Three-dimensional strong convexity and visibility

Eugene Fink

School of Computer Science
Carnegie Mellon University
Pittsburgh, PA 15213, USA
eugene@cs.cmu.edu

Derick Wood

Department of Computer Science
Hong Kong Univ. of Science and Technology
Clear Water Bay, Kowloon, Hong Kong
dwood@cs.ust.hk

ABSTRACT

We define the notions of *strong convexity* and *strong visibility*. These notions generalize standard convexity and visibility, as well as several types of nontraditional convexity, such as iso-oriented rectangles and C-oriented polygons. We explore the properties of strong convexity and strong visibility in two and three dimensions. In particular, we establish analogs of the following properties of standard convex sets:

- Every two points of a convex set are visible to each other.
- The intersection of convex sets is a convex set.
- For every point in the boundary of a convex set, there exists a supporting plane through this point.
- A closed convex set in three dimensions is the intersection of all halfspaces that contain it.

Keywords: computational geometry, mathematical foundations, convexity, visibility, restricted orientations, three dimensions.

1 INTRODUCTION

Convex sets are a comparatively recent yet fruitful concept in geometry, which has applications in optimization, statistics, geometric number theory, functional analysis, and combinatorics,[6,11] as well as in more practical areas, such as VLSI design, computer graphics, architectural databases, and geographic databases. Researchers have studied many notions of nontraditional convexity along with standard convexity, such as orthogonal convexity,[8–10] finitely oriented convexity,[4,13,20] restricted-orientation convexity,[12,16,17] NESW convexity,[7,18,20] and link convexity.[1,17,19]

Rawlins has introduced the notions of strong $\mathcal{O}$-convexity and strong $\mathcal{O}$-visibility as a part of his research on restricted-orientation visibility.[12] These notions are stronger than standard convexity and visibility, hence the name. Rawlins and Wood studied the properties of strongly convex sets and demonstrated that strong convexity generalizes not only standard convexity but also the notions of iso-oriented rectangles (that is, rectangles whose edges are parallel to the coordinate axes) and C-oriented polygons described by Güting.[3,5] The work on strong convexity adds to our understanding of convexity in general and helps us to develop simpler and more efficient algorithms.

The research on nontraditional notions of convexity and visibility has so far been restricted to two dimensions. The work reported here is the first step in exploring nontraditional convexity and visibility in higher dimensions. For simplicity of presentation, we restrict our attention to three dimensions; however, the results that we present also hold in higher dimensions.[2] We extend the notion of strong convexity to three dimensions. This extension is a generalization of planar strong convexity and standard three-dimensional convexity. We establish analogs of the following basic properties of convex sets:

Visibility Every two points of a convex set are visible to each other; that is, the straight segment joining these points is contained in the set.

Intersection The intersection of convex sets is a convex set.

Supporting planes For every point in the boundary of a convex set, there exists a plane through this point that supports the set.

Halfspace intersection A closed convex set is the intersection of all halfspaces that contain it.

Except for "intersection," these properties are defining characteristics of convex sets.

We should give a word of warning here. Our results are not completely unexpected, yet their proofs are often surprisingly complex;[2] *intuition serves us badly. Because of space limitations, we give only a few proof sketches.*

In Section 2, we briefly describe the notions of strong visibility and strong convexity in two dimensions. In Section 3, we generalize these notions to three dimensions. In Section 4, we explore the properties of strongly convex lines and planes and characterize strongly convex sets in terms of supporting planes. In Section 5, we describe strongly convex halfspaces and present an analog of the "halfspace-intersection" property for strongly convex sets. Finally, we conclude, in Section 6, with a summary of the results and notes on their extension to higher-dimensional spaces.

