Automata Theory: Solutions 1

```
                                  X
                               X  X
                      X     X  X  X
number of             X  X  X  X  X
homeworks             X  X  X  X  X
                   X  X  X  X  X  X
          X        X  X  X  X  X  X
          X     X  X  X  X  X  X  X
          --------------------------
          2  3  4  5  6  7  8  9  10
                     grades
```

The histogram shows the distribution of grades, from 0 to 10.

Problem 1
Consider the following sets of integer numbers:

$S_1 = \{4, 5, 6\}$
$S_2 = \{i : \ i \text{ is even}\}$
$S_3 = \{i : \ i \text{ is divisible by } 3\}$

For each set below, specify its elements and determine whether it is finite or infinite:

$S_4 = S_1 \times S_1 = \{(4,4), (4,5), (4,6), (5,4), (5,5), (5,6), (6,4), (6,5), (6,6)\}$
$S_5 = 2^{S_1} = \{\emptyset, \{4\}, \{5\}, \{6\}, \{4,5\}, \{4,6\}, \{5,6\}, \{4,5,6\}\}$
$S_6 = S_1 \cap S_2 = \{4, 6\}$
$S_7 = S_2 \cap S_3 = \{i : \ i \text{ is divisible by } 6\}$

The sets S_4, S_5, and S_6 are finite, whereas S_7 is infinite.

Problem 2
Prove that, if $S_1 \subseteq S_2$, then $\overline{S_2} \subseteq \overline{S_1}$.

By the definition of subsets, we need to show that every element x of $\overline{S_2}$ is also an element of $\overline{S_1}$. The proof is as follows:

$$
\begin{aligned}
x \in \overline{S_2} \quad & \Rightarrow \quad x \notin S_2 \quad && \text{(by definition of } \overline{S_2}\text{)} \\
& \Rightarrow \quad x \notin S_1 \quad && \text{(because } S_1 \subseteq S_2\text{)} \\
& \Rightarrow \quad x \in \overline{S_1} \quad && \text{(by definition of } \overline{S_1}\text{)}
\end{aligned}
$$

Problem 3
(a) $1+2+3+4+...+n = \frac{n\cdot(n+1)}{2}$.

We use a proof by induction. Clearly, the equality holds for $n=1$; we now show that, if it holds for n, then it also holds for $n+1$:

$$\begin{aligned}
1+2+...+n+(n+1) &= (1+2+...+n)+(n+1) \\
&= \frac{n\cdot(n+1)}{2}+(n+1) \\
&= \frac{n\cdot(n+1)+2\cdot(n+1)}{2} \\
&= \frac{(n+2)\cdot(n+1)}{2} \\
&= \frac{(n+1)\cdot((n+1)+1)}{2}
\end{aligned}$$

(b) $1+x+x^2+x^3+...+x^n = \frac{x^{n+1}-1}{x-1}$ (where $x \neq 1$).

We observe that the equality holds for $n=1$, and apply induction to "step" from n to $n+1$:

$$\begin{aligned}
1+x+...+x^n+x^{n+1} &= \frac{x^{n+1}-1}{x-1}+x^{n+1} \\
&= \frac{x^{n+1}-1+x^{n+1}\cdot(x-1)}{x-1} \\
&= \frac{x^{n+1}-1+x^{n+2}-x^{n+1}}{x-1} \\
&= \frac{x^{(n+1)+1}-1}{x-1}
\end{aligned}$$