# Restricted-Orientation Halfspaces

**Eugene Fink**

School of Computer Science
Carnegie Mellon University
Pittsburgh, PA 15213, USA
eugene@cs.cmu.edu
http://www.cs.cmu.edu/∼eugene

**Derick Wood**

Department of Computer Science
Hong Kong Univ. of Science and Technology
Clear Water Bay, Kowloon, Hong Kong
dwood@cs.ust.hk
http://www.cs.ust.hk/∼dwood

## ABSTRACT

Restricted-orientation convexity, also called $\mathcal{O}$-convexity, is the study of geometric objects whose intersection with lines from some fixed set is empty or connected. We introduce restricted-orientation halfspaces, which are an $\mathcal{O}$-convexity analog of halfspaces, explore their properties, and demonstrate their relationship to restricted-orientation convex sets.

**Keywords:** computational geometry, mathematical foundations, generalized convexity, restricted orientations, three dimensions.

## 1 INTRODUCTION

The notion of a convex set is one of the fundamental notions in geometry.[6,8] The study of convexity has a long and interesting history; some forms of proto-convexity date back to the works of Archimedes.[1]

In recent years, researchers have studied computational properties of convex sets. Convexity theory has found its use in many practical areas of computer science, such as motion planning and computer graphics.[12] The application of convexity to practical problems led to the exploration of nontraditional notions of convexity, which are often more appropriate for specific applications and enable researchers to design more efficient algorithms.

**Restricted-orientation convexity** is one of these nontraditional convexities, which was introduced by Rawlins in his doctoral dissertation.[13] The notion of restricted-orientation convexity, which is also called **$\mathcal{O}$-convexity**, is a generalization of standard convexity. It also generalizes some earlier notions of nontraditional convexity, such as orthogonal convexity[9–11] and finitely oriented convexity.[7,17] Rawlins, Wood, and Schuierer presented an extensive study of planar $\mathcal{O}$-convex sets and generalized several major results of standard convexity theory to $\mathcal{O}$-convexity.[14–16]

In our previous work, we generalized $\mathcal{O}$-convexity to three and higher dimensions.[3,5] We presented several major properties of three-dimensional $\mathcal{O}$-convex sets and demonstrated that their properties are much richer than that of planar $\mathcal{O}$-convex sets.

We now further develop the theory of three-dimensional restricted-orientation convexity. We present a restricted-orientation analog of halfspaces, called **$\mathcal{O}$-halfspaces**.

We define $\mathcal{O}$-halfspaces in terms of their intersection with lines from some fixed set, by analogy with Rawlins' definition of $\mathcal{O}$-convex sets (Section 3). We then explore basic properties of $\mathcal{O}$-halfspaces and compare them with standard halfspaces. In particular, we demonstrate that $\mathcal{O}$-halfspaces are connected (Section 4) and that the closure of the complement of an $\mathcal{O}$-halfspace is an $\mathcal{O}$-halfspace (Section 5). We also study properties of the boundaries of $\mathcal{O}$-halfspaces (Section 6). We conclude with discussion of open problems and notes on extending our results to higher dimensions (Section 7).

We restrict our attention to the exploration of closed sets. We conjecture that most results hold for nonclosed sets as well; however, some of our proofs work only for closed sets.

> **Closed-Set Assumption:** We consider only closed geometric objects. An object is **closed** if, for every convergent sequence of its points, the limit of the sequence belongs to the object.

In another conference article,[4] we explore an alternative analog of halfspaces in restricted-orientation convexity, whose properties differ from the properties of the halfspaces described here. In the full paper,[2] we extend both notions of restricted-orientation halfspaces to multidimensional space and present complete proofs of all our results.

## 2 PLANAR $\mathcal{O}$-CONVEXITY AND $\mathcal{O}$-HALFPLANES

We first review the notion of restricted-orientation convexity, also called $\mathcal{O}$-convexity, in two dimensions[13] and define an $\mathcal{O}$-convexity analog of halfplanes.

We can describe standard convex sets through their intersection with straight lines: a set of points is convex if its intersection with every line is empty or connected. $\mathcal{O}$-convex sets are defined through their intersection with lines from a *certain set* (rather than all lines). The definition of this restricted collection of lines is based on the notion of an orientation set.

An **orientation set** $\mathcal{O}$ in two dimensions is a (finite or infinite) set of lines through some fixed point $o$. An example of a finite orientation set is shown in Figure 1(a). A line parallel to one of the lines of $\mathcal{O}$ is called an **$\mathcal{O}$-line.** For example, the dotted lines in Figure 1 are $\mathcal{O}$-lines.

