The Sample Complexity of Revenue Maximization in the Hierarchy of Deterministic Combinatorial Auctions

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Joint work with Nina Balcan and Tuomas Sandholm

Theory Lunch
27 April 2016
Combinatorial (multi-item) auctions allow bidders to express preferences for bundles of goods.
Real-world examples

- US Government wireless spectrum auctions [FCC]
- Sourcing auctions [Sandholm 2013]
- Airport time slot allocation [Rassenti 1982]
- Building development, e.g. office space in GHC (no money)
- Property sales
Mechanism designer must determine:

– **Allocation function**: Who gets what?
– **Payment function**: What does the auctioneer charge?

**Goal**: design *strategy-proof* mechanisms

– Easy for the bidders to compute the optimal strategy
– Easy for designer to analyze possible outcomes
Warm-up: single-item auctions

Second-price auction: the classic strategy-proof, single-item auction.

**Allocation** (N: $5, T: $3) = give carrot to Nina

**Payment** (N: $5, T: $3) = charge Nina $3
Revenue-maximizing combinatorial auctions

- **Standard assumptions:** bidders’ valuations drawn from distribution $D$, mechanism designer knows $D$
  - Allocation and payment rules often depend on $D$
### Revenue-maximizing combinatorial auctions

<table>
<thead>
<tr>
<th>Design Challenges</th>
<th>Feasible Solutions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Support of $D$ might be doubly-exponential</td>
<td>Draw samples from $D$ instead</td>
</tr>
<tr>
<td>NP-hard to determine the revenue-maximizing deterministic auction with respect to $D$ [Conitzer and Sandholm 2002]</td>
<td>Fix a rich class of auctions. Can we learn the revenue-maximizing combinatorial auction in that class with respect to $D$ given samples drawn from $D$?</td>
</tr>
</tbody>
</table>

- **Central problem in Automated Mechanism Design**

No theory that relates the performance of the designed mechanism on the samples to that mechanism’s expected performance on $D$, until now.
Outline

- Introduction
- Hierarchy of deterministic combinatorial auction classes
- Our contribution: how many samples are needed to learn over the hierarchy of auctions?
- Affine maximizer auctions and Rademacher complexity
- Mixed-bundling auctions and pseudo-dimension
- Summary and future directions
Combinatorial auctions

\( NINA \)

\( \text{carrot} : $1 \)

\( \text{tomato} : $0 \)

\( \text{carrot} : $1 \)

\( \text{tomato} : $1 \)

\( TUOMAS \)

\( \text{carrot} : 50\text{¢} \)

\( \text{tomato} : 50\text{¢} \)

\( \text{carrot} : 50\text{¢} \)

\( \text{tomato} : 50\text{¢} \)

- \( 3^2 \) possible outcomes \( o = (o_1, o_2) \)

- For example, \( o = (\{ \text{carrot} \}, \{ \text{tomato} \}) \)
A natural generalization of second price

\[ o^* \text{ maximizes } SW(o) \]
\[ o^{-i} \text{ maximizes } SW_{-i}(o) \]

- **Social Welfare** \( (o) \)
  \[ = SW(o) = \sum_{i \in Bidders} v_i(o) \]
- **\( SW_{-i}(o) \)**
  \[ = \sum_{j \in Bidders - \{i\}} v_j(o) \]
- **Allocation:** \( o^* \)
- **Payment:** Nina pays \( SW_{-Nina}(o^{-Nina}) - SW_{-Nina}(o^*) \)

The “Vickrey-Clarke-Groves mechanism” (VCG).
VCG in action

<table>
<thead>
<tr>
<th>Nina</th>
<th>Tuomas</th>
</tr>
</thead>
<tbody>
<tr>
<td>🥕: $1</td>
<td>🥕: 50¢</td>
</tr>
<tr>
<td>🍅: $0</td>
<td>🍅: 50¢</td>
</tr>
<tr>
<td>🥕: $1</td>
<td>🥕: 50¢</td>
</tr>
</tbody>
</table>

- $o^* = (\{🥕\}, \{🍅\})$
- $o^{−Nina} = (\emptyset, \{🥕, 🍅\})$
- Nina pays $v_{Tuomas}(\{🥕, 🍅\}) - v_{Tuomas}(\{🍅\}) = 0$

How do we get the bidders to pay more?
# Outcome boosting

## Nina

<table>
<thead>
<tr>
<th>Item</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Carrot</td>
<td>$1</td>
</tr>
<tr>
<td>Tomato</td>
<td>$0</td>
</tr>
<tr>
<td>Carrot</td>
<td>$1</td>
</tr>
</tbody>
</table>

