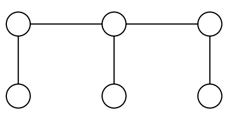
# Automata Theory: Solutions 1

```
                   X
                   X
                   X
                   X
                   X X
                   X X
                   X X
               X   X X
number of      X X X X
homeworks  X   X X X X
           X   X X X X
           X   X X X X
     X   X X   X X X X
     ------------------
     2 3 4 5 6 7 8 9 10
           grades
```

The histogram shows the distribution of grades.

**Problem 1**
Draw an example of an undirected connected acyclic graph with six vertices.



If an undirected acyclic graph has six vertices, it has five edges.

**Problem 2**
Prove that, if $S_1 \subseteq S_2$ and $S_3 \subseteq S_4$, then $(S_1 \cap S_3) \subseteq (S_2 \cap S_4)$.

By the definition of subsets, we need to show that every element $x$ of $S_1 \cap S_3$ is also an element of $S_2 \cap S_4$. The proof is as follows:

$$\begin{aligned} x \in S_1 \cap S_3 \quad &\Rightarrow \quad x \in S_1 \text{ and } x \in S_3 \quad \text{(by definition of intersection)} \\ &\Rightarrow \quad x \in S_2 \text{ and } x \in S_4 \quad \text{(because } S_1 \subseteq S_2 \text{ and } S_3 \subseteq S_4\text{)} \\ &\Rightarrow \quad x \in S_2 \cap S_4 \quad \text{(by definition of set intersection)} \end{aligned}$$

**Problem 3**
Prove that the following results hold for every natural number $n$.

**(a)** $1 + 3 + 5 + 7 + ... + (2 \cdot n - 1) = n^2$

We use a proof by induction. Clearly, the equality holds for $n = 1$; we now show that, if it holds for $n$, then it also holds for $n + 1$:

$$\begin{aligned} 1 + 3 + ... + (2 \cdot n - 1) + (2 \cdot (n+1) - 1) &= (1 + 3 + ... + (2 \cdot n - 1)) + (2 \cdot n + 1) \\ &= n^2 + 2 \cdot n + 1 \\ &= (n+1)^2 \end{aligned}$$

**(b)** $\frac{n}{n+1} < \frac{n+1}{n+2}$

We simplify the inequality as follows:

$$\begin{aligned} \frac{n}{n+1} < \frac{n+1}{n+2} &\Leftrightarrow n \cdot (n+2) < (n+1)^2 \\ &\Leftrightarrow n^2 + 2 \cdot n < n^2 + 2 \cdot n + 1 \\ &\Leftrightarrow 0 < 1 \end{aligned}$$