Automata Theory: Solutions 2

```
                        X
                        X
                        X
                        X
                        X
                        X
                        X
                        X
                  X     X
number of         X     X
homeworks         X  X  X
                  X  X  X
                  X  X  X
                  X  X  X
                  X  X  X
                  X  X  X  X
      X        X  X  X  X  X
   X  X        X  X  X  X  X
  --------------------------
   2  3  4  5  6  7  8  9  10
              grades
```

The histogram shows the distribution of grades.

Problem 1
Consider the following sets of integer numbers:

$S_1 = \{1, 2, 3\}$
$S_2 = \{i : \ i \text{ is odd}\}$
$S_3 = \{i : \ i \text{ is divisible by } 3\}$

For each set below, specify its elements:

$S_4 = 2^{S_1} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$
$S_5 = 2^{S_1} \cap 2^{S_2} = \{\emptyset, \{1\}, \{3\}, \{1, 3\}\}$
$S_6 = S_3 - S_2 = \{i : \ i \text{ is divisible by } 6\}$

The sets S_4 and S_5 are finite, whereas S_6 is infinite.

Problem 2
Prove that, if $S_1 \subseteq S_2$, then $2^{S_1} \subseteq 2^{S_2}$.

By the definition of subsets, we need to show that every element s of 2^{S_1} is also an element of 2^{S_2}. The proof is as follows:

$$\begin{aligned} s \in 2^{S_1} \quad &\Leftrightarrow \quad s \subseteq S_1 \quad \text{(by definition of powerset)} \\ &\Rightarrow \quad s \subseteq S_2 \quad \text{(because } S_1 \subseteq S_2\text{)} \\ &\Leftrightarrow \quad s \in 2^{S_2} \quad \text{(by definition of powerset)} \end{aligned}$$

Problem 3
Prove the following equality by induction:

$$1^2 + 2^2 + 3^2 + 4^2 + ... + n^2 = \tfrac{n\cdot(n+1)\cdot(2\cdot n+1)}{6}.$$

Clearly, it holds for $n = 1$; we now show that, if it holds for n, then it also holds for $n + 1$:

$$\begin{aligned}
1^2 + 2^2 + ... + n^2 + (n+1)^2 &= \frac{n\cdot(n+1)\cdot(2\cdot n+1)}{6} + (n+1)^2 \\
&= \frac{n\cdot(n+1)\cdot(2\cdot n+1)}{6} + \frac{6\cdot(n+1)^2}{6} \\
&= \frac{(n+1)}{6}\cdot(n\cdot(2\cdot n+1) + 6\cdot(n+1)) \\
&= \frac{(n+1)}{6}\cdot(2\cdot n^2 + 7\cdot n + 6) \\
&= \frac{(n+1)}{6}\cdot(n+2)(2\cdot n+3) \\
&= \frac{(n+1)\cdot((n+1)+1)\cdot(2\cdot(n+1)+1)}{6}
\end{aligned}$$