Geometry for Programming Competitions

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Available online:
- **Word version: www.cs.cmu.edu/~eugene/research/talks/compete-geom.doc**
- **PDF version: www.cs.cmu.edu/~eugene/research/talks/compete-geom.pdf**

Let no one who is ignorant of geometry enter here.
— Inscription on the entrance to Plato's academy

We humans are good at visual reasoning, but computers are not;
intuitive visual operations are often hard to program.

Overview

- *Basic geometry:* points and lines, triangle, circle, polygon area
- *Convex hull:* gift wrapping, Graham scan

Readings:
- Steven S. Skiena and Miguel A. Revilla. Programming Challenges
- Joseph O'Rourke. Computational geometry in C
- Wikipedia (*wikipedia.org*)
- Wolfram MathWorld (*mathworld.wolfram.com*)

Points and lines

Representation in Cartesian coordinates:
- Point: (x_1, y_1)
- Line:
  - $y = m \cdot x + b$ (more intuitive but does not include vertical lines)
  - $a \cdot x + b \cdot y + c = 0$ (more general but non-unique and less intuitive)
- Line segment: two endpoints

Yes/No tests:
- Point (x_1, y_1) is on line (m, b) iff $y_1 = m \cdot x_1 + b$ (available in Java)
- Lines (m_1, b_1) and (m_2, b_2) are…
  - identical iff $m_1 = m_2$ and $b_1 = b_2$
  - parallel iff $m_1 = m_2$
  - orthogonal iff $m_1 = -1 / m_2$
- Counterclockwise predicate *ccw* (available in Java):
  Consider points (x_1, y_1), (x_2, y_2), and (x_3, y_3),
  and let $A = x_1 \cdot y_2 + x_2 \cdot y_3 + x_3 \cdot y_1 - y_1 \cdot x_2 - y_2 \cdot x_3 - y_3 \cdot x_1$.
  - If $A > 0$, the points are in a counterclockwise order
  - If $A < 0$, the points are in a clockwise order
  - If $A = 0$, the points are collinear

Basic computations:

- Distance between (x_1, y_1) and (x_2, y_2):
  $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$
- Line through points (x_1, y_1) and (x_2, y_2):
  $m = (y_1 - y_2) / (x_1 - x_2)$
  $b = y_1 - m \cdot x_1$
- Intersection of (m_1, b_1) and (m_2, b_2), where $m_1 \neq m_2$:
  $x_1 = (b_2 - b_1) / (m_1 - m_2)$
  $y_1 = m_1 \cdot x_1 + b_1$
- Angle between (m_1, b_1) and (m_2, b_2):
  $\arctan\big((m_2 - m_1) / (m_1 \cdot m_2 + 1)\big)$

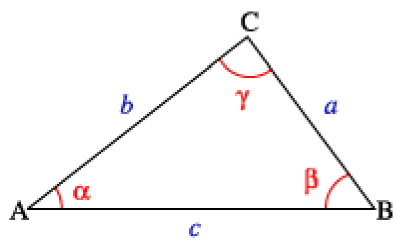
Other useful operations:

- Intersection of two segments (available in Java)
- Distance from a point to a line or a segment (available in Java)
- Point on a line closest to a given point

Triangle

A triangle specification may include three sides and three angles.
Given three of these six values, we can find the other three.

- $\alpha + \beta + \gamma = \pi$ (or 180˚)
- Law of sines:
  $a / \sin \alpha = b / \sin \beta = c / \sin \gamma$
- Law of cosines (generalized Pythagorean theorem):
  $c^2 = a^2 + b^2 - 2 \cdot a \cdot b \cdot \cos \gamma$
- Law of tangents (less useful):
  $(a - b) / (a + b) = \tan\big((\alpha - \beta) / 2\big) / \tan\big((\alpha + \beta) / 2\big)$



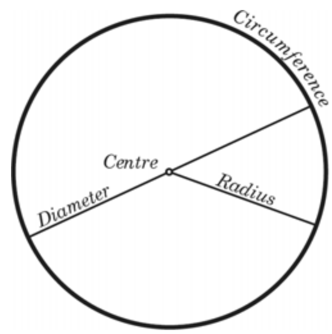
Area:

