc-Pack).
Case (T:Type-Open) - Analogous to (T:Loc-Open).
Case (T:Cap-Elim) - We have:

$\Gamma, t : \mathbf{loc}; \Delta_0, x : A_1 :: A_2 \vdash e : A_0 \dashv \Delta_1$ (1)

$\rho : \mathbf{loc} \in \Gamma$ (2)

by hypothesis.

$\Gamma, t : \mathbf{loc}; \Delta_0, x : A_1, A_2 \vdash e : A_0 \dashv \Delta_1$ (3)

by inversion on (T:Cap-Elim) with (1).

$\Gamma\{\rho/t\}; (\Delta_0, x : A_1, A_2)\{\rho/t\} \vdash e\{\rho/t\} : A_0\{\rho/t\} \dashv \Delta_1\{\rho/t\}$ (4)

by induction hypothesis with (2) and (3).

$\Gamma\{\rho/t\}; \Delta_0\{\rho/t\}, x : A_1\{\rho/t\}, A_2\{\rho/t\} \vdash e\{\rho/t\} : A_0\{\rho/t\} \dashv \Delta_1\{\rho/t\}$ (5)

by (LS:4.3), (LS:4.2) on (4).

$\Gamma\{\rho/t\}; \Delta_0\{\rho/t\}, x : A_1\{\rho/t\} :: A_2\{\rho/t\} \vdash e\{\rho/t\} : A_0\{\rho/t\} \dashv \Delta_1\{\rho/t\}$ (6)

by (T:Cap-Elim) with (5).

$\Gamma\{\rho/t\}; (\Delta_0, x : A_1 :: A_2)\{\rho/t\} \vdash e\{\rho/t\} : A_0\{\rho/t\} \dashv \Delta_1\{\rho/t\}$ (7)

by (LS:4.2), (LS:2.6) on (6).

Thus, we conclude.

Case (T:Cap-Stack), (T:Cap-Unstack) - Analogous to (T:Cap-Elim).

Case (T:Frame) - We have:

$\Gamma, t : \mathbf{loc}; \Delta_0, \Delta_2 \vdash e : A \dashv \Delta_1, \Delta_2$ (1)

$\rho : \mathbf{loc} \in \Gamma$ (2)

by hypothesis.

$\Gamma, t : \mathbf{loc}; \Delta_0 \vdash e : A \dashv \Delta_1$ (3)

by inversion on (T:Frame) with (1).

$\Gamma\{\rho/t\}; \Delta_0\{\rho/t\} \vdash e\{\rho/t\} : A\{\rho/t\} \dashv \Delta_1\{\rho/t\}$ (4)

by induction hypothesis with (2) and (3).

$\Gamma, t : \mathbf{loc} \vdash \Delta_0, \Delta_2$ (5)

by typing on (1).

$\Gamma\{\rho/t\} \vdash (\Delta_0\{\rho/t\}), (\Delta_2\{\rho/t\})$ (6)

by (Well-Formed Type Substitution - Delta) on (5) and (2)
and by (LS:4.*).

$\Gamma\{\rho/t\} \vdash \Delta_2\{\rho/t\}$ (7)

by (Well-Formed Delta) on (6).

$\Gamma\{\rho/t\}; \Delta_0\{\rho/t\}, \Delta_2\{\rho/t\} \vdash e\{\rho/t\} : A\{\rho/t\} \dashv \Delta_1\{\rho/t\}, \Delta_2\{\rho/t\}$ (8)

by (T:Frame) on (7) and (4).

$\Gamma\{\rho/t\}; (\Delta_0, \Delta_2)\{\rho/t\} \vdash e\{\rho/t\} : A\{\rho/t\} \dashv (\Delta_1, \Delta_2)\{\rho/t\}$ (9)

and by (LS:4.*).

Thus, we conclude.

Case (T:Subsumption) - We have:

$\Gamma, t : \mathbf{loc}; \Delta_0 \vdash e : A_1 \dashv \Delta_1$ (1)

$\rho : \mathbf{loc} \in \Gamma$ (2)

by hypothesis.

$\Delta_0 <: \Delta'_0$ (3)

$\Gamma, t : \mathbf{loc}; \Delta'_0 \vdash e : A_0 \dashv \Delta'_1$ (4)

$A_0 <: A_1$ (5)

$\Delta'_1 <: \Delta_1$ (6)

by inversion on (T:Subsumption) with (1).

$\Gamma\{\rho/t\}; \Delta'_0\{\rho/t\} \vdash e\{\rho/t\} : A_0\{\rho/t\} \dashv \Delta'_1\{\rho/t\}$ (7)

by induction hypothesis on (4) with (2).

$\Gamma, t : \mathbf{loc} \vdash \Delta_0$ (8)

by typing on (1).

$\Gamma\{\rho/t\} \vdash \Delta_0\{\rho/t\}$ (9)

by (Well-Formed Type Substitution - Gamma) on (8), (2).

$\Delta_0\{\rho/t\} <: \Delta'_0\{\rho/t\}$ (10)

by (3) and (9).

$A_0\{\rho/t\} <: A_1\{\rho/t\}$ (11)
$\Delta_1'\{\rho/t\} <: \Delta_1\{\rho/t\}$ (12)
analogous reasoning using
(Well-Formed Type Substitution - Delta) on (5) and (6).
$\Gamma\{\rho/t\}; \Delta_0\{\rho/t\} \vdash e\{\rho/t\} : A_1\{\rho/t\} \dashv \Delta_1'\{\rho/t\}$ (13)
by (t:Subsumption) on (7), (10), (11) and (12).
Thus, we conclude.

Case (t:Tag) - We have:
$\Gamma, t : \mathbf{loc}; \Delta_0 \vdash \mathtt{l}\#v : \mathtt{l}\#A \dashv \Delta_1$ (1)
$\rho : \mathbf{loc} \in \Gamma$ (2)
by hypothesis.
$\Gamma, t : \mathbf{loc}; \Delta_0 \vdash v : A \dashv \Delta_1$ (3)
by inversion on (t:Tag) with (1).
$\Gamma\{\rho/t\}; \Delta_0\{\rho/t\} \vdash v\{\rho/t\} : A\{\rho/t\} \dashv \Delta_1\{\rho/t\}$ (4)
by induction hypothesis on (3) and (2).
$\Gamma\{\rho/t\}; \Delta_0\{\rho/t\} \vdash \mathtt{l}\#v\{\rho/t\} : \mathtt{l}\#A\{\rho/t\} \dashv \Delta_1\{\rho/t\}$ (5)
by (t:Tag) on (4).
$\Gamma\{\rho/t\}; \Delta_0\{\rho/t\} \vdash (\mathtt{l}\#v)\{\rho/t\} : (\mathtt{l}\#A)\{\rho/t\} \dashv \Delta_1\{\rho/t\}$ (6)
by (ls:1.19), (ls:2.18) on (5).
Thus, we conclude.

Case (t:Case) - We have:
$\Gamma, t : \mathbf{loc}; \Delta_0 \vdash \mathsf{case}\ v\ \mathsf{of}\ \overline{\mathtt{l}_j\#x_j \to e_j}\ \mathsf{end} : A \dashv \Delta_2$ (1)
$\rho : \mathbf{loc} \in \Gamma$ (2)
by hypothesis.
$\Gamma, t : \mathbf{loc}; \Delta_1 \vdash v : \sum_i \mathtt{l}_i\#A_i' \dashv \Delta'$ (3)
$\overline{\Gamma, t : \mathbf{loc}; \Delta', x_i : A_i' \vdash e_i : A \dashv \Delta_2}$ (4)
$i \leq j$ (5)
by inversion (t:Case) with (1).
$\Gamma\{\rho/t\}; \Delta_1\{\rho/t\} \vdash v\{\rho/t\} : (\sum_i \mathtt{l}_i\#A_i')\{\rho/t\} \dashv \Delta'\{\rho/t\}$ (6)
by induction hypothesis on (3) and (2).
$\Gamma\{\rho/t\}; \Delta_1\{\rho/t\} \vdash v\{\rho/t\} : \sum_i \mathtt{l}_i\#(A_i'\{\rho/t\}) \dashv \Delta'\{\rho/t\}$ (7)
by (ls:2.18) on (6).
$\overline{\Gamma\{\rho/t\}; (\Delta', x_i : A_i')\{\rho/t\} \vdash e_i\{\rho/t\} : A\{\rho/t\} \dashv \Delta_2\{\rho/t\}}$ (8)
by induction hypothesis on (4) and (2).
$\overline{\Gamma\{\rho/t\}; \Delta', x_i : A_i'\{\rho/t\} \vdash e_i\{\rho/t\} : A\{\rho/t\} \dashv \Delta_2\{\rho/t\}}$ (9)
by (ls:4.2) on (8).
$\Gamma\{\rho/t\}; \Delta_0\{\rho/t\} \vdash \mathsf{case}\ v\{\rho/t\}\ \mathsf{of}\ \overline{\mathtt{l}_j\#x_j \to e_j\{\rho/t\}}\ \mathsf{end} : A\{\rho/t\} \dashv \Delta_2\{\rho/t\}$ (10)
by (t:Case) on (5), (7) and (9).
$\Gamma\{\rho/t\}; \Delta_0\{\rho/t\} \vdash (\mathsf{case}\ v\ \mathsf{of}\ \overline{\mathtt{l}_j\#x_j \to e_j}\ \mathsf{end})\{\rho/t\} : A\{\rho/t\} \dashv \Delta_2\{\rho/t\}$ (11)
by (ls:1.20) on (10).
Thus, we conclude.

Case (t:Alternative-Left) - Immediate by applying the induction hypothesis on the inversion and then re-applying the rule.
Case (t:Let) - Analogous to (t:Loc-Open).

□

4. (Type Variable), analogous to the (Location Variable) proof.

□

B.9 Values Lemma

Lemma 9 (Values Lemma). If v is a closed value such that:

$$\widehat{\Gamma}; \widehat{\Delta} \vdash v : A \dashv \widehat{\Delta'}$$

then:

$$\widehat{\Delta} <: \widehat{\Delta_v}, \widehat{\Delta'} \qquad \widehat{\Gamma}; \widehat{\Delta_v} \vdash v : A \dashv \cdot$$

Proof. By induction on the typing derivation of $\widehat{\Gamma}; \widehat{\Delta} \vdash v : A \dashv \widehat{\Delta'}$.

Case (t:Ref) - We have:

$\widehat{\Gamma}, \rho : \mathbf{loc}; \cdot \vdash \rho : \mathbf{ref}\ \rho \dashv \cdot$ (1)
by hypothesis.
Thus, by making:
$\widehat{\Delta_v} = \cdot$ (2)
$\widehat{\Delta'} = \cdot$ (3)
We immediately conclude.

Case (t:Pure) - We have:

$\widehat{\Gamma}; \cdot \vdash v : !A \dashv \cdot$ (1)
by hypothesis.
Thus, by making:
$\widehat{\Delta_v} = \cdot$ (2)
$\widehat{\Delta'} = \cdot$ (3)
We immediately conclude.