## Tuomas

<table>
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</tr>
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<tbody>
<tr>
<td>Carrot</td>
<td>50¢</td>
</tr>
<tr>
<td>Tomato</td>
<td>50¢</td>
</tr>
<tr>
<td>Carrot</td>
<td>50¢</td>
</tr>
</tbody>
</table>

\[
\text{value}(\emptyset, \{ \text{Carrot, Tomato} \}) = v_{Nina}(\emptyset) + v_{Tuomas}(\{ \text{Carrot, Tomato} \}) = 50¢
\]
Outcome boosting

\[
\begin{align*}
\text{Nina} & : \$1 \\
\text{Tuomas} & : 50\text{¢} \\
\text{o} & : 50\text{¢} \\
\end{align*}
\]

- \text{value}(\emptyset, \{\text{carrot}, \text{tomato}\}) = \nu_{\text{Nina}}(\emptyset) + \nu_{\text{Tuomas}}(\{\text{carrot}, \text{tomato}\}) = 50\text{¢} + 99\text{¢}
- o^* = (\{\text{carrot}\}, \{\text{tomato}\})
- o^{-\text{Nina}} = (\emptyset, \{\text{carrot}, \text{tomato}\})
Outcome boosting

\[ \text{value}(\emptyset, \{\text{carrot}, \text{tomato}\}) = v_{Nina}(\emptyset) + v_{Tuomas}(\{\text{carrot}, \text{tomato}\}) = 50\text{¢} + 99\text{¢} \]

\[ o^* = (\{\text{carrot}\}, \{\text{tomato}\}) \]

\[ o^{-Nina} = (\emptyset, \{\text{carrot}, \text{tomato}\}) \]

Nina pays \[ v_{Tuomas}(\{\text{carrot}, \text{tomato}\}) + 99\text{¢} - v_{Tuomas}(\{\text{tomato}\}) = 99\text{¢} \]
Affine maximizer auctions (AMAs)

- Boost outcomes: $\lambda(o)$
- Take bids $\nu$
- Compute outcome:

\[ o^* = \underset{o}{\operatorname{argmax}} \{SW(o) + \lambda(o)\} \]

- Compute Bidder $i$’s payment:

\[ SW_{-i}(o^{-i}) + \lambda(o^{-i}) - (SW_{-i}(o^*) + \lambda(o^*)) \]
Affine maximizer auctions (AMAs)

- Boost outcomes: $\lambda(o)$
- Take bids $\nu$
- Compute outcome:
  $$o^* = \text{argmax}_o \left\{ \sum_{j \in \text{Bidders}}^{n} \nu_j(o) + \lambda(o) \right\}$$
- Compute Bidder $i$’s payment:
  $$\left[ \left( \sum_{j \in \text{Bidders} - \{i\}}^{n} \nu_j(o^{-i}) + \lambda(o^{-i}) \right) - \left( \sum_{j \in \text{Bidders} - \{i\}}^{n} \nu_j(o^*) + \lambda(o^*) \right) \right]$$

Boost outcomes: $\lambda(o)$
- Take bids $\nu$
- Compute outcome:
  $$o^* = \text{argmax}_o \left\{ \sum_{j \in \text{Bidders}}^{n} \nu_j(o) + \lambda(o) \right\}$$
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Affine maximizer auctions (AMAs)

- Boost outcomes: $\lambda(o)$; Weight bidders: $w_i$
- Take bids $v$
- Compute outcome:

$$o^* = \arg \max_o \left\{ \sum_{j \in \text{Bidders}}^n v_j(o) + \lambda(o) \right\}$$

- Compute Bidder $i$’s payment:

$$\left[ \left( \sum_{j \in \text{Bidders} \setminus \{i\}} v_j(o^{-i}) + \lambda(o^{-i}) \right) - \left( \sum_{j \in \text{Bidders} \setminus \{i\}} v_j(o^*) + \lambda(o^*) \right) \right]$$
Affine maximizer auctions (AMAs)

- Boost outcomes: $\lambda(o)$; Weight bidders: $w_i$
- Take bids $\nu$
- Compute outcome:
  \[
o^* = \arg\max_o \left\{ \sum_{j\in\text{Bidders}} w_j \nu_j(o) + \lambda(o) \right\}
  \]
- Compute Bidder $i$'s payment:
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$$o^* = \arg \max_o \left\{ \sum_{j \in \text{Bidders}} w_j v_j(o) + \lambda(o) \right\}$$

