Generalizing Halfspaces[1]

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Abstract *Restricted-orientation convexity* is the study of geometric objects whose intersection with lines from some fixed set is empty or connected. We have studied the properties of restricted-orientation convex sets and demonstrated that this notion is a generalization of standard convexity. We now describe a restricted-orientation generalization of halfspaces and explore properties of these generalized halfspaces. In particular, we establish analogs of the following properties of standard halfspaces:

- The intersection of a halfspace with every line is empty, a ray, or a line
- Every halfspace is convex
- A closed set with nonempty interior and convex boundary is a halfspace
- The closure of the complement of a halfspace is a halfspace

1 Introduction

The study of convex sets is a branch of geometry that has applications in optimization, statistics, geometric number theory, and combinatorics [8], as well as in more practical areas, such as VLSI design, computer graphics, architectural databases, geographic databases, and motion planning. Researchers have explored a number of nonstandard notions of convexity, driven by application areas. Some examples are: orthogonal convexity [6, 7], NESW convexity [5, 13], finitely oriented convexity [4, 10], and link convexity [1, 12].

Rawlins introduced the notion of *restricted-orientation convexity,* also called $\mathcal{O}$*-convexity*, in his doctoral thesis, as a generalization of standard convexity [9]. Rawlins, Wood, and Schuierer studied planar $\mathcal{O}$-convex sets and demonstrated that their properties are similar to the properties of standard convex sets [11, 12].

Research on nontraditional convexities has so far been limited to two dimensions. The purpose of our work is to study nontraditional convexities in three dimensions. We demonstrated that restricted-orientation convexity can be extended to three dimensions and described major properties of this extension [3].

We now present a restricted-orientation generalization of halfspaces, explore properties of these generalized halfspaces, and describe their relation to $\mathcal{O}$-convex sets. In particular, we establish analogs of the following properties of standard halfspaces:

- **Line intersection** The intersection of a halfspace with every line is empty, a ray, or a line
- **Convexity** Every halfspace is convex
- **Boundary convexity** A closed set with nonempty interior and convex boundary is a halfspace
- **Complementation** The closure of the complement of a halfspace is a halfspace

We restrict our attention to the exploration of closed sets. We conjecture that most of the results also hold for nonclosed sets; however, some of our proofs work only for closed sets.

We should give a word of warning. The results are not completely unexpected, yet their proofs are often surprisingly hard [3]; intuition serves us badly. We present the proofs in the full paper [2].

2 Planar $\mathcal{O}$-convexity and $\mathcal{O}$-halfplanes

We begin by reviewing the notion of $\mathcal{O}$-convexity in two dimensions [9] and defining an $\mathcal{O}$-convexity analog of halfplanes.

We can describe standard convex sets through their intersection with straight lines: a set of points is convex if its intersection with every line is empty or connected. We define $\mathcal{O}$-convexity by considering the

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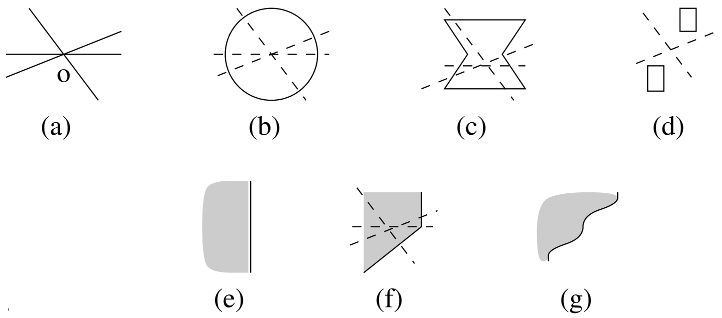

Figure 1: Planar $\mathcal{O}$-convex sets (b–d) and $\mathcal{O}$-halfplanes (e–g).

intersection of a set of points with lines from a *certain set* (rather than all lines). In other words, we pick some collection of lines and say that a set is $\mathcal{O}$-convex if its intersection with every line from this collection is empty or connected.