Case (t:Unit) - We have:

$\widehat{\Gamma}; \cdot \vdash v : [] \dashv \cdot$ (1)
by hypothesis.
Thus, by making:
$\widehat{\Delta_v} = \cdot$ (2)
$\widehat{\Delta'} = \cdot$ (3)
We immediately conclude.

Case (t:Pure-Read), (t:Linear-Read) - value not closed.
Case (t:Pure-Elim) - Environment not closed.
Case (t:New),(t:Delete), (t:Assign), (t:Dereference-Linear), (t:Dereference-Pure) - Not a value.
Case (t:Record) - We have:

$\widehat{\Gamma}; \widehat{\Delta_0} \vdash \{\overline{\mathtt{f} = v}\} : [\overline{\mathtt{f} : A}] \dashv \widehat{\Delta_1}$ (1)
by hypothesis.
$\overline{\widehat{\Gamma}; \widehat{\Delta_0} \vdash v : A \dashv \widehat{\Delta_1}}$ (2)
by inversion on (t:Record) with (1).
$\widehat{\Delta_0} <: \widehat{\Delta_v}, \widehat{\Delta_1}$ (3)
$\overline{\widehat{\Gamma}; \widehat{\Delta_v} \vdash v : A \dashv \cdot}$ (4)
by induction hypothesis on (2).
$\widehat{\Gamma}; \widehat{\Delta_v} \vdash \{\overline{\mathtt{f} = v}\} : [\overline{\mathtt{f} : A}] \dashv \cdot$ (5)
by (t:Record) on (4).
Therefore, by (3) and (5) we conclude.

Case (t:Selection) - Not a value.
Case (t:Application) - Not a value.
Case (t:Function) - We have:

$\widehat{\Gamma}; \widehat{\Delta} \vdash \mathsf{fun}(x : A_0).e : A_0 \multimap A_1 \dashv \cdot$ (1)
by hypothesis.
Thus, by making:
$\widehat{\Delta'} = \cdot$ (2)
We immediately conclude.

Case (t:Forall-Loc) - We have:

$\widehat{\Gamma}; \widehat{\Delta} \vdash \langle t \rangle\, e : \forall t.A \dashv \cdot$ (1)
by hypothesis.
Thus, by making:
$\widehat{\Delta'} = \cdot$ (2)
We immediately conclude.

Case (t:Loc-App) - Not a value.
Case (t:Loc-Pack) - We have:

$\widehat{\Gamma};\widehat{\Delta} \vdash \langle p, v\rangle : \exists t.A \dashv \cdot$ (1)
by hypothesis.

$\widehat{\Gamma};\widehat{\Delta} \vdash v : A\{p/t\} \dashv \cdot$ (2)
by inversion on (t:Loc-Pack) with (1).

$\widehat{\Delta} <: \widehat{\Delta_v}, \cdot$ (3)
$\widehat{\Gamma};\widehat{\Delta_v} \vdash v : A\{p/t\} \dashv \cdot$ (4)
by induction hypothesis on (2).

$\widehat{\Gamma};\widehat{\Delta_v} \vdash \langle p, v\rangle : \exists t.A \dashv \cdot$ (5)
by (t:Loc-Pack) on (4).

Therefore, by (3) and (5) we conclude.

Case (t:Loc-Open) - Not a value.
Case (t:Forall-Type) - We have:

$\widehat{\Gamma};\widehat{\Delta} \vdash \langle X\rangle\, e : \forall X.A \dashv \cdot$ (1)
by hypothesis.

Thus, by making:
$\widehat{\Delta'} = \cdot$ (2)
We immediately conclude.

Case (t:Type-App) - Not a value.
Case (t:Type-Pack) - We have:

$\widehat{\Gamma};\widehat{\Delta} \vdash \langle A_1, v\rangle : \exists X.A_0 \dashv \cdot$ (1)
by hypothesis.

$\widehat{\Gamma};\widehat{\Delta} \vdash v : A_0\{A_1/X\} \dashv \cdot$ (2)
by inversion on (t:Type-Pack) with (1).

$\widehat{\Delta} <: \widehat{\Delta_v}, \cdot$ (3)
$\widehat{\Gamma};\widehat{\Delta_v} \vdash v : A_0\{A_1/X\} \dashv \cdot$ (4)
by induction hypothesis on (2).

$\widehat{\Gamma};\widehat{\Delta_v} \vdash \langle A_1, v\rangle : \exists X.A_0 \dashv \cdot$ (5)
by (t:Type-Pack) on (4).

Therefore, by (3) and (5) we conclude.

Case (t:Type-Open) - Not a value.
Case (t:Cap-Elim) - Environment not closed.
Case (t:Cap-Stack) - We have:

$\widehat{\Gamma};\widehat{\Delta_0} \vdash v : A_0 :: A_1 \dashv \widehat{\Delta_1}$ (1)
by hypothesis.

$\widehat{\Gamma};\widehat{\Delta_0} \vdash v : A_0 \dashv \widehat{\Delta_1}, A_1$ (2)
by inversion on (t:Cap-Stack) with (1).

$\widehat{\Delta_0} <: \widehat{\Delta_v}, \widehat{\Delta_1}, A_1$ (3)
$\widehat{\Gamma};\widehat{\Delta_v} \vdash v : A_0 \dashv \cdot$ (4)
by induction hypothesis on (2).

$\widehat{\Gamma};\widehat{\Delta_v}, A_1 \vdash v : A_0 \dashv A_1$ (5)
by (t:Frame) on (4) using A_1.

Note that this application of (t:Frame) can be applied directly since $\widehat{\Delta_v}$.

$\widehat{\Gamma};\widehat{\Delta_v}, A_1 \vdash v : A_0 :: A_1 \dashv \cdot$ (6)
by (t:Cap-Stack) on (5).

Therefore, by (3) and (6) we conclude.
(note that A_1 is immediate since a defocus-guarantee is not a type)

Case (t:Cap-Unstack) - We have:

$\widehat{\Gamma};\widehat{\Delta_0} \vdash v : A_0 \dashv \widehat{\Delta_1}, A_1$ (1)
by hypothesis.

$\widehat{\Gamma};\widehat{\Delta_0} \vdash v : A_0 :: A_1 \dashv \widehat{\Delta_1}$ (2)
by inversion on (t:Cap-Unstack) with (1).

$\widehat{\Delta_0} <: \widehat{\Delta_v}, \widehat{\Delta_1}$ (3)
$\widehat{\Gamma};\widehat{\Delta_v} \vdash v : A_0 :: A_1 \dashv \cdot$ (4)
by induction hypothesis on (2).

$\widehat{\Gamma};\widehat{\Delta_v} \vdash v : A_0 \dashv A_1$ (5)
by (t:Cap-Unstack) with (4).

$\widehat{\Delta_v} <: \widehat{\Delta'_v}, A_1$ (6)
$\widehat{\Gamma};\widehat{\Delta'_v} \vdash v : A_0 \dashv \cdot$ (7)
by induction hypothesis on (5).

$\widehat{\Delta_0} <: \widehat{\Delta'_v}, A_1, \widehat{\Delta_1}$ (8)
by transitivity of subtyping with (3) and (6).

Therefore, by (7) and (8) we conclude.

Case (t:Frame) - We have:

$\widehat{\Gamma};\widehat{\Delta_0}, \widehat{\Delta_2} \vdash v : A \dashv \widehat{\Delta_1}, \widehat{\Delta_2}$ (1)
by hypothesis.

$\widehat{\Gamma};\widehat{\Delta_0} \vdash v : A \dashv \widehat{\Delta_1}$ (2)
by inversion on (t:Frame) with (1).

$\widehat{\Delta_0} <: \widehat{\Delta_v}, \widehat{\Delta_1}$ (3)
$\widehat{\Gamma};\widehat{\Delta_v} \vdash v : A \dashv \cdot$ (4)
by induction hypothesis on (2).

$\widehat{\Delta_0}, \widehat{\Delta_2} <: \widehat{\Delta_v}, \widehat{\Delta_1}, \widehat{\Delta_2}$ (5)
by (3) with $\widehat{\Delta_2}$.

Therefore, by (4) and (5) we immediately conclude.

Case (t:Subsumption) - We have:

$\widehat{\Gamma};\widehat{\Delta_0} \vdash v : A_1 \dashv \widehat{\Delta_1}$ (1)
by hypothesis.

$\widehat{\Delta_0} <: \widehat{\Delta'_0}$ (2)
$\widehat{\Gamma};\widehat{\Delta'_0} \vdash v : A_0 \dashv \widehat{\Delta'_1}$ (3)
$A_0 <: A_1$ (4)
$\widehat{\Delta'_1} <: \widehat{\Delta_1}$ (5)
by inversion on (t:Subsumption) with (1).

$\widehat{\Delta'_0} <: \widehat{\Delta_v}, \widehat{\Delta'_1}$ (6)
$\widehat{\Gamma};\widehat{\Delta_v} \vdash v : A_0 \dashv \cdot$ (7)
by induction hypothesis on (3).

$\widehat{\Delta'_0} <: \widehat{\Delta_v}, \widehat{\Delta_1}$ (8)
by transitivity of subtyping with (5) and (6).

$\widehat{\Delta_0} <: \widehat{\Delta_v}, \widehat{\Delta_1}$ (9)
by transitivity of subtyping with (2) and (8).

$\widehat{\Gamma};\widehat{\Delta_v} \vdash v : A_1 \dashv \cdot$ (10)
by (t:Subsumption) with (sd:Symmetry) and (4) on (7).

Therefore, by (9) and (10) we conclude.

Case (t:Tag) - We have:

$\widehat{\Gamma};\widehat{\Delta_0} \vdash \mathtt{l}\#v : \mathtt{l}\#A \dashv \cdot$ (1)
by hypothesis.

$\widehat{\Gamma};\widehat{\Delta_0} \vdash v : A \dashv \cdot$ (2)
by inversion on (t:Tag) with (1).

$\widehat{\Delta_0} <: \widehat{\Delta_v}, \widehat{\Delta_1}$ (3)
$\widehat{\Gamma};\widehat{\Delta_v} \vdash v : A \dashv \cdot$ (4)
by induction hypothesis on (2).

$\widehat{\Gamma};\widehat{\Delta_v} \vdash \mathtt{l}\#v : \mathtt{l}\#A \dashv \cdot$ (5)
by (t:Tag) on (4).

Therefore, by (5) and (3) we conclude.

Case (t:Case) - Not a value.
Case (t:Alternative-Left) - We have:

$\widehat{\Gamma};\widehat{\Delta_0}, A_0 \oplus A_1 \vdash v : A_2 \dashv \widehat{\Delta_1}$ (1)
by hypothesis.