- Compute Bidder $i$’s payment:

$$\frac{1}{w_i} \left[ \left( \sum_{j \in \text{Bidders} - \{i\}} w_j v_j(o^{-i}) + \lambda(o^{-i}) \right) - \left( \sum_{j \in \text{Bidders} - \{i\}} w_j v_j(o^*) + \lambda(o^*) \right) \right]$$
Hierarchy of parameterized auction classes

Affine maximizer auctions [R79] \( w_i, \lambda(o) \in \mathbb{R} \)

Virtual valuation combinatorial auctions [SL03] \( \lambda(o) = \sum_{i \in \text{Bidders}} \lambda_i(o) \)

\( \lambda \)-auctions [J07] \( \begin{align*} &w_i = 1 \\
&\lambda(o) \in \mathbb{R} \end{align*} \)

Mixed bundling auctions with reserve prices [TS12] \( \begin{align*} &w_i = 1 \\
&\lambda(o) = 0 \text{ except any outcome where a bidder gets all items} \\
&\text{item reserve prices} \end{align*} \)

Mixed bundling auctions [J07] \( \begin{align*} &w_i = 1 \\
&\lambda(o) = 0 \text{ except outcome where a bidder gets all items} \end{align*} \)
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Our contribution

- Optimize $\lambda(o)$ and $w$ given a sample $S \sim D^N$
  - (Automated Mechanism Design)
- We want:
  - The auction with best revenue over the sample has almost optimal expected revenue
  - Any approximately revenue-maximizing auction over the sample will have approximately optimal expected revenue
- For any auction we output, we want $|S|$ large enough such that:
  $$|\text{empirical revenue} - \text{expected revenue}| < \epsilon$$
- In other words, how many samples $|S| = N$ do we need to ensure that
  $$\left| \frac{1}{N} \sum_{v \in S} \text{rev}_A(v) - \mathbb{E}_{v \sim D}[\text{rev}_A(v)] \right| < \epsilon$$
  for all auctions $A$ in the class?
- (We can only do this with high probability.)
How many samples do we need?

Affine maximizer auctions [R79]

\[ N = O \left( \left[ U n^m \sqrt{m} (U + n^{m/2}) / \varepsilon \right]^2 \right) \]

Virtual valuation combinatorial auctions [SL03]

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\[ \lambda \text{-auctions [J07]} \]

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Mixed bundling auctions with reserve prices [TS12]

\[ N = \tilde{O} \left( (U / \varepsilon)^2 m^3 \right) \]

Mixed bundling auctions [J07]

\[ N = \tilde{O} \left( (U / \varepsilon)^2 \right) \]

Variables

- \( N \): sample size
- \( n \): number of bidders
- \( m \): number of items
- \( U \): maximum revenue achievable over the support of the bidders’ valuation distributions
How many samples do we need?

Affine maximizer auctions [R79]
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Variables
- \( N \): sample size
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Mixed bundling auctions with reserve prices [TS12]
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Nearly-matching exponential lower bounds.
How many samples do we need?

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**Learning theory tool: Rademacher complexity**
How many samples do we need?

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\[ N = O \left( \left[ Un^m \sqrt{m(U + n^{m/2})}/\epsilon \right]^2 \right) \]

**Mixed bundling auctions with reserve prices** [TS12]

\[ N = \tilde{O} \left( \frac{U}{\epsilon} m^3 \right) \]

**Mixed bundling auctions** [J07]

\[ N = \tilde{O} \left( \frac{U}{\epsilon} \right) \]

---

**Variables**

- \( N \): sample size
- \( n \): # bidders
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- \( U \): maximum revenue achievable over the support of the bidders’ valuation distributions

**Learning theory tool:** Pseudo-dimension
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Variables

- \( N \): sample size
- \( n \): # bidders
- \( m \): # items
- \( U \): maximum revenue achievable over the support of the bidders’ valuation distributions
Key challenge

Our problem...

- Boost outcomes: $\lambda(o)$; Weight bidders: $w_i$
- Take bids $v$
- Compute outcome:
  \[
  o^* = \arg\max_o \left\{ \sum_{j \in \text{Bidders}} w_j v_j(o) + \lambda(o) \right\}
  \]
- Compute Bidder $i$’s payment:
  \[
  \frac{1}{w_i} \left[ \left( \sum_{j \in \text{Bidders} - \{i\}} w_j v_j(o^{-i}) + \lambda(o^{-i}) \right) - \left( \sum_{j \in \text{Bidders} - \{i\}} w_j v_j(o^*) + \lambda(o^*) \right) \right]
  \]

Whereas typically in machine learning...
More expressive function classes need more samples to learn

How to measure expressivity?
  – How well do functions from the class fit random noise?