2 STRONG VISIBILITY AND STRONG CONVEXITY IN TWO DIMENSIONS

We begin by presenting the notions of strong visibility and strong convexity in two dimensions and describing the basic properties of planar strongly convex sets.[12]

Standard convex sets can be defined in terms of visibility: a set is convex if every two points of the set are visible to each other; that is, for every two points of the set, the straight segment joining them

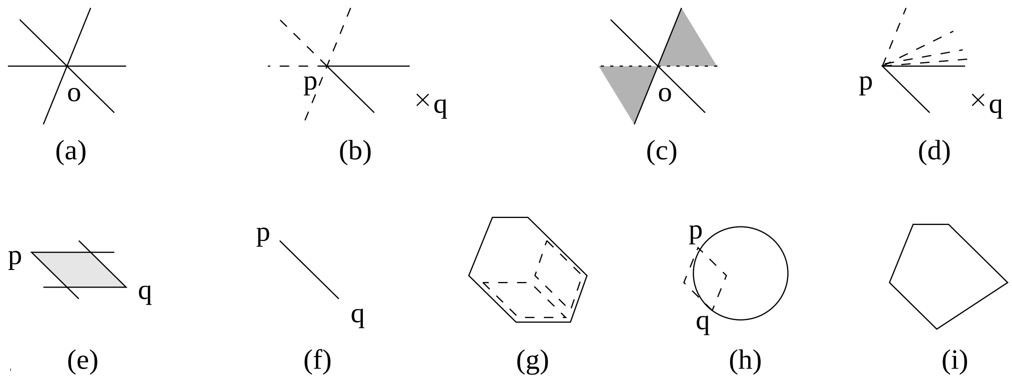

Figure 1: Planar strong visibility and strong convexity

is contained in the set. We introduce a new type of visibility by replacing straight segments with a different type of objects, called $\mathcal{O}$-*blocks*. We then define strong convexity in terms of this new visibility.

To describe the new visibility, we first introduce an *orientation set* $\mathcal{O}$, which is a (finite or infinite) set of lines through some fixed point o. We show an example of a finite orientation set in Figure 1(a). A straight line is called an $\mathcal{O}$-*oriented* line if it is parallel to one of the lines of $\mathcal{O}$.

Next, we define the $\mathcal{O}$-*block* of two points, say p and q. $\mathcal{O}$-blocks will play the same role in defining strong visibility as straight segments in defining standard visibility. Let us draw all $\mathcal{O}$-oriented rays with endpoint p and pick the two of them closest to q (see Figure 1b). The two selected rays, with the common endpoint p, are the boundaries of an angle with vertex p; this angle contains q.

If $\mathcal{O}$ is an infinite set, it may not be closed and, therefore, we may not be able to pick the ray "closest" to q. For example, consider the orientation set in Figure 1(c): here all lines in the shaded area are elements of $\mathcal{O}$ and the dotted horizontal line is not in $\mathcal{O}$; this orientation set is not closed. If $\mathcal{O}$ is not closed, we have to use a limit in selecting our two rays: we pick the ray with endpoint p such that (1) there is a sequence of $\mathcal{O}$-oriented rays convergent to the selected ray and (2) there is no $\mathcal{O}$-oriented rays between this ray and the point q (see Figure 1d). The two selected rays, with the common endpoint p, are again the boundaries of an angle with vertex p; this angle contains q.

Similarly, we draw the $\mathcal{O}$-oriented rays from q closest to p and get the angle with vertex q whose boundaries are these rays (Figure 1e). The $\mathcal{O}$-*block* of p and q is defined as the intersection of these two angles (the shaded parallelogram in Figure 1e). As a special case, if the line through p and q is $\mathcal{O}$-oriented, then the $\mathcal{O}$-block of p and q is just the straight segment joining p and q (Figure 1f).

We now define strong visibility and strong convexity, based on the notion of $\mathcal{O}$-blocks.