$\widehat{\Gamma};\widehat{\Delta_0}, A_0 \vdash v : A_2 \dashv \widehat{\Delta_1}$ (2)
$\widehat{\Gamma};\widehat{\Delta_0}, A_1 \vdash v : A_2 \dashv \widehat{\Delta_1}$ (3)
by inversion on (t:Alternative-Left) with (1).

$\widehat{\Delta_0}, A_0 <: \widehat{\Delta_v}, \widehat{\Delta_1}$ (4)
$\widehat{\Gamma};\widehat{\Delta_v} \vdash v : A_2 \dashv \cdot$ (5)
by induction hypothesis on (2).

$\widehat{\Delta_0}, A_1 <: \widehat{\Delta_v}, \widehat{\Delta_1}$ (6)
$\widehat{\Gamma};\widehat{\Delta_v} \vdash v : A_2 \dashv \cdot$ (7)
by induction hypothesis on (3).

$\widehat{\Delta_0}, A_0 \oplus A_1 <: \widehat{\Delta_v}, \widehat{\Delta_1}$ (8)
by (sd:Alternative-L) on (4) and (6).

Therefore, by (8) and (5) we conclude.

Case (t:Let) Not a value.

□

B.10 Preservation

Theorem 1 (Preservation)**.** If e_0 is a closed expression such that:

$$\widehat{\Gamma_0};\widehat{\Delta_0} \vdash e_0 : A \dashv \widehat{\Delta}$$

$$\widehat{\Gamma_0};\widehat{\Delta_0} \vdash H_0 \qquad \langle\, H_0 \parallel e_0 \,\rangle \mapsto \langle\, H_1 \parallel e_1 \,\rangle$$

then:

$$\widehat{\Gamma_0},\widehat{\Gamma_1};\widehat{\Delta_1} \vdash H_1 \qquad \widehat{\Gamma_0},\widehat{\Gamma_1};\widehat{\Delta_1} \vdash e_1 : A \dashv \widehat{\Delta}$$

for some $\widehat{\Delta_1},\widehat{\Gamma_1}$.

Proof. By induction on the typing derivation of $\widehat{\Gamma_0};\widehat{\Delta_0} \vdash e_0 : A \dashv \widehat{\Delta}$.

Case (t:Ref), (t:Pure), (t:Unit) - are values.

Case (t:Pure-Read), (t:Linear-Read), (t:Pure-Elim) - not applicable, environments not closed.

Case (t:New) - We have:

$\widehat{\Gamma_0};\widehat{\Delta_0} \vdash \mathsf{new}\ v : \exists t.(\mathbf{ref}\ t :: \mathbf{rw}\ t\ A) \dashv \widehat{\Delta}$ (1)
$\widehat{\Gamma_0};\widehat{\Delta_0} \vdash H$ (2)
$\langle\, H \parallel \mathsf{new}\ v \,\rangle \mapsto \langle\, H\ ,\ \rho \hookrightarrow v \parallel \langle \rho,\rho \rangle \,\rangle$ (3)
by hypothesis, with (d:New).

$\widehat{\Gamma_0};\widehat{\Delta_0} \vdash v : A \dashv \widehat{\Delta}$ (4)
by inversion on (t:New) with (1).

$\widehat{\Delta_0} <: \widehat{\Delta_v},\widehat{\Delta}$ (5)
$\widehat{\Gamma_0};\widehat{\Delta_v} \vdash v : A \dashv \cdot$ (6)
by (Values Lemma) with (4).

ρ **fresh** (7)
by inversion on (d:New) with (3).

$\widehat{\Gamma_0};\widehat{\Delta_v},\widehat{\Delta} \vdash H$ (8)
by (Subtyping Store Typing) with (2) and (5).

Thus, if we make:
$\widehat{\Gamma_1} = \rho : \mathbf{loc}$ (9)
We have that:
$\widehat{\Gamma_0},\widehat{\Gamma_1};\widehat{\Delta_v} \vdash v : A \dashv \cdot$ (10)
by (Weakening) (6) with $\widehat{\Gamma_1}$.
(note that weakening is only valid in the lexical environments, Γ)
$\widehat{\Gamma_0},\widehat{\Gamma_1};\widehat{\Delta_v},\widehat{\Delta} \vdash H$ (11)
by (str:Loc) with $\widehat{\Gamma_1}$ (that contains ρ) on (8).
$\widehat{\Gamma_0},\widehat{\Gamma_1};\widehat{\Delta},\mathbf{rw}\ \rho\ A \vdash H\ ,\ \rho \hookrightarrow v$ (12)
by (str:Binding) with (10) and (11) with ρ.
Thus, if we make:
$\widehat{\Delta_1} = \widehat{\Delta},\mathbf{rw}\ \rho\ A$ (13)
We have that:
$\widehat{\Gamma_0},\widehat{\Gamma_1};\cdot \vdash \rho : \mathbf{ref}\ \rho \dashv \cdot$ (14)
by (t:Ref) with ρ.
$\widehat{\Gamma_0},\widehat{\Gamma_1};\widehat{\Delta_1} \vdash \rho : \mathbf{ref}\ \rho \dashv \widehat{\Delta_1}$ (15)
by (t:Frame) on (14) with $\widehat{\Delta_1}$ (since $\cdot$ is empty, frame is immediate).
$\widehat{\Gamma_0},\widehat{\Gamma_1};\widehat{\Delta_1} \vdash \rho : \mathbf{ref}\ \rho :: \mathbf{rw}\ \rho\ A \dashv \widehat{\Delta}$ (16)
by (t:Cap-Stack) on (15) noting that (13).
If t **fresh** then:
$\widehat{\Gamma_0},\widehat{\Gamma_1};\widehat{\Delta_1} \vdash \rho : (\mathbf{ref}\ \rho :: \mathbf{rw}\ \rho\ A)\{\rho/t\} \dashv \widehat{\Delta}$ (17)
by type substitution on (16).

Note that, by (4), ρ cannot occur in A since it is a fresh location constant not present in $\widehat{\Gamma_0}$.

$\widehat{\Gamma_0},\widehat{\Gamma_1};\widehat{\Delta_1} \vdash \langle \rho,\rho \rangle : \exists t.(\mathbf{ref}\ t :: \mathbf{rw}\ t\ A) \dashv \widehat{\Delta}$ (18)
by (t:Loc-Pack) on (17).
Thus:
$\widehat{\Gamma_0},\widehat{\Gamma_1};\widehat{\Delta_1} \vdash \langle \rho,\rho \rangle : \exists t.(\mathbf{ref}\ t :: \mathbf{rw}\ t\ A) \dashv \widehat{\Delta}$ (19)
for some $\widehat{\Delta_1},\widehat{\Gamma_1}$.
by (18).

Therefore, by (12) and (19) we conclude.

Case (t:Delete) - We have:

$\widehat{\Gamma_0};\widehat{\Delta_0} \vdash \mathsf{delete}\ \langle \rho,\rho \rangle : \exists t.A \dashv \widehat{\Delta}$ (1)
$\widehat{\Gamma_0};\widehat{\Delta_0} \vdash H\ ,\ \rho \hookrightarrow v$ (2)
$\langle\, H\ ,\ \rho \hookrightarrow v \parallel \mathsf{delete}\ \langle \rho,\rho \rangle \,\rangle \mapsto \langle\, H \parallel \langle \rho,v \rangle \,\rangle$ (3)
by hypothesis, with (d:Delete).
$\widehat{\Gamma_0};\widehat{\Delta_0} \vdash \langle \rho,\rho \rangle : \exists t.(\mathbf{ref}\ t :: \mathbf{rw}\ t\ A) \dashv \widehat{\Delta}$ (4)
by inversion on (t:Delete) with (1).
$\widehat{\Delta_0} <: \widehat{\Delta_\rho},\widehat{\Delta}$ (5)
$\widehat{\Gamma_0};\widehat{\Delta_\rho} \vdash \langle \rho,\rho \rangle : \exists t.(\mathbf{ref}\ t :: \mathbf{rw}\ t\ A) \dashv \cdot$ (6)
by (Values Lemma) with (4).
(note that we will omit the G syntax until relevant, for clarity)
$\widehat{\Gamma_0};\widehat{\Delta_\rho} \vdash \rho : (\mathbf{ref}\ t :: \mathbf{rw}\ t\ A)\{\rho/t\} \dashv \cdot$ (7)
by (Values Inversion Lemma) with (6).
$\widehat{\Gamma_0};\widehat{\Delta_\rho} \vdash \rho : \mathbf{ref}\ \rho :: \mathbf{rw}\ \rho\ A\{\rho/t\} \dashv \cdot$ (8)
by (ls:2.6), (ls:2.10), (ls:2.1), (ls:2.12) with (7).
$\widehat{\Gamma_0};\widehat{\Delta_\rho} \vdash \rho : \mathbf{ref}\ \rho \dashv \mathbf{rw}\ \rho\ A\{\rho/t\}$ (9)
by (Values Inversion Lemma) with (8).
$\widehat{\Delta_\rho} <: \widehat{\Delta'_\rho},\mathbf{rw}\ \rho\ A\{\rho/t\}$ (10)
$\widehat{\Gamma_0};\widehat{\Delta'_\rho} \vdash \rho : \mathbf{ref}\ \rho \dashv \cdot$ (11)
by (Values Lemma) with (9).
$\widehat{\Delta'_\rho} = \cdot$ (12)
by inversion on (t:Ref) with (11).
Therefore:
$\widehat{\Gamma_0};\widehat{\Delta'_\rho},\mathbf{rw}\ \rho\ A\{\rho/t\},\widehat{\Delta} \vdash H\ ,\ \rho \hookrightarrow v$ i.e.:
$\widehat{\Gamma_0};\mathbf{rw}\ \rho\ A\{\rho/t\},\widehat{\Delta} \vdash H\ ,\ \rho \hookrightarrow v$ (13)
by (Subtyping Store Typing) using (2), (10) and (12).
$\widehat{\Gamma_0};\widehat{\Delta_v},\widehat{\Delta} \vdash H$ (14)
$\widehat{\Gamma_0};\widehat{\Delta_v} \vdash v : A\{\rho/t\} \dashv \cdot$ (15)
by (Store Typing Inversion Lemma) with (13).
$\widehat{\Gamma_0};\widehat{\Delta_v} \vdash \langle \rho,v \rangle : \exists t.A \dashv \cdot$ (16)
by (t:Loc-Pack) with (15) using ρ.
$\widehat{\Gamma_0};\widehat{\Delta_v},\widehat{\Delta} \vdash \langle \rho,v \rangle : \exists t.A \dashv \widehat{\Delta}$ (17)
by (t:Frame) with (16) using $\widehat{\Delta}$.
Using:
$\widehat{\Gamma_1} = \cdot$ (18)
$\widehat{\Delta_1} = \widehat{\Delta_v},\widehat{\Delta}$ (19)
We have:
$\widehat{\Gamma_0},\widehat{\Gamma_1};\widehat{\Delta_1} \vdash \langle \rho,v \rangle : \exists t.A \vdash \widehat{\Delta}$ (20)
by (17) with (18) and (19).
$\widehat{\Gamma_0},\widehat{\Gamma_1};\widehat{\Delta_1} \vdash H$ (21)
by (14) with (18) and (19).
Therefore, by (20) and (21) we conclude.

