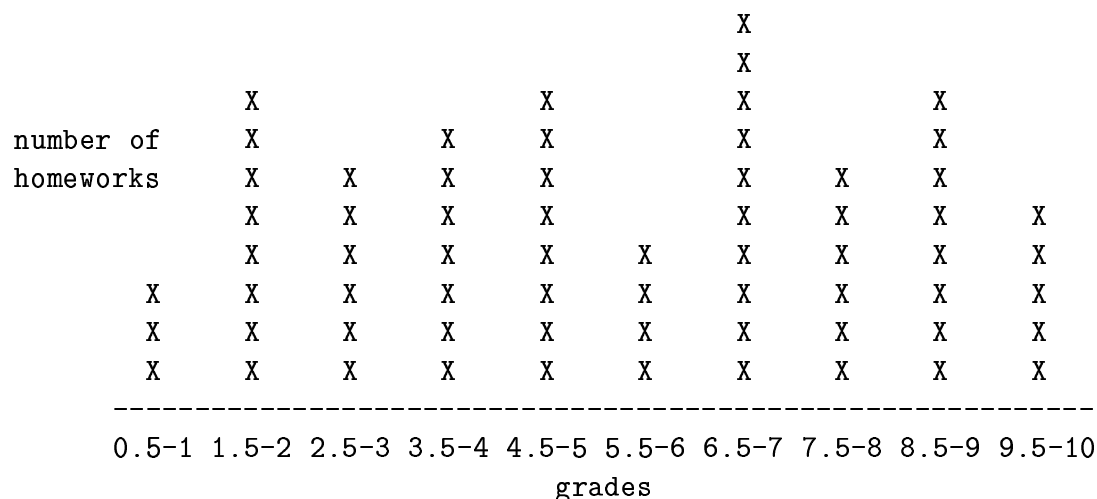


Automata Theory: Solutions 2



This histogram shows the distribution of grades for the homeworks submitted on time.

Problem 1

Prove the following equality by induction:

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = (1 + 2 + 3 + \cdots + n)^2.$$

Basis: If $n = 1$, then the equality trivially holds: $1^3 = (1)^2$.

Step: Suppose that $1^3 + 2^3 + 3^3 + \cdots + n^3 = (1 + 2 + 3 + \cdots + n)^2$ is true. Then, we may prove the equality for $n + 1$ as follows:

$$\begin{aligned}
 1^3 + \cdots + n^3 + (n + 1)^3 &= (1 + 2 + 3 + \cdots + n)^2 + (n + 1)^3 \\
 &= \left(\frac{n(n + 1)}{2} \right)^2 + (n + 1)^3 \\
 &= \left(\frac{n^2 + n}{2} \right)^2 + n^3 + 3n^2 + 3n + 1 \\
 &= \frac{n^4 + 2n^3 + n^2}{4} + \frac{4n^3 + 12n^2 + 12n + 4}{4} \\
 &= \frac{n^4 + 6n^3 + 13n^2 + 12n + 4}{4} \\
 &= \left(\frac{n^2 + 3n + 2}{2} \right)^2 \\
 &= \left(\frac{(n + 1)(n + 2)}{2} \right)^2 \\
 &= (1 + 2 + 3 + \cdots + n + (n + 1))^2
 \end{aligned}$$

Problem 2

Consider the following two languages on the alphabet $\Sigma = \{a, b\}$:

$$L_1 = \{a^n : n \geq 1\}$$

$$L_2 = \{b^n : n \geq 1\}$$

Describe the languages below, using either the set notation or precise definitions in English:

- $L_3 = L_1^* = L_1^0 \cup L_1^1 \cup L_1^2 \cup \dots = \{a^n : n \geq 0\}$
(all strings that have no b 's)
- $L_4 = \overline{L_1} = \{\lambda\} \cup \{w : w \text{ includes at least one } b\}$
(the empty string and all strings that have some b 's)
- $L_5 = L_1 \cup L_2 = \{a^n, b^n : n \geq 1\}$
(all strings that have only a 's or only b 's)
- $L_6 = L_1 L_2 = \{a^m b^n : m \geq 1, n \geq 1\}$
- $L_7 = (L_1^2)(L_2^2)(L_1^2) = \{a^m b^n a^k : m, n, k \geq 2\}$
- $L_8 = (L_1 \cup L_2)^* = \Sigma^*$
(all strings on the alphabet)
- $L_9 = (L_1 L_2)^* = \{awb : w \text{ is any string on } \Sigma\}$
(all strings that begin with a and end with b)

Problem 3

Consider the alphabet $\Sigma = \{a, b\}$. Is there any language L on this alphabet for which $(\overline{L})^* = \overline{L^*}$? If yes, give an example of such a language; if no, explain why.

By definition, the star-closure of any language includes the empty string λ . In particular, the $(\overline{L})^*$ language, which is the star-closure of \overline{L} , includes λ . On the other hand, since L^* includes λ , we conclude that its complement, $\overline{L^*}$, does *not* include λ .

Thus, for every language L , the empty string is included into $(\overline{L})^*$, and it is not included into $\overline{L^*}$; hence, $(\overline{L})^* \neq \overline{L^*}$.