Empirical Rademacher complexity:

\[(x_1, ..., x_N) \sim \{-1, 1\}^N, \quad S = \{v^1, ..., v^N\}\]

\[R_S(\mathcal{A}) = \mathbb{E}_x \left[ \sup_{A \in \mathcal{A}} \frac{1}{N} \sum x_i \cdot rev_A(v^i) \right], \text{ where}\]

Rademacher complexity:

\[R_N(\mathcal{A}) = \mathbb{E}_{S \sim D^N}[R_S(\mathcal{A})]\]

With probability at least 1 − δ, for all \(A \in \mathcal{A}\),

\[|\text{empirical revenue – expected revenue}| \leq 2R_N(\mathcal{A}) + U \sqrt{\frac{2 \ln(2/\delta)}{N}}\]

\*\(U\) is the maximum revenue achievable over the support of the bidders’ valuation distributions
• More expressive function classes need more samples to learn
• How to measure expressivity?
  – How well do functions from the class fit random noise?

• **Empirical Rademacher complexity:**
  \[(x_1, \ldots, x_N) \sim \{-1, 1\}^N, \quad S = \{v^1, \ldots, v^N\}\]
  \[R_S(\mathcal{A}) = \mathbb{E}_x \left[ \sup_{A \in \mathcal{A}} \frac{1}{N} \sum x_i \cdot \text{rev}_A(v^i) \right], \text{ where}\]

• **Rademacher complexity:**
  \[R_N(\mathcal{A}) = \mathbb{E}_{S \sim \mathcal{D}^N}[R_S(\mathcal{A})]\]

<table>
<thead>
<tr>
<th>(\mathcal{A}) = all binary valued functions</th>
<th>(R_N(\mathcal{A}) = \frac{1}{2})</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\mathcal{A}) = one binary valued function</td>
<td>(R_N(\mathcal{A}) = 0)</td>
</tr>
</tbody>
</table>
Rademacher complexity of AMAs

<table>
<thead>
<tr>
<th>Theorem</th>
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<tbody>
<tr>
<td>Let $\mathcal{A}$ be the class of $n$-bidder, $m$-item AMA revenue functions. If $N = O\left(\left[Un^m\sqrt{m(U + n^{m/2})}/\epsilon\right]^2\right)$, then with high probability over a sample $S \sim D^N$, $</td>
</tr>
</tbody>
</table>

- **Key idea:** split revenue function into its simpler components
  - Weighted social welfare without any one bidder’s participation ($n$ components)
  - Amount of revenue subtracted out to maintain strategy-proof property
- Then use compositional properties of Rademacher complexity and other tricks, for example:
  \[
  \text{If } F = \{f \mid f = g + h, g \in G, h \in H\}, \text{ then } R_N(F) \leq R_N(G) + R_N(H)
  \]
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How many samples do we need?

**Affine maximizer auctions [R79]**

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**Mixed bundling auctions with reserve prices [TS12]**

\[ N = \tilde{O} \left( \frac{(U/\epsilon)^2 m^3}{m_3} \right) \]

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---

**Variables**

- \( N \): sample size
- \( n \): # bidders
- \( m \): # items
- \( U \): maximum revenue achievable over the support of the bidders’ valuation distributions
Mixed bundling auctions (MBAs)

• Class of auctions parameterized by a scalar $c$
• Boost the allocations where one bidder gets all goods by $c$
• $\text{value}(\emptyset, \{\text{carrot}, \text{tomato}\}) = v_{\text{Nina}}(\emptyset) + v_{\text{Tuomas}}(\{\text{carrot}, \text{tomato}\}) = 50\text{c} + 99\text{c}$
• $\text{value}(\{\text{carrot}, \text{tomato}\}, \emptyset) = v_{\text{Nina}}(\{\text{carrot}, \text{tomato}\}) + v_{\text{Tuomas}}(\emptyset) = 50\text{c} + 99\text{c}$
• How large must the sample $S$ be in order to ensure that for all MBAs, $|\text{empirical revenue} - \text{expected revenue}| < \epsilon$?
Structural properties of MBA revenue functions

**Lemma**

Fix $v \in S$. Then $rev_v(c)$ is piecewise linear with at most $n + 1$ discontinuities.
VC-dimension

- Complexity measure for binary-valued functions only
- Example: $F = \{\text{single interval on the real line}\}$
- No set of size 3 can be labeled in all $2^3$ ways by $F$