Case (t:Assign) - We have:

$\widehat{\Gamma_0};\widehat{\Delta_0} \vdash \rho := v_1 : A_1 \dashv \widehat{\Delta},\mathbf{rw}\ \rho\ A_0$ (1)
$\widehat{\Gamma_0};\widehat{\Delta_0} \vdash H\ ,\ \rho \hookrightarrow v_0$ (2)
$\langle\, H\ ,\ \rho \hookrightarrow v_0 \parallel \rho := v_1 \,\rangle \mapsto \langle\, H\ ,\ \rho \hookrightarrow v_1 \parallel v_0 \,\rangle$ (3)
by hypothesis.
$\widehat{\Gamma_0};\widehat{\Delta_0} \vdash v_1 : A_0 \dashv \widehat{\Delta'}$ (4)
$\widehat{\Gamma_0};\widehat{\Delta'} \vdash \rho : \mathbf{ref}\ \rho \dashv \widehat{\Delta},\mathbf{rw}\ \rho\ A_1$ (5)
by inversion on (t:Assign) with (1).
$\widehat{\Delta_0} <: \widehat{\Delta_{v_1}},\widehat{\Delta'}$ (6)
$\widehat{\Gamma_0};\widehat{\Delta_{v_1}} \vdash v_1 : A_0 \dashv \cdot$ (7)
by (Values Lemma) on (4).
$\widehat{\Delta'} <: \widehat{\Delta_\rho},\widehat{\Delta},\mathbf{rw}\ \rho\ A_1$ (8)
$\widehat{\Gamma_0};\widehat{\Delta_\rho} \vdash \rho : \mathbf{ref}\ \rho \dashv \cdot$ (9)
by (Values Lemma) on (5).
$\widehat{\Delta_\rho} = \cdot$ (10)
by inversion on (t:Ref) with (9).
$\widehat{\Gamma_0};\widehat{\Delta_{v_1}},\widehat{\Delta},\mathbf{rw}\ \rho\ A_1 \vdash H\ ,\ \rho \hookrightarrow v_0$ (11)
by (Subtyping Store Typing) with (2), (6) and (8).
$\widehat{\Gamma_0};\widehat{\Delta_{v_1}},\widehat{\Delta_{v_0}},\widehat{\Delta} \vdash H$ (12)
$\widehat{\Gamma_0};\widehat{\Delta_{v_0}} \vdash v_0 : A_1 \dashv \cdot$ (13)
by (Store Typing Inversion Lemma) on (11).
$\widehat{\Gamma_0};\widehat{\Delta_{v_0}},\widehat{\Delta},\mathbf{rw}\ \rho\ A_0 \vdash H\ ,\ \rho \hookrightarrow v_1$ (14)
by (str:Binding) with ρ on (7) and (12).
by making:
$\widehat{\Gamma_1} = \cdot$ (15)
$\widehat{\Gamma_0},\widehat{\Gamma_1};\widehat{\Delta_{v_0}},\widehat{\Delta},\mathbf{rw}\ \rho\ A_0 \vdash H\ ,\ \rho \hookrightarrow v_1$ (16)
by (Weakening) with (14).
$\widehat{\Gamma_0},\widehat{\Gamma_1};\widehat{\Delta_{v_0}} \vdash v_0 : A_1 \dashv \cdot$ (17)
by (Weakening) on (13).
$\widehat{\Gamma_0},\widehat{\Gamma_1};\widehat{\Delta_{v_0}},\widehat{\Delta},\mathbf{rw}\ \rho\ A_0 \vdash v_0 : A_1 \dashv \widehat{\Delta},\mathbf{rw}\ \rho\ A_0$ (18)
by (t:Frame) using $\widehat{\Delta},\mathbf{rw}\ \rho\ A_0$ with (17).
Therefore, by (16) and (18) we conclude.

Case (t:Dereference-Linear) - We have:

$\widehat{\Gamma_0};\widehat{\Delta_0} \vdash\ !\rho : A \dashv \widehat{\Delta},\mathbf{rw}\ \rho\ []$ (1)
$\widehat{\Gamma_0};\widehat{\Delta_0} \vdash H\ ,\ \rho \hookrightarrow v$ (2)
$\langle\, H\ ,\ \rho \hookrightarrow v \parallel\ !\rho \,\rangle \mapsto \langle\, H\ ,\ \rho \hookrightarrow v \parallel v \,\rangle$ (3)
by hypothesis, (d:Dereference).
$\widehat{\Gamma_0};\widehat{\Delta_0} \vdash \rho : \mathbf{ref}\ \rho \dashv \widehat{\Delta},\mathbf{rw}\ \rho\ []$ (4)
by inversion on (t:Dereference-Linear) with (1).

$\widehat{\Delta_0} <: \widehat{\Delta_\rho}, \widehat{\Delta}, \mathbf{rw}\,\rho\,A$ (5)

$\widehat{\Gamma_0}; \widehat{\Delta_\rho} \vdash \rho : \mathbf{ref}\,\rho \dashv \cdot$ (6)

by (Values Lemma) on (4).

$\widehat{\Delta_\rho} = \cdot$ (7)

by (Values Inversion Lemma) on (6).

$\widehat{\Delta_0} <: \widehat{\Delta}, \mathbf{rw}\,\rho\,A$ (8)

by rewriting (5) with (7).

$\widehat{\Gamma_0}; \widehat{\Delta}, \mathbf{rw}\,\rho\,A \vdash H\ ,\ \rho \hookrightarrow v$ (9)

by (Subtyping Store Typing) with (8) and (2).

$\widehat{\Gamma_0}; \widehat{\Delta_v} \vdash v : A \dashv \cdot$ (10)

$\widehat{\Gamma_0}; \widehat{\Delta}, \widehat{\Delta_v} \vdash H$ (11)

by (Store Typing Inversion Lemma) on (9).

$\widehat{\Gamma_0}; \cdot \vdash v : [] \dashv \cdot$ (12)

by (T:UNIT) with value v.

$\widehat{\Gamma_0}; \widehat{\Delta}, \widehat{\Delta_v}, \mathbf{rw}\,\rho\,[] \vdash H\ ,\ \rho \hookrightarrow v$ (13)

by (STR:BINDING) using ρ, (11) and (12).

by making:

$\widehat{\Gamma_1} = \cdot$ (14)

$\widehat{\Gamma_0}, \widehat{\Gamma_1}; \widehat{\Delta}, \widehat{\Delta_v}, \mathbf{rw}\,\rho\,[] \vdash H\ ,\ \rho \hookrightarrow v$ (15)

by (Weakening) using $\widehat{\Gamma_1}$ on (13).

$\widehat{\Gamma_0}, \widehat{\Gamma_1}; \widehat{\Delta_v} \vdash v : A \dashv \cdot$ (16)

by (Weakening) using $\widehat{\Gamma_1}$ on (10).

$\widehat{\Gamma_0}, \widehat{\Gamma_1}; \widehat{\Delta_v}, \widehat{\Delta}, \mathbf{rw}\,\rho\,[] \vdash v : A \dashv \widehat{\Delta}, \mathbf{rw}\,\rho\,[]$ (17)

by (T:FRAME) using $\widehat{\Delta}, \mathbf{rw}\,\rho\,[]$ on (16).

Therefore, by (15) and (17) we conclude.

Case (T:DEREFERENCE-PURE) - We have:

$\widehat{\Gamma_0}; \widehat{\Delta_0} \vdash !\rho : !A \dashv \widehat{\Delta}, \mathbf{rw}\,\rho\ \ !A$ (1)

$\widehat{\Gamma_0}; \widehat{\Delta_0} \vdash H\ ,\ \rho \hookrightarrow v$ (2)

$\langle\ H\ ,\ \rho \hookrightarrow v \parallel !\rho\ \rangle \mapsto \langle\ H\ ,\ \rho \hookrightarrow v \parallel v\ \rangle$ (3)

by hypothesis, with (D:DEREFERENCE).

$\widehat{\Gamma_0}; \widehat{\Delta_0} \vdash \rho : \mathbf{ref}\,\rho \dashv \widehat{\Delta}, \mathbf{rw}\,\rho\ \ !A$ (4)

by inversion on (T:DEREFERENCE-PURE) with (1).

$\widehat{\Delta_0} <: \widehat{\Delta_\rho}, \widehat{\Delta}, \mathbf{rw}\,\rho\ \ !A$ (5)

$\widehat{\Gamma_0}; \widehat{\Delta_\rho} \vdash \rho : \mathbf{ref}\,\rho \dashv \cdot$ (6)

by (Values Lemma) on (4).

$\widehat{\Delta_\rho} = \cdot$ (7)

by (Values Inversion Lemma) on (6).

$\widehat{\Delta_0} <: \widehat{\Delta}, \mathbf{rw}\,\rho\ \ !A$ (8)

by rewriting (5) with (7).

$\widehat{\Gamma_0}; \widehat{\Delta}, \mathbf{rw}\,\rho\ \ !A \vdash H\ ,\ \rho \hookrightarrow v$ (9)

by (Subtyping Store Typing) with (8) and (2).

$\widehat{\Gamma_0}; \widehat{\Delta_v} \vdash v : !A \dashv \cdot$ (10)

$\widehat{\Gamma_0}; \widehat{\Delta}, \widehat{\Delta_v} \vdash H$ (11)

by (Store Typing Inversion Lemma) with (9).

$\widehat{\Delta_v} = \cdot$ (12)

$\widehat{\Gamma_0}; \cdot \vdash v : !A \dashv \cdot$ (13)

by (Values Inversion Lemma) on (10).

$\widehat{\Gamma_0}; \widehat{\Delta} \vdash H$ (14)

by rewriting (11) with (12).

by making:

$\widehat{\Gamma_1} = \cdot$ (15)

$\widehat{\Gamma_0}, \widehat{\Gamma_1}; \widehat{\Delta}, \mathbf{rw}\,\rho\ \ !A \vdash H\ ,\ \rho \hookrightarrow v$ (16)

by (Weakening) using $\widehat{\Gamma_1}$ on (9).

$\widehat{\Gamma_0}, \widehat{\Gamma_1}; \cdot \vdash v : !A \dashv \cdot$ (17)

by (Weakening) using $\widehat{\Gamma_1}$ on (13).

$\widehat{\Gamma_0}, \widehat{\Gamma_1}; \widehat{\Delta}, \mathbf{rw}\,\rho\ \ !A \vdash v : !A \dashv \widehat{\Delta}, \mathbf{rw}\,\rho\ \ !A$ (18)

by (T:FRAME) using $\widehat{\Delta}, \mathbf{rw}\,\rho\ \ !A$ on (17).

Therefore, by (16) and (18) we conclude.

Case (T:RECORD) - is a value.