To define this restricted collection of lines, we introduce the notion of an orientation set. We define an **orientation set** $\mathcal{O}$ as a (finite or infinite) set of lines through a fixed point o. An example of a finite orientation set is shown in Figure 1(a). A line parallel to one of the lines of $\mathcal{O}$ is called an **$\mathcal{O}$-line**. For example, the dashed lines in Figure 1 are $\mathcal{O}$-lines. We define $\mathcal{O}$-convex sets in terms of their intersection with $\mathcal{O}$-lines.

Definition 1 ($\mathcal{O}$-Convex sets) *A closed set is $\mathcal{O}$-**convex** if its intersection with every $\mathcal{O}$-line is empty or connected.*

For the orientation set in Figure 1(a), the sets in Figures 1(b)–(d) are $\mathcal{O}$-convex (some $\mathcal{O}$-lines intersecting these sets are shown by dashed lines). Note that, unlike standard convex sets, $\mathcal{O}$-convex sets may be disconnected (see Figure 1d).

We next observe that halfplanes can also be characterized in terms of their intersection with lines: a set is a halfplane if and only if its intersection with every line is empty, a ray, or a line. We use this observation to define an $\mathcal{O}$-convexity analog of halfplanes in terms of their intersection with $\mathcal{O}$-lines. We call a closed set an **$\mathcal{O}$-halfplane** if its intersection with every $\mathcal{O}$-line is empty, a ray, or a line. We show examples of $\mathcal{O}$-halfplanes in Figures 1(e)–(g). Note that the *empty set* and the *whole plane* are considered $\mathcal{O}$-halfplanes. This convention simplifies some of the results.

We now give basic properties of $\mathcal{O}$-halfplanes, which readily follow from the definition [2]:

1. Every translation of an $\mathcal{O}$-halfplane is an $\mathcal{O}$-halfplane.
2. Every standard halfplane is an $\mathcal{O}$-halfplane.
3. **(Convexity)** Every $\mathcal{O}$-halfplane is $\mathcal{O}$-convex.
4. If an orientation set $\mathcal{O}$ contains two lines, then an $\mathcal{O}$-halfplane is either connected or consists of two connected components.
5. If $\mathcal{O}$ contains three or more lines, then every $\mathcal{O}$-halfplane is connected.

3 $\mathcal{O}$-halfspaces

We now extend the notion of $\mathcal{O}$-convexity to three dimensions and describe an $\mathcal{O}$-convexity analog of halfspaces, called $\mathcal{O}$-halfspaces. An orientation set $\mathcal{O}$ in three dimensions is a set of lines through some fixed point o (just like in two dimensions). We assume that no plane contains all elements of $\mathcal{O}$; that is, the elements of $\mathcal{O}$ are not "coplanar." This assumption is essential for most of the results [3, 2].

Definition 2 (Orientation set and $\mathcal{O}$-lines) *An **orientation set** $\mathcal{O}$ in three dimensions is a nonempty set of lines through a fixed point o such that no plane contains all these lines. A line parallel to one of the elements of $\mathcal{O}$ is called an $\mathcal{O}$-**line.***

