Automata Theory: Solutions 2

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                                                X
                                                X
                  X                 X           X           X
number of         X           X     X           X           X
homeworks         X     X     X     X           X     X     X
                  X     X     X     X           X     X     X     X
                  X     X     X     X     X     X     X     X     X
            X     X     X     X     X     X     X     X     X     X
            X     X     X     X     X     X     X     X     X     X
            X     X     X     X     X     X     X     X     X     X
          ------------------------------------------------------------
          0.5-1 1.5-2 2.5-3 3.5-4 4.5-5 5.5-6 6.5-7 7.5-8 8.5-9 9.5-10
                                                       grades
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This histogram shows the distribution of grades for the homeworks submitted on time.

Problem 1
Prove the following equality by induction:

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = (1 + 2 + 3 + \cdots + n)^2.$$

Basis: If $n = 1$, then the equality trivially holds: $1^3 = (1)^2$.

Step: Suppose that $1^3 + 2^3 + 3^3 + \cdots + n^3 = (1 + 2 + 3 + \cdots + n)^2$ is true. Then, we may prove the equality for $n + 1$ as follows:

$$\begin{aligned}
1^3 + \cdots + n^3 + (n+1)^3 &= (1 + 2 + 3 + \cdots + n)^2 + (n+1)^3 \\
&= \left(\frac{n(n+1)}{2}\right)^2 + (n+1)^3 \\
&= \left(\frac{n^2+n}{2}\right)^2 + n^3 + 3n^2 + 3n + 1 \\
&= \frac{n^4 + 2n^3 + n^2}{4} + \frac{4n^3 + 12n^2 + 12n + 4}{4} \\
&= \frac{n^4 + 6n^3 + 13n^2 + 12n + 4}{4} \\
&= \left(\frac{n^2 + 3n + 2}{2}\right)^2 \\
&= \left(\frac{(n+1)(n+2)}{2}\right)^2 \\
&= (1 + 2 + 3 + \cdots + n + (n+1))^2
\end{aligned}$$

Problem 2
Consider the following two languages on the alphabet $\Sigma = \{a, b\}$:

$$L_1 = \{a^n : \ n \geq 1\}$$
$$L_2 = \{b^n : \ n \geq 1\}$$

Describe the languages below, using either the set notation or precise definitions in English:

- $L_3 = L_1^* = {L_1}^0 \cup {L_1}^1 \cup {L_1}^2 \cup \ldots = \{a^n : \ n \geq 0\}$
  (all strings that have no b's)

- $L_4 = \overline{L_1} = \{\lambda\} \cup \{w : \ w \text{ includes at least one } b\}$
  (the empty string and all strings that have some b's)

- $L_5 = L_1 \cup L_2 = \{a^n, b^n : \ n \geq 1\}$
  (all strings that have only a's or only b's)

- $L_6 = L_1 L_2 = \{a^m b^n : \ m \geq 1, \ n \geq 1\}$

- $L_7 = ({L_1}^2)({L_2}^2)({L_1}^2) = \{a^m b^n a^k : \ m, n, k \geq 2\}$

- $L_8 = (L_1 \cup L_2)^* = \Sigma^*$
  (all strings on the alphabet)

- $L_9 = (L_1 L_2)^* = \{awb : \ w \text{ is any string on } \Sigma\}$
  (all strings that begin with a and end with b)

Problem 3
Consider the alphabet $\Sigma = \{a, b\}$. Is there any language L on this alphabet for which $(\overline{L})^* = \overline{L^*}$? If yes, give an example of such a language; if no, explain why.

By definition, the star-closure of any language includes the empty string λ. In particular, the $(\overline{L})^*$ language, which is the star-closure of $\overline{L}$, includes λ. On the other hand, since L^* includes λ, we conclude that its complement, $\overline{L^*}$, does *not* include λ.

Thus, for every language L, the empty string is included into $(\overline{L})^*$, and it is not included into $\overline{L^*}$; hence, $(\overline{L})^* \neq (\overline{L})^*$.