Case (T:SELECTION) - We have:

$\widehat{\Gamma_0}; \widehat{\Delta_0} \vdash \{\overline{\mathtt{f} = v}\}.\mathtt{f}_i : A_i \dashv \widehat{\Delta}$ (1)

$\widehat{\Gamma_0}; \widehat{\Delta_0} \vdash H$ (2)

$\left\langle\ H \parallel \{\overline{\mathtt{f} = v}\}.\mathtt{f}_i\ \right\rangle \mapsto \langle\ H \parallel v_i\ \rangle$ (3)

by hypothesis, with (D:SELECTION).

$\widehat{\Gamma_0}; \widehat{\Delta_0} \vdash \{\overline{\mathtt{f} = v}\} : [\overline{\mathtt{f} : A}] \dashv \widehat{\Delta}$ (4)

by inversion on (T:SELECTION) with (1).

$\widehat{\Delta_0} <: \widehat{\Delta'}, \widehat{\Delta}$ (5)

$\widehat{\Gamma_0}; \widehat{\Delta'} \vdash \{\overline{\mathtt{f} = v}\} : [\overline{\mathtt{f} : A}] \dashv \cdot$ (6)

by (Values Lemma) on (4).

$\widehat{\Gamma_0}; \widehat{\Delta'} \vdash v_i : A_i \dashv \cdot$ (7)

by (Values Inversion Lemma) with (6) .

$\widehat{\Gamma_0}; \widehat{\Delta'}, \widehat{\Delta} \vdash v_i : A_i \dashv \widehat{\Delta}$ (8)

by (T:FRAME) with $\widehat{\Delta}$ with (7) ($\widehat{\Delta'}$ by (Values Lemma)).

$\widehat{\Gamma_0}; \widehat{\Delta'}, \widehat{\Delta} \vdash H$ (9)

by (Subtyping Store Typing) with (2) and (5).

Therefore, by making:

$\widehat{\Gamma_1} = \cdot$ (10)

$\widehat{\Delta_1} = \widehat{\Delta'}, \widehat{\Delta}$ (11)

$\widehat{\Gamma_0}, \widehat{\Gamma_1}; \widehat{\Delta_1} \vdash H$ (12)

by (Weakening) with (10) on (9) and rewriting (9) using (11).

$\widehat{\Gamma_0}, \widehat{\Gamma_1}; \widehat{\Delta_1} \vdash v_i : A_i \dashv \widehat{\Delta}$ (13)

by (Weakening) with (10) on (8) and rewriting (8) using (11).

Therefore, by (12) and (13) we conclude.

Case (T:APPLICATION) - We have:

$\widehat{\Gamma_0}; \widehat{\Delta_0} \vdash (\mathsf{fun}(x : A_0).e)\ v : A_1 \dashv \widehat{\Delta}$ (1)

$\widehat{\Gamma_0}; \widehat{\Delta_0} \vdash H_0$ (2)

$\langle\ H_0 \parallel (\mathsf{fun}(x : A_0).e)\ v\ \rangle \mapsto \langle\ H_0 \parallel e\{v/x\}\ \rangle$ (3)

by hypothesis.

$\widehat{\Gamma_0}; \widehat{\Delta_0} \vdash v : A_0 \dashv \widehat{\Delta'}$ (4)

$\widehat{\Gamma_0}; \widehat{\Delta'} \vdash \mathsf{fun}(x : A_0).e : A_0 \multimap A_1 \dashv \widehat{\Delta}$ (5)

by inversion on (T:APPLICATION) with (1).

$\widehat{\Delta_0} <: \widehat{\Delta'}, \widehat{\Delta_v}$ (6)

$\widehat{\Gamma_0}; \widehat{\Delta_v} \vdash v : A_0 \dashv \cdot$ (7)

by (Values Lemma) on (4).

$\widehat{\Delta'} <: \widehat{\Delta}, \widehat{\Delta'_v}$ (8)

$\widehat{\Gamma_0}; \widehat{\Delta'_v} \vdash \mathsf{fun}(x : A_0).e : A_0 \multimap A_1 \dashv \cdot$ (9)

by (Values Lemma) on (5).

$\widehat{\Gamma_0}; \widehat{\Delta'_v}, x : A_0 \vdash e : A_1 \dashv \cdot$ (10)

$v = \mathsf{fun}(x : A_0).e$ (11)

$A_0 <: A_0$ (12)

by (Values Inversion Lemma) with (9).

$\widehat{\Gamma_0}; \widehat{\Delta_v}, \widehat{\Delta'_v}, \widehat{\Delta} \vdash v : A_0 \dashv \widehat{\Delta'_v}, \widehat{\Delta}$ (13)

by (T:FRAME) on (7) with $\widehat{\Delta'_v}, \widehat{\Delta}$.

$\widehat{\Gamma_0}; \widehat{\Delta'_v}, x : A_0, \widehat{\Delta} \vdash e : A_1 \dashv \widehat{\Delta}$ (14)

by (T:FRAME) on (10) with $\widehat{\Delta}$.

$\widehat{\Gamma_0}; \widehat{\Delta_v}, \widehat{\Delta'_v}, \widehat{\Delta} \vdash e\{v/x\} : A_1 \dashv \widehat{\Delta}$ (15)

by (Substitution Lemma - Linear) with (13) and (14).

By making:

$\widehat{\Gamma_1} = \cdot$

$\widehat{\Delta_1} = \widehat{\Delta_v}, \widehat{\Delta'_v}, \widehat{\Delta}$

We immediately have:

$\widehat{\Gamma_0}, \widehat{\Gamma_1}; \widehat{\Delta_1} \vdash e\{v/x\} : A_1 \dashv \widehat{\Delta}$ (16)

with (15).

$\widehat{\Gamma_0}, \widehat{\Gamma_1}; \widehat{\Delta'}, \widehat{\Delta_v} \vdash H_0$ (17)

by (Subtyping Store Typing) with (2) and (6).

$\widehat{\Gamma_0}, \widehat{\Gamma_1}; \widehat{\Delta}, \widehat{\Delta'_v}, \widehat{\Delta_v} \vdash H_0$ (18)

by (Subtyping Store Typing) with (17) and (8).

$\widehat{\Gamma_0}, \widehat{\Gamma_1}; \widehat{\Delta_1} \vdash H_0$ (19)

by renaming the environment.

Therefore, by (16) and (19) we conclude.

Case (T:FUNCTION) - is a value.

Case (T:FORALL-LOC) - is a value.

Case (T:LOC-APP) - We have:

$\widehat{\Gamma_0}; \widehat{\Delta_0} \vdash (\langle t \rangle\, e)[\rho] : A\{\rho/t\} \dashv \widehat{\Delta}$ (1)

$\widehat{\Gamma_0}; \widehat{\Delta_0} \vdash H_0$ (2)

$\langle\ H_0 \parallel (\langle t \rangle\, e)[\rho]\ \rangle \mapsto \langle\ H_0 \parallel e\{\rho/t\}\ \rangle$ (3)

by hypothesis, with (D:LOCAPP).

$\widehat{\Gamma_0}; \widehat{\Delta_0} \vdash \langle t \rangle\, e : \forall t.A \dashv \widehat{\Delta}$ (4)

$\rho : \mathbf{loc} \in \widehat{\Gamma_0}$ (5)

by inversion on (T:LOC-APP) with (1).

$\widehat{\Delta_0} <: \widehat{\Delta}, \widehat{\Delta_v}$ (6)

$\widehat{\Gamma_0}; \widehat{\Delta_v} \vdash \langle t \rangle\, e : \forall t.A \dashv \cdot$ (7)

by (Values Lemma) on (4).

$\widehat{\Gamma_0}, t : \mathbf{loc}; \widehat{\Delta_v} \vdash e : A \dashv \cdot$ (8)

by (Values Inversion Lemma) with (7).

$\widehat{\Gamma_0}, t : \mathbf{loc}; \widehat{\Delta_v}, \widehat{\Delta} \vdash e : A \dashv \widehat{\Delta}$ (9)

by (T:FRAME) with $\widehat{\Delta}$ on (8).

$\widehat{\Gamma_0}\{\rho/t\}; \widehat{\Delta_v}\{\rho/t\}, \widehat{\Delta}\{\rho/t\} \vdash e\{\rho/t\} : A\{\rho/t\} \dashv \widehat{\Delta}\{\rho/t\}$ (10)

by (Substitution Lemma - Location Variable) on (5) and (9).

$\widehat{\Gamma_0}; \widehat{\Delta_v}, \widehat{\Delta} \vdash e\{\rho/t\} : A\{\rho/t\} \dashv \widehat{\Delta}$ (11)

since t cannot occur in $\widehat{\Gamma_0}, \widehat{\Delta_v}, \widehat{\Delta}$ (is fresh in conclusion) and (10).

By making:

$\widehat{\Gamma_1} = \cdot$
$\widehat{\Delta_1} = \widehat{\Delta_v}, \widehat{\Delta}$
We trivially have:
$\widehat{\Gamma_0}, \widehat{\Gamma_1}; \widehat{\Delta_1} \vdash e\{\rho/t\} : A\{\rho/t\} \dashv \widehat{\Delta}$ (12)
with (11).
$\widehat{\Gamma_0}, \widehat{\Gamma_1}; \widehat{\Delta_1} \vdash H_0$ (13)
by (Subtyping Store Typing) using with (2) and (6).
Therefore, by (12) and (13) we conclude.