DEFINITION 1.
Strong visibility: *Two points of a set are* strongly visible *to each other if the $\mathcal{O}$-block of these two points is contained in the set.*
Strong convexity: *A set is* strongly convex *if every two points of the set are strongly visible to each other.*

For example, the polygon in Figure 1(g) is strongly convex for the orientation set in Figure 1(a) as well as for the orientation set in Figure 1(c) (two $\mathcal{O}$-blocks contained in this polygon are shown by dashed lines). On the other hand, the circle in Figure 1(h) is not strongly convex for either orientation set: the $\mathcal{O}$-block shown by dashed lines is not in the circle and, therefore, the points p and q of the circle are not strongly visible to each other. Finally, the polygon in Figure 1(i) is strongly convex for the orientation set in Figure 1(c) but not for the orientation set in Figure 1(a).

The following properties of strong visibility and strong convexity readily follow from the definition (Properties 1–4 and 6 were stated by Rawlins[12]):

1. A translation of a strongly convex set is strongly convex.
2. **(Intersection)** If C is a collection of strongly convex sets, then the intersection $\bigcap C$ of these sets is also a strongly convex set.
3. For every orientation set $\mathcal{O}$:
   - if two points of a set P are strongly visible to each other in P, then they are "standardly" visible to each other;
   - if a set P is strongly convex, then P is convex.
4. For every two orientation sets $\mathcal{O}_1$ and $\mathcal{O}_2$ such that $\mathcal{O}_1 \subseteq \mathcal{O}_2$:
   - if two points of a set P are strongly visible to each other with respect to $\mathcal{O}_1$, then they are also strongly visible to each other with respect to $\mathcal{O}_2$;
   - if a set P is strongly convex for the orientation set $\mathcal{O}_1$, then P is also strongly convex for $\mathcal{O}_2$.
5. Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be two orientation sets through the same point o; then, the following three statements are equivalent:
   ∘ strong visibility for $\mathcal{O}_1$ is equivalent to strong visibility for $\mathcal{O}_2$;
   ∘ strong convexity for $\mathcal{O}_1$ is equivalent to strong convexity for $\mathcal{O}_2$;
   ∘ $\text{Closure}(\mathcal{O}_1) = \text{Closure}(\mathcal{O}_2)$.
6. For closed $\mathcal{O}$, a polygon is strongly convex if and only if it is convex and its edges are $\mathcal{O}$-oriented.

3 STRONG VISIBILITY AND STRONG CONVEXITY IN THREE DIMENSIONS

We now extend the notions of strong visibility and strong convexity to three-dimensional space. We introduce a set $\mathcal{O}$ of planes through a fixed point o, define $\mathcal{O}$-blocks in three dimensions, and use $\mathcal{O}$-blocks to define strong visibility and strongly convex sets. Then, we explore basic properties of three-dimensional strongly convex sets.

Definition 2. *An* orientation set $\mathcal{O}$ *in three dimensions is a set of planes through a fixed point* o. *A plane parallel to one of the elements of* $\mathcal{O}$ *is called an* $\mathcal{O}$-oriented *plane.*

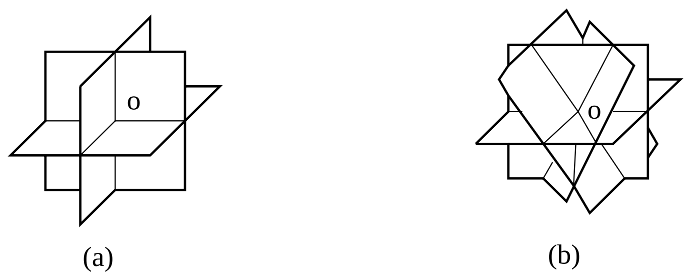

Figure 2: Orientation sets in three dimensions

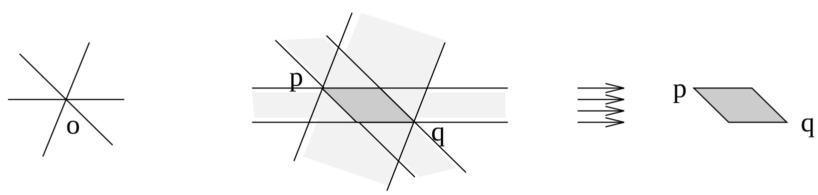

Figure 3: In two dimensions, the intersection of all $\mathcal{O}$-oriented layers is the $\mathcal{O}$-block

Note that every translation of an $\mathcal{O}$-oriented plane is an $\mathcal{O}$-oriented plane and a particular choice of the point o is not important. Two examples of finite orientation sets are shown in Figure 2.