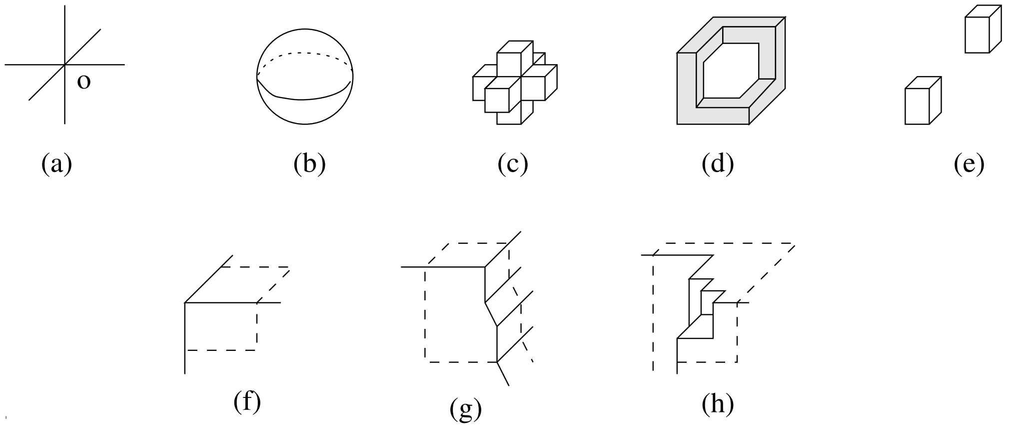

Figure 2: $\mathcal{O}$-convex sets (b–e) and $\mathcal{O}$-halfspaces (f–h) in three dimensions.

Note that every translation of an $\mathcal{O}$-line is an $\mathcal{O}$-line and a particular choice of the point o is not important.

In Figure 2(a), we give an example of a three-dimensional orientation set. This set comprises three mutually orthogonal lines; we call it the **orthogonal-orientation set.**

We define $\mathcal{O}$-convex sets in three dimensions in the same way as in two dimensions: a closed set is $\mathcal{O}$-convex if its intersection with every line is empty or connected. For example, the sets in Figures 2(b)–(e) are $\mathcal{O}$-convex for the orthogonal-orientation set shown in Figure 2(a). The notion of $\mathcal{O}$-halfspaces is the three-dimensional analog of $\mathcal{O}$-halfplanes.

Definition 3 **($\mathcal{O}$-halfspaces)** *A closed set is an $\mathcal{O}$-**halfspace** if its intersection with every $\mathcal{O}$-line is empty, a ray, or a line.*

In Figures 1(f)–(h), we give examples of $\mathcal{O}$-halfspaces for the orthogonal-orientation set. We use dashed lines to show infinite planar regions in the boundaries of these $\mathcal{O}$-halfspaces.

4 Basic properties of $\mathcal{O}$-halfspaces

We next present some basic properties of $\mathcal{O}$-halfspaces and compare them with properties of $\mathcal{O}$-halfplanes.

We begin by looking back at the properties of $\mathcal{O}$-halfplanes listed in the end of Section 2. We readily conclude that Properties 1, 2, and 3 hold in three dimensions: a translation of an $\mathcal{O}$-halfspace is an $\mathcal{O}$-halfspace, every standard halfspace is an $\mathcal{O}$-halfspace, and every $\mathcal{O}$-halfspace is $\mathcal{O}$-convex. The third property is a generalization of the "convexity" property of standard halfspaces (see Section 1). In the next result, we state a necessary and sufficient condition under which an $\mathcal{O}$-convex set is an $\mathcal{O}$-halfspace.

Lemma 1 *A set P is an $\mathcal{O}$-halfspace if and only if*

1. *P is an $\mathcal{O}$-convex set*
2. *and, for every point x in P and every $\mathcal{O}$-line l, one of the two parallel-to-l rays with endpoint x is contained in P.*

According to Property 4 of $\mathcal{O}$-halfplanes, a disconnected $\mathcal{O}$-halfplane consists of two components. We show that an $\mathcal{O}$-halfspace may have up to four connected components and characterize an $\mathcal{O}$-halfspace in terms of its components. In Figure 3, we give an example of a four-component disconnected $\mathcal{O}$-halfspace for the orthogonal-orientation set. The components of this $\mathcal{O}$-halfspace are rectangular polyhedral angles ("quadrants") vertical to the angles of the dotted cube.

