Three-Dimensional Restricted-Orientation Convexity[1]

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Abstract A *restricted-orientation convex set* is a set of points whose intersection with lines from some fixed set is empty or connected. This notion generalizes both standard convexity and orthogonal convexity. We explore basic properties of restricted-orientation convex sets in three dimensions. In particular, we establish analogs of the following properties of standard convex sets:

- The intersection of a convex set with every line is empty or connected
- The intersection of a collection of convex sets is a convex set
- For every two points of a convex set, the straight segment joining them is contained in the set
- Convex sets are contractable

1 Introduction

The study of convex sets is a branch of geometry that has numerous connections with other areas of mathematics, including analysis, linear algebra, statistics, and combinatorics [4]. Its importance stems from the fact that convex sets arise in many areas of mathematics and are often amenable to rather elementary reasoning. The concept of convexity serves to unify a wide range of mathematical phenomena.

The application of convexity theory to practical problems led to the exploration of nontraditional notions of convexity, such as orthogonal convexity [6], finitely oriented convexity [5], and link convexity [1, 9]. These nontraditional convexities are used in pixel graphics, VLSI design, motion planning, and other areas.

Rawlins introduced the notion of *restricted-orientation convexity* as a generalization of standard convexity and orthogonal convexity [7]. Rawlins, Wood, and Schuierer studied restricted-orientation convex sets and demonstrated that their properties are similar to the properties of standard convex sets [8, 9].

The research on nontraditional convexities has so far been restricted to two dimensions. Our goal is to study nontraditional convexities in three and higher dimensions. We have generalized two different types of restricted-orientation convexity, called *strong convexity* and *$\mathcal{O}$-convexity* [7], to higher-dimensional spaces and presented an extensive study of restricted-orientation convex sets in higher dimensions [2, 3].

In this paper, we describe $\mathcal{O}$-convex sets in three dimensions and give some of their basic properties. In particular, we establish analogs of the following properties of standard convex sets:

- **Line intersection** The intersection of a convex set with every line is empty or connected
- **Intersection** The intersection of a collection of convex sets is a convex set
- **Visibility** For every two points of a convex set, the straight segment joining them is in the set
- **Contractability** Convex sets are contractable

We restrict our attention to the exploration of closed sets. We conjecture that most of the results presented in the paper hold for nonclosed sets as well; however, some of our proofs work only for closed sets.

We should give a word of warning here. The results are not completely unexpected, yet their proofs are sometimes surprisingly complex; intuition serves us badly. We give only a few proof sketches and present complete proofs in the full paper [3].

2 $\mathcal{O}$-convexity in two and three dimensions

We begin by reviewing the notion of $\mathcal{O}$-convexity in two dimensions [7] and presenting basic properties of planar $\mathcal{O}$-convex sets.

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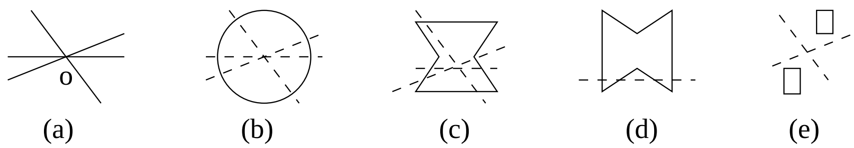


Figure 1: Planar $\mathcal{O}$-convexity.

Figure 2: Orientation sets.

We can describe standard convex sets in terms of their intersection with straight lines: a set of points is convex if its intersection with every line is either empty or connected. We define $\mathcal{O}$-convexity by considering the intersection of a set of points with lines from a *certain set* (rather than all lines). In other words, we select some collection of lines and say that a set is $\mathcal{O}$-convex if its intersection with every line from this collection is empty or connected.

To define this restricted collection of lines, we first introduce the notion of an orientation set. An *orientation set* $\mathcal{O}$ is a (finite or infinite) closed set of lines through some fixed point o. An example of a finite orientation set is shown in Figure 1(a).

A straight line is called an *$\mathcal{O}$-line* if it is parallel to one of the lines of $\mathcal{O}$. Note that every translation of an $\mathcal{O}$-line is also an $\mathcal{O}$-line. We use the collection of all $\mathcal{O}$-lines in defining $\mathcal{O}$-convexity.