Case (t:Loc-Pack) - Is a value.
Case (t:Loc-Open) - We have:

$\widehat{\Gamma_0}; \widehat{\Delta_0} \vdash \mathsf{open}\ \langle t, x\rangle = \langle \rho, v\rangle\ \mathsf{in}\ e\ \mathsf{end} : A_1 \dashv \widehat{\Delta}$ (1)
$\widehat{\Gamma_0}; \widehat{\Delta_0} \vdash H_0$ (2)
$\langle\ H_0 \parallel \mathsf{open}\ \langle t, x\rangle = \langle \rho, v\rangle\ \mathsf{in}\ e\ \mathsf{end}\ \rangle \mapsto \langle\ H_0 \parallel e\{\rho/t\}\{v/x\}\ \rangle$ (3)
by hypothesis, (d:LocOpen).
$\widehat{\Gamma_0}; \widehat{\Delta_0} \vdash \langle \rho, v\rangle : \exists t.A_0 \dashv \widehat{\Delta'}$ (4)
$\widehat{\Gamma_0}, t : \mathbf{loc}; \widehat{\Delta'}, x : A_0 \vdash e : A_1 \dashv \widehat{\Delta}$ (5)
by inversion on (t:Loc-Open) with (1).
$\widehat{\Delta_0} <: \widehat{\Delta_v}, \widehat{\Delta'}$ (6)
$\widehat{\Gamma_0}; \widehat{\Delta_v} \vdash \langle \rho, v\rangle : \exists t.A_0 \dashv \cdot$ (7)
by (Values Lemma) with (4).
$\widehat{\Gamma_0}; \widehat{\Delta_v} \vdash v :\ A_0\{\rho/t\} \dashv \cdot$ (8)
by (Values Inversion Lemma) with (7).
$\rho : \mathbf{loc} \in \widehat{\Gamma_0}$ (9)
by well-formed types of (8).
$\widehat{\Gamma_0}\{\rho/t\}; \widehat{\Delta'}\{\rho/t\}, x : A_0\{\rho/t\} \vdash e\{\rho/t\} : A_1\{\rho/t\} \dashv \widehat{\Delta}\{\rho/t\}$ (10)
by (Substitution Lemma - Location Variable) with (5) and (9).
$\widehat{\Gamma_0}; \widehat{\Delta'}, \widehat{\Delta_v} \vdash v :\ A_0\{\rho/t\} \dashv \widehat{\Delta'}$ (11)
by (t:Frame) with $\widehat{\Delta'}$ on (8).
$\widehat{\Gamma_0}; \widehat{\Delta_0} \vdash v :\ A_0\{\rho/t\} \dashv \widehat{\Delta'}$ (12)
by (t:Subsumption) with (6) and (11).
$\widehat{\Gamma_0}\{\rho/t\}; \widehat{\Delta_0}\{\rho/t\} \vdash v\{\rho/t\} :\ A_0\{\rho/t\} \dashv \widehat{\Delta'}\{\rho/t\}$ (13)
by (Substitution Lemma - Location Variable) with (9) and (12).
$\widehat{\Gamma_0}\{\rho/t\}; \widehat{\Delta_0}\{\rho/t\} \vdash e\{\rho/t\}\{v/x\} : A_1\{\rho/t\} \dashv \widehat{\Delta}\{\rho/t\}$ (14)
by (Substitution Lemma - Linear) with (13) and (10).
By making:
$\widehat{\Gamma_1} = \cdot$
$\widehat{\Delta_1} = \widehat{\Delta_0}$
We immediately have:
$\widehat{\Gamma_0}\{\rho/t\}, \widehat{\Gamma_1}; \widehat{\Delta_1}\{\rho/t\} \vdash e\{\rho/t\}\{v/x\} : A_1\{\rho/t\} \dashv \widehat{\Delta}\{\rho/t\}$ (15)
with (14).
$\widehat{\Gamma_0}, \widehat{\Gamma_1}; \widehat{\Delta_1} \vdash e\{\rho/t\}\{v/x\} : A_1 \dashv \widehat{\Delta}$ (16)
since $\widehat{\Gamma_0}$, $\widehat{\Delta_1}$ and $\widehat{\Delta}$ are closed, t is fresh in the conclusion and (14).
$\widehat{\Gamma_0}, \widehat{\Gamma_1}; \widehat{\Delta_1} \vdash H_0$ (17)
by (Weakening) with $\widehat{\Gamma_1}$ on (2).
Therefore, by (16) and (17) we conclude.

Case (t:Forall-Type) - is a value.
Case (t:Type-App) - Analogous to (t:Loc-App).
Case (t:Type-Pack) - is a value.
Case (t:Type-Open) - Analogous to (t:Loc-Open).
Case (t:Cap-Elim) - Not applicable, environment not closed.
Case (t:Cap-Stack) - We have:

$\widehat{\Gamma_0}; \widehat{\Delta_0} \vdash e_0 : A_0 :: A_1 \dashv \widehat{\Delta}$ (1)
$\widehat{\Gamma_0}; \widehat{\Delta_0} \vdash H_0$ (2)
$\langle\ H_0 \parallel e_0\ \rangle \mapsto \langle\ H_1 \parallel e_1\ \rangle$ (3)
by hypothesis.
$\widehat{\Gamma_0}; \widehat{\Delta_0} \vdash e_0 : A_0 \dashv \widehat{\Delta}, A_1$ (4)
by inversion on (t:Cap-Stack) on (1).
$\widehat{\Gamma_0}, \widehat{\Gamma_1}; \widehat{\Delta_1} \vdash H_1$ (5)
$\widehat{\Gamma_0}, \widehat{\Gamma_1}; \widehat{\Delta_1} \vdash e_1 : A_0 \dashv \widehat{\Delta}, A_1$ (6)
for some $\widehat{\Delta_1}, \widehat{\Gamma_1}$.
by induction hypothesis on (2), (3) and (4).
$\widehat{\Gamma_0}, \widehat{\Gamma_1}; \widehat{\Delta_1} \vdash e_1 : A_0 :: A_1 \dashv \widehat{\Delta}$ (7)
by (t:Cap-Stack) on (6).
Therefore, by (5) and (7) we conclude.

Case (t:Cap-Unstack) - We have:

$\widehat{\Gamma_0}; \widehat{\Delta_0} \vdash e_0 : A_0 \dashv \widehat{\Delta}, A_1$ (1)
$\widehat{\Gamma_0}; \widehat{\Delta_0} \vdash H_0$ (2)
$\langle\ H_0 \parallel e_0\ \rangle \mapsto \langle\ H_1 \parallel e_1\ \rangle$ (3)
by hypothesis.
$\widehat{\Gamma_0}; \widehat{\Delta_0} \vdash e_0 : A_0 :: A_1 \dashv \widehat{\Delta}$ (4)
by inversion on (t:Cap-Unstack) on (1).
$\widehat{\Gamma_0}, \widehat{\Gamma_1}; \widehat{\Delta_1} \vdash H_1$ (5)
$\widehat{\Gamma_0}, \widehat{\Gamma_1}; \widehat{\Delta_1} \vdash e_1 : A_0 :: A_1 \dashv \widehat{\Delta}$ (6)
for some $\widehat{\Delta_1}, \widehat{\Gamma_1}$.
by induction hypothesis on (2), (3) and (4).
$\widehat{\Gamma_0}, \widehat{\Gamma_1}; \widehat{\Delta_1} \vdash e_1 : A_0 \dashv \widehat{\Delta}, A_1$ (7)
for some $\widehat{\Delta_1}, \widehat{\Gamma_1}$.
by (t:Cap-Unstack) on (6).
Therefore, by (5) and (7) we conclude.

Case (t:Subsumption) - We have:

$\widehat{\Gamma_0}; \widehat{\Delta_0} \vdash e_0 : A_1 \dashv \widehat{\Delta}$ (1)
$\widehat{\Gamma_0}; \widehat{\Delta_0} \vdash H_0$ (2)
$\langle\ H_0 \parallel e_0\ \rangle \mapsto \langle\ H_1 \parallel e_1\ \rangle$ (3)
by hypothesis.
$\widehat{\Delta_0} <: \widehat{\Delta'_0}$ (4)
$\widehat{\Gamma_0}; \widehat{\Delta'_0} \vdash e_0 : A_0 \dashv \widehat{\Delta'}$ (5)
$A_0 <: A_1$ (6)
$\widehat{\Delta'} <: \widehat{\Delta}$ (7)
by inversion on (t:Subsumption) with (1).
$\widehat{\Gamma_0}; \widehat{\Delta'_0} \vdash H_0$ (8)
by (Subtyping Store Typing) with (2) and (4).
$\widehat{\Gamma_0}, \widehat{\Gamma_1}; \widehat{\Delta_1} \vdash H_1$ (9)
$\widehat{\Gamma_0}, \widehat{\Gamma_1}; \widehat{\Delta_1} \vdash e_1 : A_0 \dashv \widehat{\Delta'}$ (10)
for some $\widehat{\Delta_1}, \widehat{\Gamma_1}$.
by induction hypothesis on (3), (5) and (8).
$\widehat{\Gamma_0}, \widehat{\Gamma_1}; \widehat{\Delta_1} \vdash e_1 : A_1 \dashv \widehat{\Delta}$ (11)
by (t:Subsumption) with (6), (7) and (10) noting that $\widehat{\Delta_1} <: \widehat{\Delta_1}$.
Therefore, by (9) and (11) we conclude.

Case (t:Tag) - is a value.
Case (t:Case) - We have:

$\widehat{\Gamma_0}; \widehat{\Delta_0} \vdash \mathsf{case}\ \mathtt{l}_i\#v_i\ \mathsf{of}\ \overline{\mathtt{l}_j\#x_j \rightarrow e_j}\ \mathsf{end} : A \dashv \widehat{\Delta}$ (1)
$\widehat{\Gamma_0}; \widehat{\Delta_0} \vdash H_0$ (2)
$\left\langle\ H_0 \parallel \mathsf{case}\ \mathtt{l}_i\#v_i\ \mathsf{of}\ \overline{\mathtt{l}_j\#x_j \rightarrow e_j}\ \mathsf{end}\ \right\rangle \mapsto \langle\ H_0 \parallel e_i\{v_i/x_i\}\ \rangle$ (3)
by hypothesis, (d:Case).
$\underline{\widehat{\Gamma_0}; \widehat{\Delta_0} \vdash \mathtt{l}_i\#v_i : \textstyle\sum_i \mathtt{l}_i\#A_i} \dashv \widehat{\Delta'}$ (4)
$\widehat{\Gamma_0}; \widehat{\Delta'}, x_i : A_i \vdash e_i : A \dashv \widehat{\Delta}$ (5)
$i \leq j$ (6)
by inversion on (d:Case) with (1).
$\widehat{\Delta_0} <: \widehat{\Delta_v}, \widehat{\Delta'}$ (7)
$\widehat{\Gamma_0}; \widehat{\Delta_v} \vdash \mathtt{l}_i\#v_i : \textstyle\sum_i \mathtt{l}_i\#A_i \dashv \cdot$ (8)
by (Values Lemma) with (4).
$\widehat{\Gamma_0}; \widehat{\Delta_v} \vdash v_i : A_i \dashv \cdot$ (9)
for some i.
by (Values Inversion Lemma) with (8).
$\widehat{\Gamma_0}; \widehat{\Delta_v}, \widehat{\Delta'} \vdash v_i : A_i \dashv \widehat{\Delta}$ (10)
by (t:Frame) on (9) with $\widehat{\Delta'}$.
$\widehat{\Gamma_0}; \widehat{\Delta_0} \vdash e_i\{v_i/x_i\} : A \dashv \widehat{\Delta}$ (11)
by (Substitution Lemma - Linear) with (10) and (5), for some i.
By making:
$\widehat{\Gamma_1} = \cdot$
$\widehat{\Delta_1} = \widehat{\Delta_0}$
We trivially have:
$\widehat{\Gamma_0}, \widehat{\Gamma_1}; \widehat{\Delta_1} \vdash e_i\{v_i/x_i\} : A \dashv \widehat{\Delta}$ (12)
by (11).
$\widehat{\Gamma_0}, \widehat{\Gamma_1}; \widehat{\Delta_1} \vdash H_0$ (13)
by (2).
Thus, by (12) and (13) we conclude.