The definition of $\mathcal{O}$-blocks in three dimensions is more complex than that of planar $\mathcal{O}$-blocks. First, we define the notion of a *layer* of two points, p and q. Let H be a plane from the orientation set $\mathcal{O}$, H_p be the plane through p parallel to H, and H_q be the plane through q parallel to H. The "layer" of space between the planes H_p and H_q is called the *H-layer* of p and q. The $\mathcal{O}$-block of p and q is defined as the intersection of all $\mathcal{O}$-oriented layers of p and q:

$$\mathcal{O}\text{-block}(p,q) = \bigcap_{H \in \mathcal{O}} H\text{-layer}(p,q).$$

In other words, a point is in the $\mathcal{O}$-block of p and q if, for every element H of $\mathcal{O}$, the point is between H_p and H_q.

In two dimensions, we may define $\mathcal{O}$-blocks in the same way: a planar layer is the "layer" between two parallel lines and the $\mathcal{O}$-block of two points is the intersection of all $\mathcal{O}$-oriented layers of these points. This definition is equivalent to the definition of planar $\mathcal{O}$-blocks in Section 2, as illustrated by Figure 3.

We show some examples of three-dimensional $\mathcal{O}$-blocks in Figure 4. For the three-element orientation set in Figure 4(a), $\mathcal{O}$-blocks are parallelepipeds with $\mathcal{O}$-oriented facets. The orientation set in Figure 4(b) comprises four planes and gives rise to more complex $\mathcal{O}$-blocks.

We define strong visibility and strong convexity in three dimensions in the same way as in two dimensions. Two points of a three-dimensional set are strongly visible to each other if the $\mathcal{O}$-block of these two points is contained in the set. A set is strongly convex if every two points of the set are strongly visible to each other.

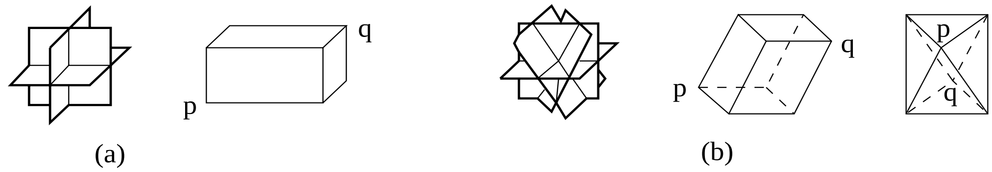

Figure 4: $\mathcal{O}$-blocks in three dimensions

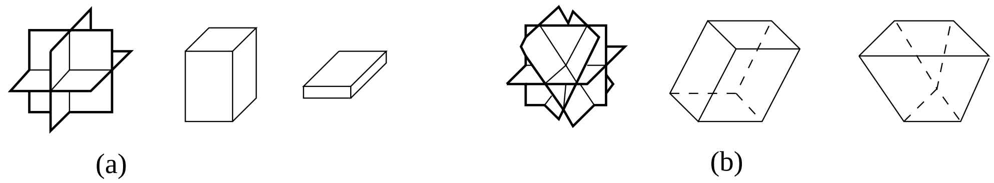

Figure 5: Strongly convex sets

We show some examples of strongly convex polytopes in Figure 5. For the orientation set in Figure 5(a), strongly convex polytopes are parallelepipeds with $\mathcal{O}$-oriented facets. The four-element orientation set of Figure 5(b) gives rise to more complex strongly convex objects; the facets of these objects are also $\mathcal{O}$-oriented, as we will show in Section 5 (see Corollary 10).

