Algorithms: Solutions 4

```
                         X
                         X
                         X
                         X  X
                         X  X
                         X  X
                         X  X
                         X  X
number of             X  X  X
homeworks             X  X  X
                      X  X  X
                      X  X  X
                X  X  X  X  X
                X  X  X  X  X
          X     X  X  X  X  X
          X     X  X  X  X  X
          X  X  X  X  X  X  X
          --------------------
          4  5  6  7  8  9  10
                grades
```

The histogram shows the distribution of grades.

Problem 1
Let $A[1..n]$ be a sorted array of n distinct integer numbers. Write an efficient algorithm INDEX-SEARCH(A, n) that finds an index i such that $A[i] = i$. If the array does not have such an element, the algorithm should return 0.

The algorithm is almost identical to BINARY-SEARCH, and its time complexity is $O(\lg n)$. It works only for integer arrays, since it is based on the assumption that, for every two indices p and r (where $p \leq r$), we have $A[r] - A[p] \geq r - p$.

INDEX-SEARCH(A, n)
$p \leftarrow 1$
$r \leftarrow n$
while $p < r$
  do $q = \lfloor (p + r)/2 \rfloor$
    if $q \leq A[q]$
      then $r \leftarrow q$
      else $p \leftarrow q + 1$
if $p = A[p]$
  then return p
  else return 0

Problem 2
Suppose that $A[1..n]$ is a heap and you need to change its ith element. Give an algorithm CHANGE-ELEMENT(A, i, n, k) that sets $A[i] \leftarrow k$ and updates the heap appropriately.

```
CHANGE-ELEMENT(A, i, n, k)
if (A[i] > k)
   then A[i] ← k
        HEAPIFY(A, i, n)
   else while i > 1 and A[PARENT(i)] < k
            do A[i] ← A[PARENT(i)]
               i ← PARENT(i)
        A[i] ← k
```

The running time of this algorithm is $O(\lg n)$.

Problem 3
A *d-ary heap* is like a binary heap, but instead of 2 children, nodes have d children.

(a) How would you represent a d-ary heap with n elements in an array? What are the expressions for determining the parent of a given element, PARENT(i), and a j-th child of a given element, CHILD(i, j), where $1 \leq j \leq d$?

The following expressions determine the parent and j-th child of element i (where $1 \leq j \leq d$):

$$\begin{aligned} \text{PARENT}(i) &= \left\lfloor \frac{i+d-2}{d} \right\rfloor, \\ \text{CHILD}(i,j) &= (i-1)d + j + 1. \end{aligned}$$

(b) Write an efficient implementation of HEAPIFY and HEAP-INSERT for a d-ary heap.

The HEAPIFY algorithm is somewhat different from the binary-heap version, whereas HEAP-INSERT is identical to the corresponding algorithm for binary heaps. The running time of HEAPIFY is $O(d \cdot \log_d n)$, and the running time of HEAP-INSERT is $O(\log_d n)$.

```
HEAPIFY(A, i, n, d)
largest ← i
for l ← CHILD(i, 1) to CHILD(i, d)      ▷ loop through all children of i
    do  if l ≤ n and A[l] > A[largest]
          then largest ← l
if largest ≠ i
    then exchange A[i] ↔ A[largest]
         HEAPIFY(A, largest)
```

```
HEAP-INSERT(A, key)
heap-size[A] ← heap-size[A] + 1
i ← heap-size[A]
while i > 1 and A[PARENT(i)] < key
      do A[i] ← A[PARENT(i)]
         i ← PARENT(i)
A[i] ← key
```

Problem 4
What is the height of a d-ary heap of n elements in terms of n and d?

The height h of a heap is *approximately* equal to $\log_d n$. The exact height is

$$h = \lceil \log_d(nd - n + 1) - 1 \rceil.$$