Theorem 2 **(Disconnected $\mathcal{O}$-halfspaces)**

1. *A disconnected $\mathcal{O}$-halfspace consists of at most four connected components.*

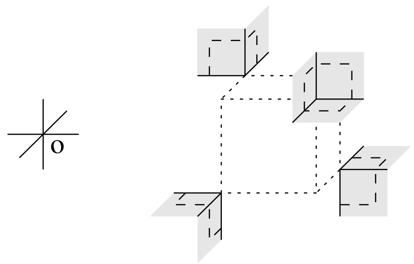

Figure 3: $\mathcal{O}$-halfspace with four connected components.

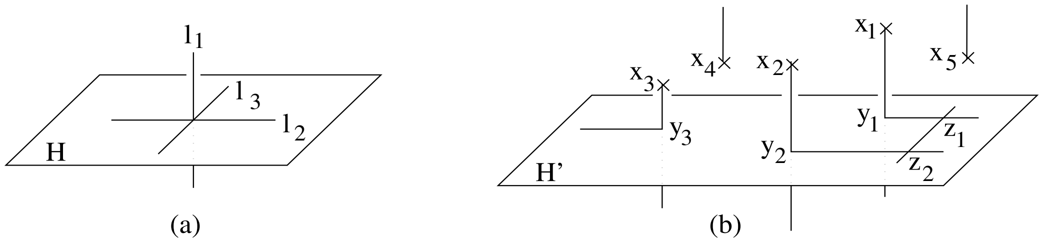

Figure 4: Proof of Theorem 2.

2. *A disconnected set is an $\mathcal{O}$-halfspace if and only if every connected component of the set is an $\mathcal{O}$-halfspace and no $\mathcal{O}$-line intersects two components.*

Sketch of a proof.

(1) We show that, for every five points x_1, x_2, x_3, x_4, x_5 of an $\mathcal{O}$-halfspace P, two of these points are in the same component. We pick three lines $l_1, l_2, l_3 \in \mathcal{O}$ that are not in the same plane (not "coplanar") and denote the plane that contains l_2 and l_3 by H (see Figure 4a). For every point x_k, one of the two parallel-to-l_1 rays with endpoint x_k is in P (see Figure 4b). Thus, we get five parallel rays in P and at least three of them point in the same direction. We assume, for convenience, that these three rays correspond to the points x_1, x_2, and x_3. We select a plane H', parallel to H, that intersects these three rays and denote the intersection points y_1, y_2, and y_3, respectively (Figure 4b).

We now pick parallel-to-l_2 rays, with endpoints y_1, y_2, and y_3, that are contained in P and select two of them that point in the same direction. We assume that these two rays correspond to y_1 and y_2. Finally, we select a parallel-to-l_3 line that intersects these two rays and denote the intersection points z_1 and z_2. Clearly, the segment between z_1 and z_2 is in P. We thus get a polygonal line $(x_1, y_1, z_1, z_2, y_2, x_2)$, contained in P, that connects two of the original five points.

(2) If P is the union of $\mathcal{O}$-halfspaces and no $\mathcal{O}$-line intersects two of them, then, clearly, P is an $\mathcal{O}$-halfspace. If some connected component of P is not an $\mathcal{O}$-halfspace, then the intersection of this component with some $\mathcal{O}$-line l is not empty, a ray, or a line; therefore, $P \cap l$ is not empty, a ray, or a line. Finally, if some $\mathcal{O}$-line l intersects two components, then $P \cap l$ is disconnected. □

5 Boundaries and complements of $\mathcal{O}$-halfspaces

We now present analogs of the "boundary-convexity" and "complementation" properties of standard half-spaces (see Section 1). We first observe that all points in the boundary of an $\mathcal{O}$-halfspace are "infinitely close" to the interior; that is, every $\mathcal{O}$-halfspace is equal to the closure of its interior.