Definition 1 ($\mathcal{O}$-convexity) *A set is $\mathcal{O}$-*convex *if its intersection with every $\mathcal{O}$-line is empty or connected.*

For the orientation set in Figure 1(a), the sets shown in Figures 1(b) and 1(c) are $\mathcal{O}$-convex (some $\mathcal{O}$-lines intersecting these sets are shown by dashed lines). On the other hand, the set in Figure 1(d) is not $\mathcal{O}$-convex, since its intersection with the dashed $\mathcal{O}$-line is disconnected. This example demonstrates that rotations do not preserve $\mathcal{O}$-convexity, since the set in Figure 1(d) is a rotation of that in Figure 1(c). Unlike standard convex sets, $\mathcal{O}$-convex sets may be disconnected. We show a disconnected $\mathcal{O}$-convex set in Figures 1(e).

The following properties of $\mathcal{O}$-convex sets readily follow from the definition [8].

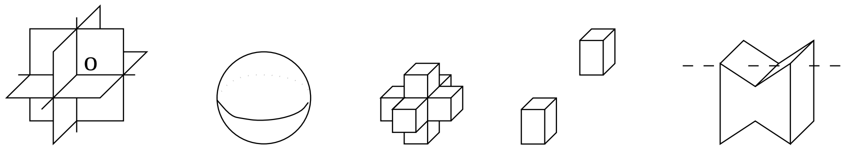
Lemma 1

1. *Every translation of an $\mathcal{O}$-convex set is $\mathcal{O}$-convex.*
2. *Every standard convex set is $\mathcal{O}$-convex.*
3. *If C is a collection of $\mathcal{O}$-convex sets, then the intersection $\bigcap C$ of these sets is also $\mathcal{O}$-convex.*
4. *A disconnected set is $\mathcal{O}$-convex if and only if every connected component of the set is $\mathcal{O}$-convex and no $\mathcal{O}$-line intersects two components.*

We now extend the notion of $\mathcal{O}$-convexity to three dimensions. An *orientation set $\mathcal{O}$* in three dimensions is a (finite or infinite) closed set of planes through a fixed point o. A plane parallel to one of the elements of $\mathcal{O}$ is called an *$\mathcal{O}$-plane*. In Figure 2, we show two finite orientation sets. The first set contains three mutually orthogonal planes; we call it an *orthogonal-orientation set*. The second set consists of four planes.

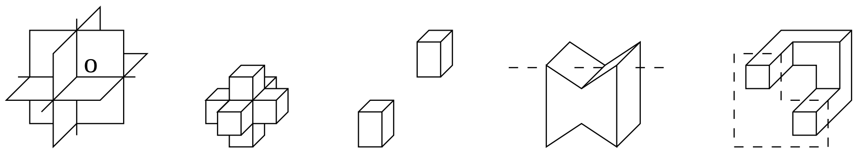
$\mathcal{O}$-lines in three dimensions are formed by the intersections of $\mathcal{O}$-planes. In other words, a straight line is an $\mathcal{O}$-line if it is the intersection of two $\mathcal{O}$-planes. Note that every translation of an $\mathcal{O}$-plane is an $\mathcal{O}$-plane and, hence, every translation of an $\mathcal{O}$-line is an $\mathcal{O}$-line.

We define $\mathcal{O}$-convexity in three dimensions in the same way as in two dimensions: *a set is $\mathcal{O}$-convex if its intersection with every $\mathcal{O}$-line is empty or connected.* For example, the sets in Figures 3(b)–(d) are



(a) (b) (c) (d) (e)

Figure 3: $\mathcal{O}$-convexity in three dimensions.



(a) (b) (c) (d) (e)

Figure 4: $\mathcal{O}$-connectedness in three dimensions.

$\mathcal{O}$-convex for the orthogonal-orientation set shown in Figure 3(a). On the other hand, the set in Figure 3(e) is not $\mathcal{O}$-convex, because its intersection with the dashed $\mathcal{O}$-line is disconnected.

The properties of $\mathcal{O}$-convex sets given in Lemma 1 hold in three dimensions as well. We next characterize three-dimensional $\mathcal{O}$-convex sets in terms of their intersection with $\mathcal{O}$-planes.

Theorem 2 *A set is* $\mathcal{O}$-convex *if and only if its intersection with every $\mathcal{O}$-plane is $\mathcal{O}$-convex.*

Sketch of a proof. Since all planes are $\mathcal{O}$-convex, the intersection of an $\mathcal{O}$-convex set with every $\mathcal{O}$-plane is $\mathcal{O}$-convex. Suppose, conversely, that the intersection of a set P with every $\mathcal{O}$-plane is $\mathcal{O}$-convex. To demonstrate that the intersection of P with every $\mathcal{O}$-line l is empty or connected, we choose some $\mathcal{O}$-plane H that contains l. Since $P \cap H$ is $\mathcal{O}$-convex, $P \cap H \cap l$ is empty or connected. We next note that $P \cap H \cap l = P \cap l$ and, hence, $P \cap l$ is empty or connected. □

3 $\mathcal{O}$-connectedness

We have seen that $\mathcal{O}$-convex sets may be disconnected (see Figure 3d), whereas all standard convex sets are connected. We now describe a subclass of $\mathcal{O}$-convex sets that has the connectedness property: all sets of this subclass are connected, just like standard convex sets.