Class of functions $F$ shatters set $S = \{x_1, \ldots, x_N\}$ if for all $b \in \{0, 1\}^N$, there exists $f \in F$ such that $f(x_i) = b_i$

VC-dimension of $F$ is the cardinality of the largest set $S$ that can be shattered by $F$

How can we extend VC-dim to real-valued functions?
Pseudo-dimension

\[ x_1 \ f(x_1) \leq r^{(1)} \ 0 \]
\[ x_2 \ f(x_2) \leq r^{(2)} \ 0 \]
\[ x_3 \ f(x_3) > r^{(3)} \ 1 \]
\[ x_4 \ f(x_4) \leq r^{(4)} \ 0 \]
\[ x_5 \ f(x_5) > r^{(5)} \ 1 \]
\[ x_6 \ f(x_6) \leq r^{(6)} \ 0 \]
\[ x_7 \ f(x_7) > r^{(7)} \ 1 \]
\[ x_8 \ f(x_8) > r^{(8)} \ 1 \]
\[ x_9 \ f(x_9) \leq r^{(9)} \ 0 \]

- Sample \( S = \{x_1, \ldots, x_N\} \)
- Class of functions \( F \) into \([-U, U]\)
- \( r = (r^{(1)}, \ldots, r^{(N)}) \in \mathbb{R}^N \)
- \( r = (r^{(1)}, \ldots, r^{(N)}) \in \mathbb{R}^N \)
  witnesses the shattering of \( S \) by \( F \)
  if for all \( T \subseteq S \), there exists \( f_T \in F \)
  such that \( f_T(x_i) \leq r^{(i)} \) iff \( x_i \in T \)
- Pseudo-dimension of \( F \) is the cardinality of the largest sample \( S \)
  that can be shattered by \( F \)

\[ \text{P-dim}(F) = \text{VC-dim}(\{(x, r) \mapsto 1_{f(x)-r>0} | f \in F\}) \]
How many samples do we need?

- Set of auction revenue functions $\mathcal{A}$ with range in $[0, U]$, distribution $D$ over valuations $v$.
- For every $\epsilon > 0$, $\delta \in (0, 1)$, if

$$N = O \left( \left( \frac{U}{\epsilon} \right)^2 \left( \text{P–dim}(\mathcal{A}) \cdot \ln \frac{U}{\epsilon} + \ln \frac{1}{\delta} \right) \right),$$

then with probability at least $1 - \delta$ over a sample $S \sim D^N$, $|\text{empirical revenue} - \text{expected revenue}| < \epsilon$

for every $\text{rev}_A \in \mathcal{A}$.

Pseudo-dimension allows us to derive strong sample complexity bounds.
How many samples do we need?

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Variables
- \( N \): sample size
- \( n \): # bidders
- \( m \): # items
- \( U \): maximum revenue achievable over the support of the bidders’ valuation distributions
### Theorem

Let $\mathcal{A} = \{rev_c\}_{c \geq 0}$ be the class of $n$-bidder, $m$-item mixed bundling auction revenue functions. Then $\text{P-dim}(\mathcal{A}) = O(\log n)$. 

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2-bidder MBA pseudo-dimension

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*Proof sketch.*

- **Fact:** there exists a set of 2 samples that is shattered by $\mathcal{A}$.

We need to show that no set of 3 samples can be shattered by $\mathcal{A}$.

- Suppose, for a contradiction, that $S = \{v^1, v^2, v^3\}$ is shatterable.
- Recall $v^1 = (v^1_1, v^1_2)$
- This means:
  - There exists $r = (r^1, r^2, r^3) \in \mathbb{R}^3$ and $2^{|S|} = 8$ MBA parameters $c = \{c_1, ..., c_8\}$ such that $\{\text{rev}_{c_1}, ..., \text{rev}_{c_8}\}$ induce all 8 binary labelings on $S$ with respect to $r$. 


Lemma

Fix $v^i \in S$. Then $rev_{v^i}(c)$ is piecewise linear with one discontinuity, with a slope of 2 followed by a constant function with value $\min\{v^i_1([m]), v^i_2([m])\}$. 
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This is impossible, so we reach a contradiction. Therefore, no set of size 3 can be shattered by the class of 2-bidder MBA revenue functions, so the pseudo-dimension is at most 2.
Summary

• Analyzed the sample complexity of learning over a hierarchy of deterministic combinatorial auctions
• Uncovered structural properties of these auctions’ revenue functions along the way
  – Of independent interest beyond sample complexity results