Case (t:Alternative-Left) - We have:

$\widehat{\Gamma_0}; \widehat{\Delta_0}, A_0 \oplus A_1 \vdash e_0 : A_2 \dashv \widehat{\Delta}$ (1)
$\widehat{\Gamma_0}; \widehat{\Delta_0}, A_0 \oplus A_1 \vdash H_0$ (2)
$\langle\ H_0 \parallel e_0\ \rangle \mapsto \langle\ H_1 \parallel e_1\ \rangle$ (3)
by hypothesis.
$\widehat{\Gamma_0}; \widehat{\Delta_0}, A_0 \vdash e_0 : A_2 \dashv \widehat{\Delta}$ (4)
$\widehat{\Gamma_0}; \widehat{\Delta_0}, A_1 \vdash e_0 : A_2 \dashv \widehat{\Delta}$ (5)
by inversion on (t:Alternative-Left) with (1).
By (Store Typing Inversion Lemma) on (2), we have that either:
- $\widehat{\Gamma_0}; \widehat{\Delta_0}, A_0 \vdash H_0$ (1.1)

by sub-case hypothesis.
$\widehat{\Gamma_0}, \widehat{\Gamma_1}; \widehat{\Delta_1} \vdash H_1$ (1.2)
$\widehat{\Gamma_0}, \widehat{\Gamma_1}; \widehat{\Delta_1} \vdash e_1 : A_2 \dashv \widehat{\Delta}$ (1.3)

for some $\widehat{\Delta_1}, \widehat{\Gamma_1}$.

by induction hypothesis with (1.1), (3) and (4).

Therefore, we conclude.

- $\widehat{\Gamma_0}; \widehat{\Delta_0}, A_1 \vdash H_0$ (2.1)

analogous to previous sub-case but using (5).

Thus, we conclude.

Case (t:Frame) - We have:

$\widehat{\Gamma_0}; \widehat{\Delta_0}, \widehat{\Delta_2} \vdash e_0 : A \dashv \widehat{\Delta}, \widehat{\Delta_2}$ (1)

$\widehat{\Gamma_0}; \widehat{\Delta_0}, \widehat{\Delta_2} \vdash H_0$ (2)

$\langle H_0 \parallel e_0 \rangle \mapsto \langle H_1 \parallel e_1 \rangle$ (3)

by hypothesis.

$\widehat{\Gamma_0}; \widehat{\Delta_0} \vdash e_0 : A \dashv \widehat{\Delta}$ (4)

by inversion on (t:Frame) with (1).

$H_0 = H_0', H_0''$ (5)

$\widehat{\Gamma_0}; \widehat{\Delta_0} \vdash H_0'$ (6)

$\widehat{\Gamma_0}; \widehat{\Delta_2} \vdash H_0''$ (7)

by store typing definition since capabilities are disjoint on (2)

$\left\langle H_0', H_0'' \parallel e_0 \right\rangle \mapsto \left\langle H_1', H_0'' \parallel e_1 \right\rangle$ (8)

by the support of the expression and (3).

$\left\langle H_0' \parallel e_0 \right\rangle \mapsto \left\langle H_1' \parallel e_1 \right\rangle$ (9)

by (8) since H_0'' part of the heap is not used.

$\widehat{\Gamma_0}, \widehat{\Gamma_1}; \widehat{\Delta_1} \vdash H_1'$ (10)

$\widehat{\Gamma_0}, \widehat{\Gamma_1}; \widehat{\Delta_1} \vdash e_1 : A \dashv \widehat{\Delta}$ (11)

for some $\widehat{\Delta_1}, \widehat{\Gamma_1}$.

by induction hypothesis on (4), (6) and (9).

$\widehat{\Gamma_0}, \widehat{\Gamma_1}; \widehat{\Delta_1}, \widehat{\Delta_2} \vdash e_1 : A \dashv \widehat{\Delta}, \widehat{\Delta_2}$ (12)

by (t:Frame) on (11) using $\widehat{\Delta_2}$.

$\widehat{\Gamma_0}, \widehat{\Gamma_1}; \widehat{\Delta_1}, \widehat{\Delta_2} \vdash H_1', H_0''$ (13)

by (Weakening) and store typing definition with (7) and (10).

$\widehat{\Gamma_0}, \widehat{\Gamma_1}; \widehat{\Delta_1}, \widehat{\Delta_2} \vdash H_1$ (14)

by rewriting (13).

Therefore, by (12) and (14) we conclude.

Case (t:Let) - We have two reductions:

1. **Sub-Case (d:LetCong):**

   $\widehat{\Gamma_0}; \widehat{\Delta_0} \vdash \mathsf{let}\ x = e_0\ \mathsf{in}\ e_2\ \mathsf{end} : A_1 \dashv \widehat{\Delta}$ (1)

   $\widehat{\Gamma_0}; \widehat{\Delta_0} \vdash H_0$ (2)

   $\langle H_0 \parallel \mathsf{let}\ x = e_0\ \mathsf{in}\ e_2\ \mathsf{end} \rangle \mapsto \langle H_1 \parallel \mathsf{let}\ x = e_1\ \mathsf{in}\ e_2\ \mathsf{end} \rangle$ (3)

   by hypothesis.

   $\langle H_0 \parallel e_0 \rangle \mapsto \langle H_1 \parallel e_1 \rangle$ (4)

   by inversion on (d:LetCong) with (3).

   $\widehat{\Gamma_0}; \widehat{\Delta_0} \vdash e_0 : A_0 \dashv \widehat{\Delta'}$ (5)

   $\widehat{\Gamma_0}; \widehat{\Delta'}, x : A_0 \vdash e_2 : A_1 \dashv \widehat{\Delta}$ (6)

   by inversion on (t:Let) with (1).

   $\widehat{\Gamma_0}, \widehat{\Gamma_1}; \widehat{\Delta_1} \vdash H_1$ (6)

   $\widehat{\Gamma_0}, \widehat{\Gamma_1}; \widehat{\Delta_0} \vdash e_1 : A_0 \dashv \widehat{\Delta'}$ (7)

   for some $\widehat{\Delta_1}, \widehat{\Gamma_1}$.

   by induction hypothesis on (2), (4) and (5).

   $\widehat{\Gamma_0}, \widehat{\Gamma_1}; \widehat{\Delta'}, x : A_0 \vdash e_2 : A_1 \dashv \widehat{\Delta}$ (8)

   by (Weakening) on (6).

   $\widehat{\Gamma_0}, \widehat{\Gamma_1}; \widehat{\Delta_1} \vdash \mathsf{let}\ x = e_1\ \mathsf{in}\ e_2\ \mathsf{end} : A_1 \dashv \widehat{\Delta}$ (9)

   by (t:Let) with (7) and (8).

   Therefore, by (9) and (6) we conclude.

2. **Sub-Case (d:Let):**

   $\widehat{\Gamma_0}; \widehat{\Delta_0} \vdash \mathsf{let}\ x = v\ \mathsf{in}\ e\ \mathsf{end} : A_1 \dashv \widehat{\Delta}$ (1)

   $\widehat{\Gamma_0}; \widehat{\Delta_0} \vdash H$ (2)

   $\langle H \parallel \mathsf{let}\ x = v\ \mathsf{in}\ e\ \mathsf{end} \rangle \mapsto \langle H \parallel e\{v/x\} \rangle$ (3)

   by hypothesis

   $\widehat{\Gamma_0}; \widehat{\Delta_0} \vdash v : A_0 \dashv \widehat{\Delta'}$ (5)

   $\widehat{\Gamma_0}; \widehat{\Delta'}, x : A_0 \vdash e : A_1 \dashv \widehat{\Delta}$ (6)

   by inversion on (t:Let) with (1).

   $\widehat{\Delta_0} <: \widehat{\Delta_v}, \widehat{\Delta'}$ (7)

   $\widehat{\Gamma_0}; \widehat{\Delta_v} \vdash v : A_0 \dashv \cdot$ (8)

   by (Values Lemma) with (4).

   $\widehat{\Gamma_0}; \widehat{\Delta_v}, \widehat{\Delta'} \vdash v : A_0 \dashv \widehat{\Delta'}$ (9)

   by (t:Frame) with (8).

   $\widehat{\Gamma_0}; \widehat{\Delta_v}, \widehat{\Delta'} \vdash e\{v/x\} : A_1 \dashv \widehat{\Delta}$ (10)

   by (Substitution Lemma - Linear) with (6) and (9).

   $\widehat{\Gamma_0}; \widehat{\Delta_v}, \widehat{\Delta'} \vdash H$ (11)

   by (Subtyping Store Typing) with (2) and (7).

   Therefore, by (Weakening) with $\widehat{\Gamma_1} = \cdot$ and by (10) and (11) we conclude.

□

B.11 Progress

Theorem 2 (Progress)**.** If e_0 is a closed expression such that

$$\widehat{\Gamma}; \widehat{\Delta_0} \vdash e_0 : A \dashv \widehat{\Delta_1}$$

then either:

(value) e_0 is a value (v), or;
(steps) if exists H_0 such that $\widehat{\Gamma}; \widehat{\Delta_0} \vdash H_0$ then $\langle\, H_0 \parallel e_0 \,\rangle \mapsto \langle\, H_1 \parallel e_1 \,\rangle$.

Proof. By induction on the typing derivation of $\widehat{\Gamma}; \widehat{\Delta_0} \vdash e_0 : A \dashv \widehat{\Delta}$.

Case (T:Ref), (T:Pure), (T:Unit), (T:Pure-Read), (T:Linear-Read), (T:Pure-Elim) - are all values or the environments are not closed.

Case (T:New) - We have:

$$\widehat{\Gamma}; \widehat{\Delta_0} \vdash \mathsf{new}\ v : \exists t.(\mathbf{ref}\ t :: \mathbf{rw}\ t\ A) \dashv \widehat{\Delta_1} \tag{1}$$
by hypothesis.

Which is not a value but transitions by (D:New).
Thus, we conclude.

Case (T:Delete) - We have:

$$\widehat{\Gamma}; \widehat{\Delta_0} \vdash \mathsf{delete}\ v : \exists t.A \dashv \widehat{\Delta_1} \tag{1}$$
by hypothesis.

$$\widehat{\Gamma}; \widehat{\Delta_0} \vdash v : \exists t.(\mathbf{ref}\ t :: \mathbf{rw}\ t\ A) \dashv \widehat{\Delta_1} \tag{2}$$
by inversion on (T:Delete) with (1).

$$v = \langle \rho, \rho \rangle \tag{3}$$
by (Values Lemma) and (Values Inversion Lemma) on (2).

Thus, by (D:Delete) the expression transitions.

Case (T:Assign) - We have:

$$\widehat{\Gamma}; \widehat{\Delta_0} \vdash v_0 := v_1 : A_1 \dashv \widehat{\Delta_2}, \mathbf{rw}\ \rho\ A_0 \tag{1}$$
by hypothesis.

$$\widehat{\Gamma}; \widehat{\Delta_0} \vdash v_1 : A_0 \dashv \widehat{\Delta_1} \tag{2}$$
$$\widehat{\Gamma}; \widehat{\Delta_1} \vdash v_0 : \mathbf{ref}\ \rho \dashv \widehat{\Delta_2}, \mathbf{rw}\ \rho\ A_1 \tag{3}$$
by inversion on (T:Assign) with (1).

$$v_0 = \rho \tag{4}$$
by (Values Lemma) and (Values Inversion Lemma) with (3).