Let us look back at the properties of planar strong visibility and strong convexity, listed at the end of Section 2. We note that Properties 1–4 hold in three dimensions (a proof is straightforward). The most important of them is Property 2, which is a generalization of the "intersection" property of standard convex sets: the intersection of strongly convex sets is a strongly convex set.

Property 5 holds only in one direction: if two orientation sets, $\mathcal{O}_1$ and $\mathcal{O}_2$, have identical closures, then the notions of strong visibility and strong convexity for $\mathcal{O}_1$ are equivalent to these notions for $\mathcal{O}_2$.

Lemma 1. *If $\mathcal{O}_1$ and $\mathcal{O}_2$ are orientation sets such that* $\text{Closure}(\mathcal{O}_1) = \text{Closure}(\mathcal{O}_2)$*, then:*
◦ *strong visibility for $\mathcal{O}_1$ is equivalent to strong visibility for $\mathcal{O}_2$;*
◦ *strong convexity for $\mathcal{O}_1$ is equivalent to strong convexity for $\mathcal{O}_2$.*

According to this result, *we may restrict our attention to the study of strong visibility and strong convexity for closed orientation sets*, since strong visibility and strong convexity for every orientation set are equivalent to that for its closure.

The converse direction of Property 5 does not hold in three dimensions: the notions of strong visibility and strong convexity for $\mathcal{O}_1$ and $\mathcal{O}_2$ may be equivalent even if $\text{Closure}(\mathcal{O}_1) \neq \text{Closure}(\mathcal{O}_2)$ (see Example 1 in Section 4).

For Property 6, we will show that its analog holds in three dimensions for *finite* orientations sets (Corollary 10): a polytope is strongly convex if and only if it is convex and its facets are $\mathcal{O}$-oriented.

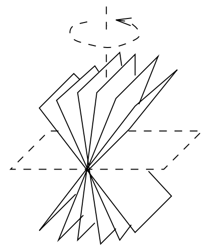

Figure 6: Construction of the orientation set $\mathcal{O}_{sc}$

4 STRONGLY CONVEX LINES AND PLANES

We now explore the properties of strongly convex lines and planes, whose role in the study of strongly convex sets is similar to the role of lines and planes in standard convexity. We characterize strongly convex lines and planes in terms of $\mathcal{O}$-oriented planes and present a strong-convexity analog of the "supporting-planes" property of convex sets (see Section 1). In Section 5, we will explore the properties of halfspaces whose boundaries are strongly convex planes.

We begin by characterizing strongly convex lines in terms of the intersections of $\mathcal{O}$-oriented planes. We note that every $\mathcal{O}$-oriented plane is strongly convex, because, for every two points of an $\mathcal{O}$-oriented plane, the $\mathcal{O}$-block of these points is contained in the plane and, therefore, every two points of the plane are strongly visible to each other. If two $\mathcal{O}$-oriented planes are not parallel, then their intersection is a strongly convex line, since the intersection of strongly convex sets is always strongly convex. For closed $\mathcal{O}$, the converse of this observation is also true: every strongly convex line is formed by the intersection of two $\mathcal{O}$-oriented planes.

THEOREM 2. *For a closed orientation set $\mathcal{O}$, a line is strongly convex if and only if it is the intersection of two $\mathcal{O}$-oriented planes.*

Sketch of a proof. We have shown that the intersection of $\mathcal{O}$-oriented planes is strongly convex. Suppose, conversely, that the line is strongly convex. Then, every two points p and q of this line are strongly visible to each other. Therefore, the $\mathcal{O}$-block of p and q is the straight segment joining them, which can happen only if there exist two different $\mathcal{O}$-oriented planes through p and q. □

We have noted that every $\mathcal{O}$-oriented plane is strongly convex. Can a plane be strongly convex if it is not $\mathcal{O}$-oriented? If $\mathcal{O}$ is a finite or closed countably infinite set, the answer to this question is negative: only $\mathcal{O}$-oriented planes are strongly convex (see Corollary 4). For uncountable orientation sets, however, we can construct an example where some planes are strongly convex and not $\mathcal{O}$-oriented.