We define sets of this subclass in terms of path-connectedness of their intersection with $\mathcal{O}$-planes. A set is *path-connected* if every two points of the set can be connected by a curve that is wholly contained in the set. (This property is stronger than usual connectedness.)

Definition 2 ($\mathcal{O}$-connectedness) *A set is* $\mathcal{O}$-connected *if it is path-connected and $\mathcal{O}$-convex, and its intersection with every $\mathcal{O}$-plane is empty or path-connected.*

For example, the set in Figure 4(b) is $\mathcal{O}$-connected for the orthogonal-orientation set shown in Figure 4(a). On the other hand, the set in Figure 4(c) is not $\mathcal{O}$-connected because it is disconnected, the set in Figure 4(d) is not $\mathcal{O}$-connected because it is not $\mathcal{O}$-convex, and the set in Figure 4(e) is not $\mathcal{O}$-connected because its intersection with the dashed $\mathcal{O}$-plane is disconnected.

Lemma 3

1. *Every translation of an $\mathcal{O}$-connected set is $\mathcal{O}$-connected.*
2. *Every standard convex set is $\mathcal{O}$-connected.*

We can characterize $\mathcal{O}$-connected sets in terms of their intersection with $\mathcal{O}$-planes, much in the same way as we characterized $\mathcal{O}$-convex sets (see Theorem 2).

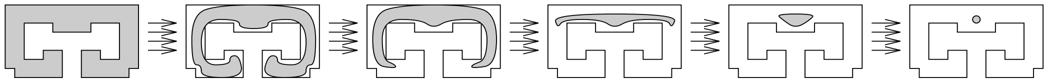
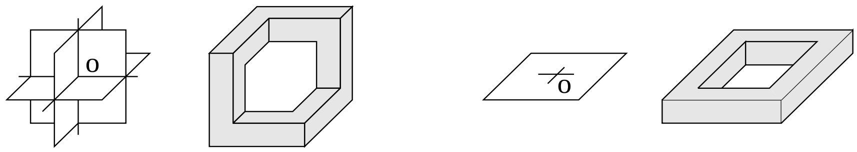


Figure 5: The contraction of a set to a point.



(a) (b)

Figure 6: Sets that are not contractable.

Theorem 4 *A set is $\mathcal{O}$-connected if and only if it is path-connected and its intersection with every $\mathcal{O}$-plane is $\mathcal{O}$-connected.*

Sketch of a proof. Suppose that P is a path-connected set whose intersection with every $\mathcal{O}$-plane is $\mathcal{O}$-connected. Then, by Theorem 2, P is $\mathcal{O}$-convex. Since the intersection of P with every $\mathcal{O}$-plane is path-connected, P is $\mathcal{O}$-connected.

Suppose, conversely, that P is $\mathcal{O}$-connected. We show that the intersection of P with an arbitrary $\mathcal{O}$-plane H is $\mathcal{O}$-connected. We note that $P \cap H$ is path-connected (by the definition of $\mathcal{O}$-connectedness) and $\mathcal{O}$-convex (by Theorem 2). To show that the intersection of $P \cap H$ with every $\mathcal{O}$-plane $H' \neq H$ is empty or path-connected, we observe that $H \cap H'$ is empty or an $\mathcal{O}$-line. Therefore, $P \cap H \cap H'$ is empty or the intersection of P with an $\mathcal{O}$-line; in the latter case, $P \cap H \cap H'$ is empty or path-connected. □

The intersection of $\mathcal{O}$-connected sets may not be $\mathcal{O}$-connected. For example, the intersection of the set in Figure 4(b) with some straight lines is disconnected, even though this set and all lines are $\mathcal{O}$-connected. Because of this "drawback," we do not consider $\mathcal{O}$-connectedness a "true" generalization of convexity.

We next establish *contractability* of $\mathcal{O}$-connected sets. Intuitively, a set is contractable if it is connected and does not have holes. For example, lines, planes, and balls are contractable. A hollow sphere is *not* contractable, because it has a cavity inside. A doughnut (torus) is *not* contractable either, because it has a hole through it. To put it more formally, a set is contractable if it can be continuously transformed (contracted) to a point in such a way that all intermediate stages of the transformation are in the original set (see Figure 5).