Thus, by (D:Assign) the expression transitions.

Case (T:Dereference-Linear) - We have:

$$\widehat{\Gamma}; \widehat{\Delta_0} \vdash !v : A \dashv \widehat{\Delta_1}, \mathbf{rw}\ \rho\ [] \tag{1}$$
by hypothesis.

$$\widehat{\Gamma}; \widehat{\Delta_0} \vdash v : \mathbf{ref}\ \rho \dashv \widehat{\Delta_1}, \mathbf{rw}\ \rho\ A \tag{2}$$
by inversion on (T:Dereference-Linear) with (1).

$$v = \rho \tag{3}$$
by (Values Lemma) and (Values Inversion Lemma) with (2).

Thus, by (D:Dereference) the expression transitions.

Case (T:Dereference-Pure) - Analogous to (T:Dereference-Linear).
Case (T:Record) - is a value.
Case (T:Selection) - We have:

$$\widehat{\Gamma}; \widehat{\Delta_0} \vdash v.\mathtt{f}_i : A_i \dashv \widehat{\Delta_1} \tag{1}$$
by hypothesis.

$$\widehat{\Gamma}; \widehat{\Delta_0} \vdash v : [\overline{\mathtt{f} : A}] \dashv \widehat{\Delta_1} \tag{2}$$
by inversion on (T:Selection) with (1).

$$v = \{\overline{\mathtt{f} = v'}\} \tag{3}$$
by (Values Lemma) and (Values Inversion Lemma) with (2).

Thus, by (D:Selection) the expression transitions.

Case (T:Application) - We have:

$$\widehat{\Gamma}; \widehat{\Delta_0} \vdash v_0\ v_1 : A_1 \dashv \widehat{\Delta_2} \tag{1}$$
by hypothesis.

$$\widehat{\Gamma}; \widehat{\Delta_0} \vdash v_1 : A_0 \dashv \widehat{\Delta_1} \tag{2}$$
$$\widehat{\Gamma}; \widehat{\Delta_1} \vdash v_0 : A_0 \multimap A_1 \dashv \widehat{\Delta_2} \tag{3}$$
by inversion on (T:Application) with (1).

$$v_0 = \mathsf{fun}(x : A'').e \qquad A_0 <: A'' \tag{4}$$
by (Values Lemma) and (Values Inversion Lemma) with (3).

Thus, by (D:Application) the expression transitions.

Case (T:Function) - is a value.
Case (T:Forall-Loc) - is a value.
Case (T:Loc-App) - We have:

$$\widehat{\Gamma}; \widehat{\Delta_0} \vdash v[\rho] : A\{\rho/t\} \dashv \widehat{\Delta_1} \tag{1}$$
by hypothesis.

$$\widehat{\Gamma}; \widehat{\Delta_0} \vdash v : \forall t.A \dashv \widehat{\Delta_1} \tag{2}$$
by inversion on (T:Loc-App) with (1).

$$v = \langle t \rangle\, e \tag{3}$$
by (Values Lemma) and (Values Inversion Lemma) with (2).

Thus, by (D:LocApp) the expression transitions.

Case (T:Loc-Open) - We have:

$$\widehat{\Gamma}; \widehat{\Delta_0} \vdash \mathsf{open}\,\langle t, x \rangle = v\ \mathsf{in}\ e\ \mathsf{end} : A_1 \dashv \widehat{\Delta_2} \tag{1}$$
by hypothesis.

$$\widehat{\Gamma}; \widehat{\Delta_0} \vdash v : \exists t.A_0 \dashv \widehat{\Delta_1} \tag{2}$$
$$\widehat{\Gamma}, t : \mathbf{loc}; \widehat{\Delta_1}, x : A_0 \vdash e : A_1 \dashv \widehat{\Delta_2} \tag{3}$$
by inversion on (T:Loc-Open) with (1).

$$v = \langle \rho, v' \rangle \tag{4}$$
by (Values Lemma) and (Values Inversion Lemma) with (2).

Thus, by (D:LocOpen) the expression transitions.

Case (T:Loc-Pack) - is a value.
Case (T:Forall-Type) - is a value.
Case (T:Type-App) - Analogous to (T:Loc-App) but using (D:TypeApp).
Case (T:Type-Open) - Analogous to (T:Loc-Open) but using (D:TypeOpen).
Case (T:Type-Pack) - is a value.
Case (T:Cap-Elim) - Environment not closed.
Case (T:Cap-Stack), (T:Cap-Unstack) - By direct application of induction hypothesis on the inversion of each of the typing rules.
Case (T:Frame) - We have:

$$\widehat{\Gamma}; \widehat{\Delta_0}, \widehat{\Delta_2} \vdash e : A_0 \dashv \widehat{\Delta_1}, \widehat{\Delta_2} \tag{1}$$
by hypothesis.

$$\widehat{\Gamma}; \widehat{\Delta_0} \vdash e : A_0 \dashv \widehat{\Delta_1} \tag{2}$$
by inversion on (T:Frame) with (1).

Then, by induction hypothesis on (2), we have that either:
- e is a value (v), or; (3)
- if exists H_0 such that $\widehat{\Gamma}; \widehat{\Delta_0} \vdash H_0$ then $\langle\, H_0 \parallel e \,\rangle \mapsto \left\langle\, H_0' \parallel e' \,\right\rangle$ (4)

Then, since we know that $\widehat{\Delta_0}, \widehat{\Delta_2}$ then exists H_2 such that:
$$\widehat{\Gamma}; \widehat{\Delta_0}, \widehat{\Delta_2} \vdash H_0, H_2 \tag{5}$$
Therefore, by (5), (3) and (4) we conclude.

Case (T:Subsumption) - We have:

$$\widehat{\Gamma}; \widehat{\Delta_0} \vdash e : A_1 \dashv \widehat{\Delta_3} \tag{1}$$
by hypothesis.

$$\widehat{\Delta_0} <: \widehat{\Delta_1} \tag{2}$$
$$\widehat{\Gamma}; \widehat{\Delta_1} \vdash e : A_0 \dashv \widehat{\Delta_2} \tag{3}$$
$$A_0 <: A_1 \tag{4}$$
$$\widehat{\Delta_2} <: \widehat{\Delta_3} \tag{5}$$
by inversion on (T:Subsumption) with (1).

If exists H_0 such that:
$$\widehat{\Gamma}; \widehat{\Delta_0} \vdash H_0 \tag{6}$$
$$\widehat{\Gamma}; \widehat{\Delta_1} \vdash H_0 \tag{7}$$
by (Subtyping Store Typing) with (6) and (2).

By induction hypothesis on (3), we have that either:
- e is a value (v), or; (8)
- or $\langle\, H_0 \parallel e \,\rangle \mapsto \langle\, H_1 \parallel e' \,\rangle$ (9)

Therefore, we conclude.

Case (T:Tag) - is a value.
Case (T:Case) - We have:

$$\widehat{\Gamma}; \widehat{\Delta_0} \vdash \mathsf{case}\ v\ \mathsf{of}\ \overline{\mathtt{l}_j\#x_j \to e_j}\ \mathsf{end} : A \dashv \widehat{\Delta_1} \tag{1}$$
by hypothesis.

$$\widehat{\Gamma}; \widehat{\Delta_0} \vdash v : \textstyle\sum_i \mathtt{l}_i\#A_i \dashv \widehat{\Delta_1} \tag{2}$$
$$\overline{\widehat{\Gamma}; \widehat{\Delta_1}, x_i : A_i \vdash e_i : A \dashv \widehat{\Delta_2}} \tag{3}$$
$$i \leq j \tag{4}$$
by inversion on (T:Case) with (1).

$$v = \mathtt{l}_i\#v_i \tag{5}$$
by (Values Lemma) and (Values Inversion Lemma) with (2).

Thus, by (D:Case) the expression transitions.

Case (T:Alternative-Left) - We have:

$$\widehat{\Gamma}; \widehat{\Delta_0}, A_0 \oplus A_1 \vdash e : A_2 \dashv \widehat{\Delta_1} \tag{1}$$
by hypothesis.

$$\widehat{\Gamma}; \widehat{\Delta_0}, A_0 \vdash e : A_2 \dashv \widehat{\Delta_1} \tag{2}$$
$$\widehat{\Gamma}; \widehat{\Delta_0}, A_1 \vdash e : A_2 \dashv \widehat{\Delta_1} \tag{3}$$
by inversion on (T:Alternative-Left) with (1).

We have that either:
- e is a value (v); (4)

Therefore the expression is a *value*.

- If exists H_0 such that $\widehat{\Gamma};\widehat{\Delta_0}, A_0 \oplus A_1 \vdash H_0$ (5)

By (Store Typing Inversion Lemma) on (5), we have that either:

$\diamond\ \widehat{\Gamma};\widehat{\Delta_0}, A_0 \vdash H_0$ (6)

Then by induction hypothesis on (2), we conclude that:

$\langle\, H_0 \parallel e \,\rangle \mapsto \left\langle\, H_0' \parallel e' \,\right\rangle$ (7)

Thus, the expression steps, since e cannot be a value.

$\diamond\ \widehat{\Gamma};\widehat{\Delta_0}, A_1 \vdash H_0$ (8)

Then by induction hypothesis on (3), we conclude that:

$\langle\, H_0 \parallel e \,\rangle \mapsto \left\langle\, H_0' \parallel e' \,\right\rangle$ (9)

Thus, the expression steps, since e cannot be a value.

Therefore, we conclude.

Case (t:Let) - We have:

$\widehat{\Gamma};\widehat{\Delta_0} \vdash \mathsf{let}\ x = e_0\ \mathsf{in}\ e_1\ \mathsf{end} : A \dashv \widehat{\Delta_1}$ (1)

by hypothesis.

$\widehat{\Gamma};\widehat{\Delta_0} \vdash e_0 : A_0 \dashv \widehat{\Delta_1}$ (2)

$\widehat{\Gamma};\widehat{\Delta_1}, x : A_0 \vdash e_1 : A_1 \dashv \widehat{\Delta_2}$ (3)

by inversion on (t:Let) with (1).

By induction hypothesis on (2), we have that either:

- e_0 is a value (v); (4)

Thus, by (d:Let) the expression transitions.

- if exists H_0 such that $\widehat{\Gamma};\widehat{\Delta_0} \vdash H_0$ (5)

$\langle\, H_0 \parallel e_0 \,\rangle \mapsto \left\langle\, H_1 \parallel e_0' \,\right\rangle$ (6)

Thus, by (d:LetCong) the expression (1) transitions.

Therefore, we conclude.

□