EXAMPLE 1: **A strongly convex plane may not be $\mathcal{O}$-oriented.**
Let $\mathcal{O}_{sc}$ be the orientation set in three dimensions that comprises all planes through o whose angles with the "horizontal" plane is at least $\pi/3$ (where any plane through o can serve as the horizontal plane). We illustrate a construction of the orientation set $\mathcal{O}_{sc}$ in Figure 6: the set comprises the (uncountably many) solid planes shown in the figure and all their rotations around the vertical axis. The index "sc" stands for "standard convexity," as we show that strong convexity for $\mathcal{O}_{sc}$ is equivalent to standard convexity.

Every straight line through o is the intersection of two elements of $\mathcal{O}_{sc}$. Since translations of elements of $\mathcal{O}_{\mathrm{sc}}$ are $\mathcal{O}_{\mathrm{sc}}$-oriented planes, we conclude that every line is the intersection of two $\mathcal{O}_{\mathrm{sc}}$-oriented planes. Thus, the $\mathcal{O}_{\mathrm{sc}}$-block of every two points is the straight segment joining these points and, therefore, strong visibility is equivalent to standard visibility. We conclude that strong convexity for $\mathcal{O}_{\mathrm{sc}}$ is equivalent to standard convexity, which implies that every plane is strongly convex. We have shown that all planes are strongly convex, whereas some planes are not $\mathcal{O}_{\mathrm{sc}}$-oriented.

Note that, if we define $\mathcal{O}'_{\mathrm{sc}}$ as the set of planes through o whose angle with some *vertical* plane is at least $\pi/3$, then strong convexity for $\mathcal{O}'_{\mathrm{sc}}$ is also equivalent to standard convexity. This example demonstrates that the notions of strong convexity for different closed orientation sets can be equivalent, which means that Property 5 of strongly convex sets (Section 2) does not hold in three dimensions. □

THEOREM 3. *A plane H is strongly convex if and only if*
(1) H is $\mathcal{O}$-oriented or
(2) every line contained in H is strongly convex.

Sketch of a proof. If H is $\mathcal{O}$-oriented, then the $\mathcal{O}$-block of every two points of H is in H. On the other hand, if every line contained in H is strongly convex, then, for every two points $p, q \in H$, the $\mathcal{O}$-block of p and q is a subset of the line through p and q, and, therefore, it is a subset of H. Thus, if H is $\mathcal{O}$-oriented or every line contained in H is strongly convex, then the $\mathcal{O}$-block of every two points of H is in H and, therefore, H is strongly convex.

To prove the converse, suppose that a plane H is strongly convex. We consider two possible cases. First, suppose that the $\mathcal{O}$-block of every two points of H is a straight segment; then, every line contained in H is strongly convex. Next, suppose that, for some points $p, q \in H$, their $\mathcal{O}$-block is *not* a straight segment. Then, this $\mathcal{O}$-block is a planar object with nonempty interior contained in H, which can happen only if H is $\mathcal{O}$-oriented. Thus, we have shown that, if H is strongly convex, then every line contained in H is strongly convex or H is $\mathcal{O}$-oriented. □

If an orientation set $\mathcal{O}$ is finite or countably infinite, then the second condition of Theorem 3 cannot be satisfied: for every point p of H, there are only countably many $\mathcal{O}$-oriented planes through p. Thus, the intersections of $\mathcal{O}$-oriented planes through p form countably many strongly convex lines and we can pick such a point $q \in H$ that a line through p and q is not strongly convex. Therefore, for a closed countable set $\mathcal{O}$, only $\mathcal{O}$-oriented planes are strongly convex.

