Ph.D. Qualifying Exam:
Math and Algorithms

Spring 2003

The exam includes eight problems;
the length of the exam is three hours.

Problem 1

Prove or disprove that, if n is a natural number, then $(n+4)^4 + 4$ is *not* a prime number.

Problem 2

The following algorithm computes the nth Fibonacci number. Determine its time complexity; you should give an asymptotically tight bound (Θ-notation) and show its derivation.

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FIBONACCI(n)
if n ≤ 2
   then return 1
   else return FIBONACCI(n − 2) + FIBONACCI(n − 1)
```

Problem 3

Give an efficient algorithm for determining the number of elements with a given key in a binary search tree. The input includes a binary search tree and a natural number k; the output is the number of elements with key k in the tree. Your algorithm should be more efficient than INORDER-TREE-WALK; thus, it should not traverse the whole tree.

Problem 4

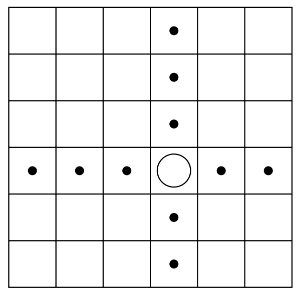
Give an efficient algorithm for scheduling n lectures using multiple lecture halls. For each lecture i, we know its start time s_i and finish time f_i. Two lectures can be in the same hall only if the corresponding time intervals do not overlap. We need to schedule all lectures using the minimal possible number of lecture halls.

Problem 5

Suppose that we need to determine whether a given point in inside a given polygon. We specify the point by its x and y co-ordinates, and the polygon by the co-ordinates of its vertices in the counterclockwise order. We assume that the boundary of the polygon has no self-intersections, and that the given point is not on the polygon's boundary. Give an efficient algorithm that inputs the co-ordinates of a given point and the co-ordinates of a polygon's vertices, and returns TRUE if the point is inside the polygon.

Problem 6

A *rook* is a chess piece that can move vertically and horizontally, as shown in the picture. Two rooks attack each other if they are in the same row or column and there is no rook between them. We consider the task of placing rooks on a $3n \times 3n$ chess board is such a way that each rook attacks at most one other rook. We need to construct an algorithm that inputs natural numbers k and n, and returns TRUE if k rooks can be placed on the $3n \times 3n$ chess board with each rook attacking at most one other. Determine whether this problem is NP-hard and justify your answer.



Problem 7

We consider a connected undirected unweighted graph. We define the length of a path as the number of its edges, and the distance between two vertices as the length of the shortest path between them. For a set of vertices in the graph, we define the diameter of the set as the maximal distance between its vertices; that is, we can find the diameter by determining the distance for every pair of vertices in the set, and taking the maximum of these distances. We need to write an algorithm that inputs a connected undirected unweighted graph, along with natural numbers k and d, and returns TRUE if the graph includes a set of k vertices with diameter less than d. Determine whether this problem is NP-hard and justify your answer.

Problem 8

Suppose that u and v are two distinct vertices of an undirected graph, and we need to determine whether there is a simple path with an *even* number of edges between u and v. That is, we have to construct an algorithm that inputs an undirected graph, along with two specific vertices of this graph, and returns TRUE if there is a simple path with an even number of edges between these vertices. Determine whether this problem is NP-hard and justify your answer.