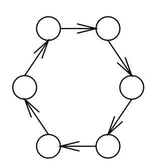
Automata Theory: Solutions 2

```
                         X
                         X
                         X
                         X
                         X
                         X
                         X
                         X
   number of             X
   homeworks             X
                         X
                   X     X
             X     X     X
             X     X  X  X
             X  X  X  X  X
          X  X  X  X  X  X
          X  X  X  X  X  X
          -----------------
          5  6  7  8  9  10
               grades
```

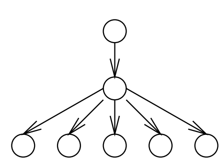
The histogram shows the distribution of grades, from 0 to 10.

Problem 1
(a) Draw a graph with six vertices and six edges, and mark all simple cycles.



This graph has one simple cycle; other examples may have a different number of cycles.

(b) Draw a tree with seven vertices, five of which are leaves. How many edges are in your tree?



A tree with n vertices always has $n-1$ edges; in particular, a seven-vertex tree has six edges.

Problem 2

Prove the following equality by induction:

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = (1 + 2 + 3 + \cdots + n)^2.$$

Basis: If $n = 1$, then the equality trivially holds: $1^3 = (1)^2$.

Step: Suppose that $1^3 + 2^3 + 3^3 + \cdots + n^3 = (1 + 2 + 3 + \cdots + n)^2$ is true. Then, we may prove the equality for $n + 1$ as follows:

$$\begin{aligned}
1^3 + \cdots + n^3 + (n+1)^3 &= (1 + 2 + 3 + \cdots + n)^2 + (n+1)^3 \\
&= \left(\frac{n(n+1)}{2}\right)^2 + (n+1)^3 \\
&= \left(\frac{n^2+n}{2}\right)^2 + n^3 + 3n^2 + 3n + 1 \\
&= \frac{n^4 + 2n^3 + n^2}{4} + \frac{4n^3 + 12n^2 + 12n + 4}{4} \\
&= \frac{n^4 + 6n^3 + 13n^2 + 12n + 4}{4} \\
&= \left(\frac{n^2 + 3n + 2}{2}\right)^2 \\
&= \left(\frac{(n+1)(n+2)}{2}\right)^2 \\
&= (1 + 2 + 3 + \cdots + n + (n+1))^2
\end{aligned}$$