Lemma 3 *Let P be an $\mathcal{O}$-halfspace and P_{int} be the interior of P. Then, Closure$(P_{\text{int}}) = P$.*

We call sets satisfying the property stated in Lemma 3 **interior-closed** sets: a set P is interior-closed if Closure$(P_{\text{int}}) = P$. Our next goal is to present an $\mathcal{O}$-convexity analog of the following "boundary-convexity" characterization of standard halfspaces:

(a) (b)

Figure 5: The boundary of this $\mathcal{O}$-halfplane (a) and $\mathcal{O}$-halfspace (b) is not $\mathcal{O}$-convex.

Lemma 4 *An interior-closed set is a halfspace if and only if its boundary is a nonempty convex set.*

We first generalize the "if" part of this characterization.

Lemma 5 *An interior-closed set with $\mathcal{O}$-convex boundary is an $\mathcal{O}$-halfspace.*

The converse of Lemma 5 does not hold: the boundary of an $\mathcal{O}$-halfplane (in two dimensions) or an $\mathcal{O}$-halfspace (in three dimensions) may not be $\mathcal{O}$-convex. In Figure 5(a), we show an $\mathcal{O}$-halfplane whose boundary is not $\mathcal{O}$-convex: the intersection of its boundary with the dotted $\mathcal{O}$-line is disconnected. Similarly, the boundary of the $\mathcal{O}$-halfspace in Figure 5(b) is not $\mathcal{O}$-convex. We now present a necessary and sufficient characterization of $\mathcal{O}$-halfspaces in terms of their boundary.

Theorem 6 (Boundary characterization) *An interior-closed set P is an $\mathcal{O}$-halfspace if and only if, for every $\mathcal{O}$-line l, one of the following two conditions holds:*

1. *The intersection of l with the boundary of P is empty or connected.*
2. *The intersection of l with the boundary of P consists of two disconnected rays and the segment of l between these rays is in P.*

Sketch of a proof. Suppose that, for every $\mathcal{O}$-line, one of the two conditions holds. Then, the intersection of P's boundary with every $\mathcal{O}$-line is empty, a segment (or point), a ray, a line, or two disconnected rays. An analysis of these five possible cases shows that, if P is interior-closed, then, in all cases, the intersection of the $\mathcal{O}$-line with P is empty, a ray, or a line.

The proof of the converse is trickier. We have to show that, if P is an $\mathcal{O}$-halfspace and the intersection of P's boundary with some $\mathcal{O}$-line l is *not* connected, then this intersection satisfies Condition 2. Since the boundary is closed, we can select two points in the intersection of l with the boundary such that all points of l between them is not in the boundary. Since the intersection of l with P is connected, the segment of l between these two points is in P. We then show that all points of l outside this segment are in P's boundary. $\square$

Observe that, if the intersection of P's boundary with a line l consists of two rays and the segment of l between these rays is in P, then l is in P. We use this observation to simplify Condition 2 in Theorem 6.

Corollary 7 *An interior-closed set P is an $\mathcal{O}$-halfspace if and only if, for every $\mathcal{O}$-line l, one of the following two conditions holds:*

1. *The intersection of l with the boundary of P is empty or connected.*
2. *The $\mathcal{O}$-line l is contained in P.*

We next characterize the closure of the complement of an $\mathcal{O}$-halfspace. We call the closure of the complement of a set the **closed complement.** We observed in Section 1 that the closed complement of a standard halfspace is a halfspace. We lose this property in the generalization to $\mathcal{O}$-convexity: the closed complement of an $\mathcal{O}$-halfplane may not be an $\mathcal{O}$-halfplane and the closed complement of an $\mathcal{O}$-halfspace may not be an $\mathcal{O}$-halfspace. For example, the closed complement of the $\mathcal{O}$-halfplane in Figure 5(a) is not an $\mathcal{O}$-halfplane, because its intersection with the dotted line in not empty, a ray, or a line. Similarly, the closed complement of the $\mathcal{O}$-halfspace in Figure 5(b) is not an $\mathcal{O}$-halfspace. We state a necessary and sufficient condition under which the closed complement of an $\mathcal{O}$-halfspace is an $\mathcal{O}$-halfspace.