- $Area = a \cdot b \cdot \sin \gamma / 2$
- $Area = | x_1 \cdot y_2 + x_2 \cdot y_3 + x_3 \cdot y_1 - y_1 \cdot x_2 - y_2 \cdot x_3 - y_3 \cdot x_1 | / 2$
- Heron's formula:
  Let $s = (a + b + c) / 2$; then $Area = \sqrt{s \cdot (s-a) \cdot (s-b) \cdot (s-c)}$
- Less useful:
  $Area = r \cdot s = a \cdot b \cdot c / (4 \cdot R)$, where r is inradius and R is circumradius

Other useful operations:

- Centers of inscribed and circumscribed circles

Circle



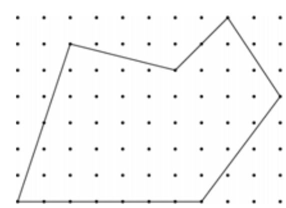
- A *circle* is the set of points located at a given distance from a given center.
- A *disk* is the set of points located no further than a given distance from a given center.

Representation:
Let r be the radius and (x_c, y_c) by the center.
- Circle: $(x - x_c)^2 + (y - y_c)^2 = r^2$
- Disk: $(x - x_c)^2 + (y - y_c)^2 \leq r^2$

Basic computations:
- Circumference: $2 \cdot \pi \cdot r$
- Area: $\pi \cdot r^2$
- Point (x_1, y_1) is inside the circle iff $(x_1 - x_c)^2 + (y_1 - y_c)^2 \leq r^2$

Other useful operations:
- Intersection of a circle and a line
- Intersection of two circles
- Tangent to a circle through a given point

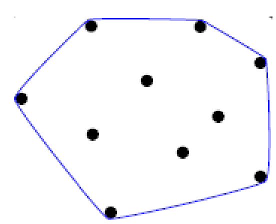
Polygon area



- Based on vertex coordinates:
  $| x_1 \cdot y_2 + x_2 \cdot y_3 + ... + x_n \cdot y_1 - y_1 \cdot x_2 - y_2 \cdot x_3 - ... - y_n \cdot x_1 | / 2$
- Pick's formula for a polygon on a lattice:
  If a polygon has i lettice points inside and b lettice points on the boundary, its area is $i + b/2 - 1$

Convex hull

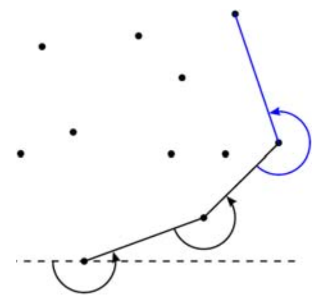
The *convex hull* of a geometric object is the minimal convex set containing the object. Intuitively, it is a rubber band pulled taut around the object.



- The hull of a finite point set is a convex polygon.
- The hull of a set of polygons is identical to the hull of their vertices.
- A polygon is convex (i.e. identical to its hull) iff all its angles are at most π (180º).

Gift wrapping (a.k.a. Jarvis march)

Intuitively, wrap a string around nails; simple but slow.
Time complexity is O($n \cdot h$), where n is the number of points and h is the number of hull vertices.

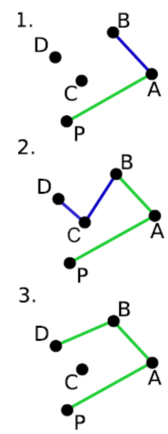


- Select the smallest-y point as the first hull vertex;
  if several, choose the largest-x point among them
- At each step, select the next hull vertex, which is the "rightmost" as seen from the previously selected vertex

Detailed description in Wikipedia: *en.wikipedia.org/wiki/Gift_wrapping_algorithm*

Graham scan

Efficient version of the gift wrapping.
Time complexity is O($n \cdot \lg n$).



- Select the smallest-y point as the first hull vertex;
  if several, choose the largest-x point among them
- Sort the other points right-to-left, as seen from the selected vertex
- Walk through the points in the sorted order;
  when making the "right turn," prune the respective point
- The remaining points are the hull vertices

Detailed description in Wikipedia: *en.wikipedia.org/wiki/Graham_scan*