All convex sets are contractable. Connected $\mathcal{O}$-convex sets in *two dimensions* are also contractable, if the orientation set $\mathcal{O}$ contains at least one line [8, 3]. This property of $\mathcal{O}$-convex sets does not hold in three dimensions. In Figure 6(a), we provide an example of a connected $\mathcal{O}$-convex set that is not contractable.

If the orientation set $\mathcal{O}$ is empty or contains only one plain, then even $\mathcal{O}$-connected sets may not be contractable. For example, consider the set shown in Figure 6(b). If a horizontal plane is the only element of $\mathcal{O}$, then this set is $\mathcal{O}$-connected; however, it is not contractable. If the orientation set $\mathcal{O}$ contains more than one plane, then $\mathcal{O}$-connected sets are contractable.

Theorem 5 *If $\mathcal{O}$ comprises at least two planes, then every $\mathcal{O}$-connected set is contractable.*

Sketch of a proof. The intersection of two elements of $\mathcal{O}$ is an $\mathcal{O}$-line and every translation of this line is also an $\mathcal{O}$-line. If P is not contractable, then either the intersection of P with one of these $\mathcal{O}$-lines is disconnected (Figure 7a) or one of the $\mathcal{O}$-lines, say l, is through a hole in P (Figure 7b). In the latter case, the intersection of P with some $\mathcal{O}$-plane H containing l is disconnected (Figure 7c). In either case, P is not $\mathcal{O}$-connected. □

4 Visibility

We present two notions of generalized visibility and characterize $\mathcal{O}$-convex and $\mathcal{O}$-connected sets in terms of this generalized visibility.

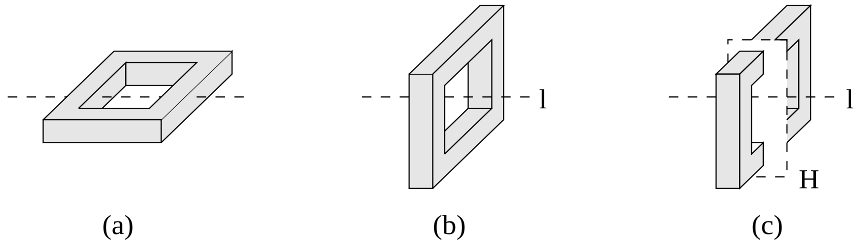


Figure 7: Proof of Theorem 5.

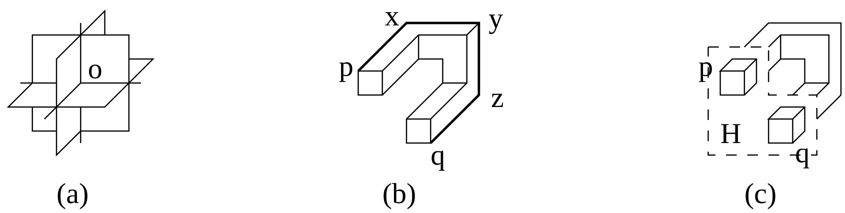


Figure 8: Generalized visibility.

In standard convexity, two points of a set are *visible* to each other if the straight segment joining them is wholly contained in the set. For example, the points p and x of the set in Figure 8(b) are visible to each other, whereas p and q are not. We can characterize convex sets in terms of visibility: a set is convex if and only if every two points of the set are visible to each other. We define a weaker visibility, which enables us to characterize of $\mathcal{O}$-convex sets, by replacing straight segments with simple $\mathcal{O}$-convex curves.

A curve c is *simple* if, for every two points p and q of c, the shortest path from p to q that is wholly contained in c is a segment of c. Informally, this definition says that the shortest way to reach p from q while remaining in c is to follow c. Self-intersecting curves are not simple: if p and q are points on different sides of a loop, the shortest path from p to q does not traverse the loop. Some unusual curves are not simple even though they are not self-intersecting. For example, a Peano curve that covers all points of a square is not simple, even though it may not be self-intersecting.

We say that two points of a set are weakly visible to each other if there is a simple $\mathcal{O}$-convex curve joining them that is wholly in the set. For example, we can join the points p and q in Figure 8(b) by the $\mathcal{O}$-convex polygonal line (p, x, y, z, q), which is contained in the set. We characterize path-connected $\mathcal{O}$-convex sets in terms of this generalized visibility.

