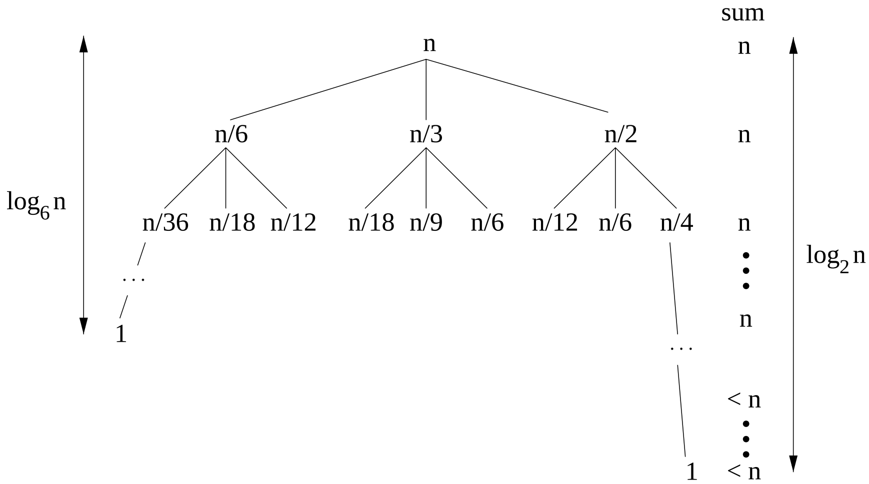
Algorithms: Solutions 5

Problem 1

Determine asymptotically tight bounds for the following recurrences.

(a) $T(n) = T(n/6) + T(n/3) + T(n/2) + n$.

We use the recursion-tree method:



The summation shows that $n{\cdot}\log_6 n < T(n) < n{\cdot}\log_2 n$, which implies that $T(n) = \Theta(n{\cdot}\lg n)$.

(b) $T(n) = T(n-1) + 1$.

$$
\begin{aligned}
T(n) &= T(n-1) + 1 \\
&= T(n-2) + 1 + 1 \;=\; T(n-2) + 2 \\
&= T(n-3) + 1 + 2 \;=\; T(n-3) + 3 \\
&\quad \cdots \\
&= T(1) + (n-1) \\
&= \Theta(n)
\end{aligned}
$$

(c) $T(n) = T(n-1) + n$.

$$
\begin{aligned}
T(n) &= T(n-1) + n \\
&= T(n-2) + (n-1) + n \\
&\quad \cdots \\
&= T(1) + 2 + 3 + ... + (n-1) + n \\
&= T(1) + \frac{(n-1)\cdot(n+2)}{2} \\
&= \Theta(n^2)
\end{aligned}
$$

(d) $T(n) = T(\sqrt{n}) + 1$.

For convenience, we assume that $n = 2^{2^k}$, for some natural value k.

$$\begin{aligned}
T(n) &= T(\sqrt{2^{2^k}}) + 1 \;=\; T(2^{2^{k-1}}) + 1 \\
&= T(\sqrt{2^{2^{k-1}}}) + 1 + 1 \;=\; T(2^{2^{k-2}}) + 2 \\
&= T(\sqrt{2^{2^{k-2}}}) + 1 + 2 \;=\; T(2^{2^{k-3}}) + 3 \\
&\quad \cdots \\
&= T(2^{2^{k-k}}) + k \\
&= T(2) + k \\
&= \Theta(k)
\end{aligned}$$

Note that $k = \lg\lg n$, which implies that $T(n) = \Theta(\lg\lg n)$.

(e) $T(n) = \sqrt{n} \cdot T(\sqrt{n}) + n$.

We assume for convenience that $n = 2^{2^k}$ and $T(4) = 4$, and use induction to prove the following equality:

$$T(2^{2^k}) = 2^{2^k} \cdot k.$$

This equality holds for $k = 1$:

$$T(2^{2^1}) = T(4) = 4 = 2^{2^1} \cdot 1,$$

and the induction step is as follows:

$$\begin{aligned}
T(2^{2^{k+1}}) &= \sqrt{2^{2^{k+1}}} \cdot T(\sqrt{2^{2^{k+1}}}) + 2^{2^{k+1}} \\
&= 2^{2^k} \cdot T(2^{2^k}) + 2^{2^{k+1}} \\
&= 2^{2^k} \cdot (2^{2^k} \cdot k) + 2^{2^{k+1}} \\
&= (2^{2^k})^2 \cdot k + 2^{2^{k+1}} \\
&= 2^{2^{k+1}} \cdot k + 2^{2^{k+1}} \\
&= 2^{2^{k+1}} \cdot (k+1)
\end{aligned}$$

Note that $k = \lg\lg n$, which implies that $T(n) = \Theta(n \cdot \lg\lg n)$.