Theorem 8 (Complementation) *The closed complement of an $\mathcal{O}$-halfspace P is an $\mathcal{O}$-halfspace if and only if the boundary of P is $\mathcal{O}$-convex.*

Sketch of a proof. We denote the closed complement of P by Q. Note that Q is interior-closed and the boundary of Q is the same as the boundary of P. If the boundary of P is $\mathcal{O}$-convex, then Q is an interior-closed set with an $\mathcal{O}$-convex boundary, which implies that Q is an $\mathcal{O}$-halfspace (Theorem 6).

Suppose, conversely, that Q is an $\mathcal{O}$-halfspace. If the boundary of P is *not* $\mathcal{O}$-convex, then there are points x, y, and z on some $\mathcal{O}$-line l such that x and z are in the boundary, whereas y, located between them, is not in the boundary. Then, y is either in the interior of P or in the interior of Q, which implies that either $P \cap l$ or $Q \cap l$ is disconnected, contradicting the fact that P and Q are $\mathcal{O}$-halfspaces. □

6 Conclusions

We have described a generalization of halfspaces in the theory of restricted-orientation convexity and demonstrated that the properties of these generalized halfspaces are similar to the properties of standard halfspaces.

The work we have presented here extends our previous study of $\mathcal{O}$-convex sets [3]. In the full paper [2], we demonstrate that the results hold not only in three dimensions but also in higher dimensions.

The work leaves some open research problems, which we are currently trying to address. For example, we have not established the contractability of $\mathcal{O}$-halfspaces. We conjecture that every connected $\mathcal{O}$-halfspace is contractable. We also plan to study computational properties of $\mathcal{O}$-convex sets and $\mathcal{O}$-halfspaces.

References

[1] C. K. Bruckner and J. B. Bruckner. On L_n-sets, the Hausdorff metric, and connectedness. *Proceedings of the American Mathematical Society*, 13:765–767, 1962.

[2] Eugene Fink and Derick Wood. Generalized halfspaces in restricted-orientation convexity. Unpublished manuscript, 1995.

[3] Eugene Fink and Derick Wood. Fundamentals of restricted-orientation convexity. *Information Sciences*, 1996. To appear.

[4] Ralf Hartmut Güting. Stabbing C-oriented polygons. *Information Processing Letters*, 16:35–40, 1983.

[5] Witold Lipski and Christos H. Papadimitriou. A fast algorithm for testing for safety and detecting deadlock in locked transaction systems. *Journal of Algorithms*, 2:211–226, 1981.

[6] Tina M. Nicholl, D. T. Lee, Y. Z. Liao, and C. K. Wong. Constructing the X-Y convex hull of a set of X-Y polygons. *BIT*, 23:456–471, 1983.

[7] Thomas Ottmann, Eljas Soisalon-Soininen, and Derick Wood. On the definition and computation of rectilinear convex hulls. *Information Sciences*, 33:157–171, 1984.

[8] Franco P. Preparata and Michael I. Shamos. *Computational Geometry*. Springer-Verlag, New York, NY, 1985.

[9] Gregory J. E. Rawlins. *Explorations in Restricted-Orientation Geometry*. PhD thesis, University of Waterloo, 1987. Technical Report CS-89-48.

[10] Gregory J. E. Rawlins and Derick Wood. Optimal computation of finitely oriented convex hulls. *Information and Computation*, 72:150–166, 1987.

[11] Gregory J. E. Rawlins and Derick Wood. Restricted-orientation convex sets. *Information Sciences*, 54:263–281, 1991.

[12] Sven Schuierer. *On Generalized Visibility*. PhD thesis, Universität Freiburg, Germany, 1991.

[13] Eljas Soisalon-Soininen and Derick Wood. Optimal algorithms to compute the closure of a set of iso-rectangles. *Journal of Algorithms*, 5:199–214, 1984.