COROLLARY 4. *If $\mathcal{O}$ is a closed countable set, then a plane is strongly convex if and only if it is $\mathcal{O}$-oriented.*

In Example 1, we described the orientation set $\mathcal{O}_{\mathrm{sc}}$, for which strong convexity is equivalent to standard convexity. We now present a condition of equivalence of strong and standard convexity, in terms of strongly convex lines.

LEMMA 5. *If every straight line is strongly convex, then every convex set is strongly convex.*

We next describe supporting planes of strongly convex sets. A plane is said to *support* a set if it "touches" the set in some of its boundary points and does not cut the set into two parts. If we put a three-dimensional object on a table, then the surface of the table is a plane supporting the object. To put it more formally, a plane H supports P if the intersection of H and the boundary of P is nonempty and P is contained in one of the two halfspaces whose boundary is H.

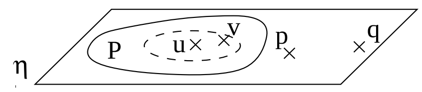

Figure 7: Proof of Theorem 7

We can describe standard convex sets in terms of supporting planes: a closed set with a nonempty interior is convex if and only if, for every point of its boundary, there exists a supporting plane through this point. We now generalize this property to strongly convex sets.

THEOREM 6. *A closed set with a nonempty interior is strongly convex if and only if, for every point in the boundary of the set, there exists a strongly convex plane through this point that supports the set.*

Sketch of a proof. Let P be a closed set with a nonempty interior. Suppose that, for every point of P's boundary, there exists a strongly convex plane through this point that supports P. The intersection of halfspaces bounded by these hyperplanes form P and each halfspace with strongly convex boundary is strongly convex (see Lemma 8); therefore, P is the intersection of strongly convex sets, which implies that P is strongly convex.

Suppose, conversely, that P is strongly convex. Then, P is convex and "small" pieces of P's boundary can be viewed as graphs of convex functions. We consider two types of points in the boundary of P: regular points, where the function is differentiable, and nonregular points. If p is a regular point of the boundary, then there exists exactly one supporting plane through p and we can prove that this plane is strongly convex. If p is not regular, we can select a sequence of regular boundary points convergent to p and consider a sequence of supporting planes in these points. This sequence has a convergent subsequence and its limit is a strongly convex supporting plane through p. □

To see that the analogous result does not hold for sets with an *empty interior*, let us consider an $\mathcal{O}$-oriented plane H and a nonconvex set P contained in H. Then, for every point in P's boundary, H is a supporting plane through this point; however, P is not strongly convex since it is not convex.

Next, we characterize the affine hulls of strongly convex sets. The *affine hull* of a set P is the minimal flat that contains P, where a *flat* is a point, a line, a plane, or the whole space. For example, the affine hull of a straight segment is a line, the affine hull of a triangle is a plane, and the affine hull of a ball is the whole space.

THEOREM 7. *The affine hull of a strongly convex set is strongly convex.*

Sketch of a proof. Let P be a strongly convex set. If the affine hull of P is a point, a line, or the whole space, then the proof is straightforward. Suppose that the affine hull is a plane H (see Figure 7). Since P is convex, its interior in the plane H is nonempty and we can pick a circle S contained in P (the dashed circle in Figure 7).

We have to show that every two points p and q of H are strongly visible to each other; that is, the $\mathcal{O}$-block of p and q is in H. We can pick points $u, v \in S$ such that the line through u and v is parallel to the line through p and q. The $\mathcal{O}$-block of u and v is in P and, therefore, it is in H. The $\mathcal{O}$-block of p and q is a scaled version of $\mathcal{O}$-block(u, v) and, therefore, it is also in H. □

5 STRONGLY CONVEX HALFSPACES

We now study the properties of strongly convex halfspaces and show that their role in strong convexity is similar to the role of halfspaces in standard convexity. We present a strong-convexity analog of the "halfspace-intersection" property of convex sets (see Section 1), which characterizes strongly convex sets in terms of halfspace intersections.

