Ph.D. Comprehensive Exam: Math and Algorithms

Fall 2001

The exam includes eight problems;
the length of the exam is three hours.

Problem 1

Prove or disprove the following asymptotic bound:

$(n+1)^n = o(n^{n+1})$.

Problem 2

Consider the following recurrence:

$$A_0 = 1$$
$$A_1 = 2$$
$$A_n = n \cdot A_{n-1} + n \cdot A_{n-2} \quad \text{(where } n \geq 2\text{)}$$

Give a formula for A_n in terms of n without the use of recurrence.

Problem 3

We consider twelve-digit strings, such as "124698356187" or "010203040506," and we say that a string is "balanced" if the sum of the first six digits is equal to the sum of the last six digits. For example, "124698356187" is balanced since $1+2+4+6+9+8 = 3+5+6+1+8+7$, whereas "010203040506" is not balanced.

The total number of distinct twelve-digit strings is 10^{12}. Write an algorithm that determines how many of them are balanced. It should allow efficient execution on a regular workstation; that is, its execution should take at most an hour. Writing an algorithm that loops through all 10^{12} strings is *not* an acceptable solution since it is impractically slow.

Problem 4

We consider a set of integers and denote the number of its elements by n. Describe a data structure for representing this set that supports the following three operations, with the specified worst-case complexity.

- Add a given integer to the set; if this integer is already in the set, do nothing. The time complexity of adding an integer must be $O(\lg n)$.
- Delete a given integer from the set; if this integer is not in the set, do nothing. The complexity of this operation must be $O(\lg n)$.
- Count the number of elements that are between two given values. For example, if the set includes the integers 1, 4, 5, 7, and 9, and the two given values are 2 and 7, then this operation returns 3. Its complexity must be $O(\lg n)$.

Problem 5

Suppose that G is an undirected connected graph with positive weights of edges. Prove or disprove the following statements:

(a) If all edge weights in G are distinct, then G has a *unique* minimum spanning tree.

(b) If G has two edges of the same weight, then it has *several* minimum spanning trees.

Problem 6

Consider a two-player game that involves *twenty-five stones* placed in a pile. The players gradually reduce the pile by taking stones from it. At each move, a player has to take one, two, or three stones (but no more than three). The game continues until the pile becomes empty; the player who has collected an *odd* number of stones is the winner. Describe an efficient strategy for playing this game; for the full credit, you should give a strategy suitable for a human player in a fast-paced game.

Problem 7

The classical Hamiltonian Cycle Problem requires to input an undirected graph and find a simple cycle through all its vertices; that is, the cycle should include all vertices of the graph and have no self-intersections. We consider a modified version of this problem, which requires to find a simple cycle that includes *at least half of the vertices;* for example, if the graph has ten vertices, we need to find a simple cycle that includes at least five of them. The classical Hamiltonian Cycle Problem is known to be NP-complete. Use this fact to prove that the modified version is also NP-complete.

Problem 8

We consider a directed graph that may have negative-weight edges. The problem is to find a minimum-weight *simple* path between two given vertices. Recall that a path is simple if it has no self-intersections. Determine whether this problem is NP-hard and justify your answer.