Theorem 6 *A path-connected set is $\mathcal{O}$-convex if and only if every two points of the set can be joined by a simple $\mathcal{O}$-convex curve that is wholly in the set.*

Sketch of a proof. Suppose that every two points of a set P can be joined by a simple $\mathcal{O}$-convex curve. If the line through two points is an $\mathcal{O}$-line, then the only simple $\mathcal{O}$-convex curve joining them is the straight segment. Therefore, the intersection of every $\mathcal{O}$-line with P is empty or connected and, thus, P is $\mathcal{O}$-convex.

The proof of the converse is trickier. To demonstrate that every two points p and q of a path-connected $\mathcal{O}$-convex set P can be joined by an $\mathcal{O}$-convex curve, we consider a shortest curve joining p and q in P. (Since P is assumed closed, there exists a shortest curve.) Then, we can show that this curve is $\mathcal{O}$-convex. □

We can characterize $\mathcal{O}$-connected sets in a similar way, if we define visibility in terms of simple $\mathcal{O}$-connected curves joining points of a set. This visibility is stronger than $\mathcal{O}$-convex visibility: two points sometimes cannot be joined by an $\mathcal{O}$-connected curve even when they can be joined by an $\mathcal{O}$-convex curve. For example, the points p and q in Figure 8(b) cannot be joined by an $\mathcal{O}$-connected curve contained in the set, because the intersection of the $\mathcal{O}$-plane H (Figure 8c) with every curve joining them is disconnected.

Theorem 7 *A set is $\mathcal{O}$-connected if and only if every two points of the set can be joined by a simple $\mathcal{O}$-connected curve that is wholly in the set.*

Sketch of a proof. Suppose that every two points of a set P can be joined by an $\mathcal{O}$-connected curve. If two points of P are in some $\mathcal{O}$-plane, then an $\mathcal{O}$-connected curve that joins these points is contained in this $\mathcal{O}$-plane; therefore, the intersection of P with every $\mathcal{O}$-plane is path-connected. We next note that P is $\mathcal{O}$-convex (by Theorem 6) and path-connected. Therefore, P is $\mathcal{O}$-connected.

The proof that every two points p and q of an $\mathcal{O}$-connected set P can be joined by a simple $\mathcal{O}$-connected curve is quite complex. We can demonstrate it in the following three steps. First, we show that there exists an $\mathcal{O}$-plane H and points $p_1, q_1 \in P \cap H$ such that p can be joined with p_1 by an $\mathcal{O}$-connected curve contained in P and q can be joined with q_1. Second, we show that p_1 and q_1 can be joined with each other. Finally, we show that the concatenation of these three curves is an $\mathcal{O}$-connected curve, which joins p and q. □

5 Conclusions

We described a generalization of convexity, called $\mathcal{O}$-convexity, and demonstrated that the properties of $\mathcal{O}$-convex sets are similar to that of standard convex sets. The main property of convex sets that we lose in $\mathcal{O}$-convexity is connectedness: every convex set is connected, whereas an $\mathcal{O}$-convex set may be disconnected. To bridge this difference, we introduced $\mathcal{O}$-connected sets, which are always connected. The following list summarizes the properties of $\mathcal{O}$-convex and $\mathcal{O}$-connected sets generalized from standard convexity:

- **Line intersection** The intersection of an $\mathcal{O}$-convex set with every $\mathcal{O}$-line is empty or connected
- **Intersection** The intersection of a collection of $\mathcal{O}$-convex sets is an $\mathcal{O}$-convex set
- **Visibility** For every two points of an $\mathcal{O}$-convex set, there exists an $\mathcal{O}$-convex curve, contained in the set, that joins these two points. Similarly, every two points of an $\mathcal{O}$-connected set can be joined by an $\mathcal{O}$-connected curve within the set
- **Contractability** If $\mathcal{O}$ comprises at least two planes, then $\mathcal{O}$-connected sets are contractable

The results of this paper, except for contractability, hold not only in three dimensions but also in higher dimensions [3]. The generalization of the contractability result to higher dimensions is still an open problem.

The work presented here leaves some unanswered questions. For example, we have not characterized the boundaries of $\mathcal{O}$-convex polytopes. In two dimensions, if the orientation set contains n lines, the boundary of every $\mathcal{O}$-convex polytope can be partitioned into at most n $\mathcal{O}$-convex polygonal lines [7]. We conjecture that, for every orientation set $\mathcal{O}$ in three dimensions, there is some fixed number n such that the boundary of every $\mathcal{O}$-convex polytope can be partitioned into at most n connected $\mathcal{O}$-convex regions.

We also plan to address computational aspects of $\mathcal{O}$-convexity, such as verifying $\mathcal{O}$-convexity and $\mathcal{O}$-connectedness of a polytope and computing the $\mathcal{O}$-convex hull.

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