We begin by characterizing strongly convex halfspaces in terms of their boundaries.

LEMMA 8. *A halfspace is strongly convex if and only if its boundary is a strongly convex plane.*

We now characterize strongly convex sets in terms of strongly convex halfspaces. A standard convex set is the intersection of all halfspaces that contain it. An analogous result holds for strongly convex sets.

THEOREM 9. *A closed set is strongly convex if and only if it is the intersection of strongly convex halfspaces.*

Sketch of a proof. The intersection of strongly convex sets is strongly convex; therefore, if a set P is the intersection of strongly convex halfspaces, then P is a strongly convex set.

Suppose, conversely, that P is strongly convex. Then, for every point of P's boundary, there exists a strongly convex plane through this point that supports P. The intersection of halfspaces bounded by these planes form P and each of these halfspaces is strongly convex. (A special care should be taken for strongly convex sets with empty interiors, for which this argument does not work.) □

This result can be readily generalized to nonclosed sets if we use *open halfspaces*, that is, halfspaces that do not contain their boundary: a set is strongly convex if and only if it is the intersection of strongly convex open halfspaces.

If an orientation set $\mathcal{O}$ is finite, then the intersection of strongly convex halfspaces is a convex polytope with $\mathcal{O}$-oriented facets. Thus, the following result describes strongly convex sets for finite $\mathcal{O}$; this result is analogous to Property 6 of planar strong convexity (see Section 2).

COROLLARY 10. *For finite $\mathcal{O}$, a set is strongly convex if and only if it is a convex polytope whose facets are $\mathcal{O}$-oriented.*

6 CONCLUSIONS

We have described a generalization of convexity and visibility in three dimensions, called strong convexity and strong visibility, and demonstrated that the properties of strongly convex sets are similar to that of standard convex sets. The following list summarizes the properties of strongly convex sets generalized from the theory of standard convexity:

Visibility Every two points of a strongly convex set are strongly visible to each other; that is, the

$\mathcal{O}$-block of these points is contained in the set.

Intersection The intersection of strongly convex sets is a strongly convex set.

Supporting planes For every point in the boundary of a strongly convex set, there exists a strongly convex plane through this point that supports the set.

Halfspace intersection A closed strongly convex set is the intersection of all strongly convex halfspaces that contain it.

Other major properties of strongly convex sets include the characterization of strongly convex planes and lines in terms of $\mathcal{O}$-oriented planes (Theorems 2 and 3) and strong convexity of the affine hull of a strongly convex set (Theorem 7).

These results hold not only in three dimensions but also in higher-dimensional spaces.[2] For d-dimensional space, we define $\mathcal{O}$ as a set of $(d-1)$-dimensional planes through a fixed point o. A layer of two points is the part of space between parallel $(d-1)$-dimensional planes through these points and the $\mathcal{O}$-block is the intersection of all $\mathcal{O}$-oriented layers. We define strong visibility in d dimensions in the same way as in three dimensions: two points of a set are strongly visible to each other if their $\mathcal{O}$-block is in the set. We then define strong convexity based on strong visibility. All results of the paper except the characterization of strongly convex planes (Theorem 3) hold in higher dimensions.

The work presented in this paper is just beginning; it leaves many unanswered questions, which we are currently trying to address. First, we have not studied computational aspects of strong convexity, such as finding strongly convex hulls. Second, we can explore other generalizations of convexity. For example, the notion of *NESW*-convexity[12,15] can be generalized to higher dimensions. We are also working on the extension of restricted-orientation convexity[14,16] to higher dimensions.[2]

7 ACKNOWLEDGMENTS

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