A closed set is **$\mathcal{O}$-convex** if its intersection with every $\mathcal{O}$-line is empty or connected. For example, the sets in Figures 1(b) and 1(c) are $\mathcal{O}$-convex for the orientation set given in Figure 1(a). We have demonstrated that many properties of $\mathcal{O}$-convex sets are similar to that of standard convex sets.[15,3]

We next observe that standard halfplanes can also be characterized in terms of their intersection with

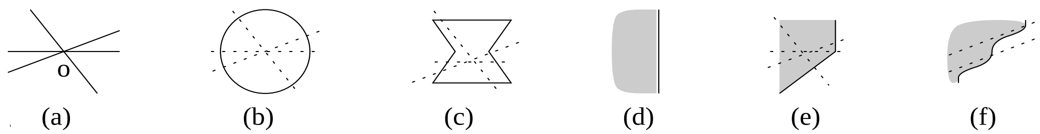


Figure 1: Planar $\mathcal{O}$-convex sets (b,c) and $\mathcal{O}$-halfplanes (d–f).

straight lines: a set is a halfplane if and only if its intersection with every line is empty, a ray, or a line. If the intersection of two parallel lines with a halfplane forms rays, then these rays point in the same direction (rather than in opposite directions).

We use this observation to define an $\mathcal{O}$-convexity analog of halfplanes. A closed set $P$ is an **$\mathcal{O}$-halfplane** if the following two conditions hold:

(1) The intersection of $P$ with every $\mathcal{O}$-line is empty, a ray, or a line.

(2) For every two parallel $\mathcal{O}$-lines whose intersection with $P$ forms rays, these rays point in the same direction.

We show examples of $\mathcal{O}$-halfplanes in Figures 1(d)–(f). Note that the *empty set* and the *whole plane* are considered to be $\mathcal{O}$-halfplanes.

We now give basic properties of $\mathcal{O}$-halfplanes.[2] The last two properties exhibit a relationship between $\mathcal{O}$-halfplanes and $\mathcal{O}$-convex sets.

1. Every translation of an $\mathcal{O}$-halfplane is an $\mathcal{O}$-halfplane.
2. Every standard halfplane is an $\mathcal{O}$-halfplane.
3. If the orientation set $\mathcal{O}$ comprises at least two lines, every $\mathcal{O}$-halfplane is connected and $\mathcal{O}$-convex.
4. Every connected $\mathcal{O}$-convex set is the intersection of the $\mathcal{O}$-halfplanes that contain it.

## 3 GENERALIZATION TO THREE DIMENSIONS

We now extend the notion of restricted-orientation convexity to three dimensions and describe a restricted-orientation analog of halfspaces, called **$\mathcal{O}$-halfspaces.** Note that, in our other papers on restricted-orientation convexity,[2,4] we use the term "directed $\mathcal{O}$-halfspaces" to refer to the analog of halfspaces described here and reserve the term "$\mathcal{O}$-halfspaces" for a weaker analog. We present the properties of the weaker analog in another conference article.[4]

We introduce a set $\mathcal{O}$ of planes in three dimensions, show how this set gives rise to $\mathcal{O}$-lines, and define $\mathcal{O}$-halfspaces in terms of their intersection with $\mathcal{O}$-lines.

Definition 3.1. **(Orientation set and $\mathcal{O}$-planes)** *An* **orientation set** $\mathcal{O}$ *in three dimensions is a set of planes through a fixed point $o$. A plane parallel to one of the elements of $\mathcal{O}$ is called an* **$\mathcal{O}$-plane.**

Note that every translation of an $\mathcal{O}$-plane is an $\mathcal{O}$-plane and a particular choice of the point $o$ is not

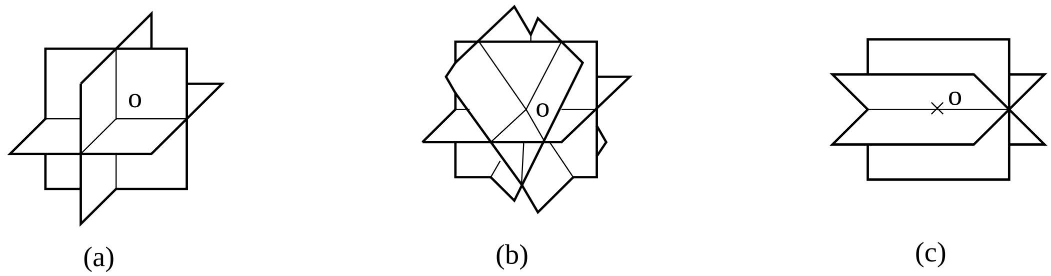


Figure 2: Examples of finite orientation sets.

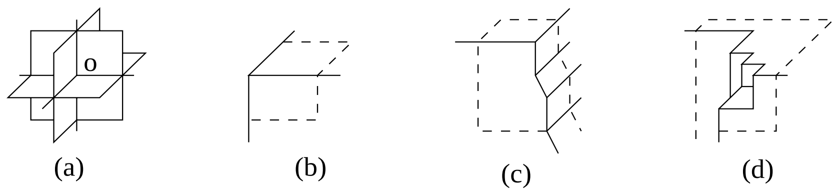


Figure 3: $\mathcal{O}$-halfspaces in three dimensions.

important. In Figure 2, we give examples of finite orientation sets in three dimensions. The set in Figure 2(a) comprises three mutually orthogonal planes; we call it the **orthogonal-orientation set.**

**$\mathcal{O}$-lines** in three dimensions are formed by the intersections of $\mathcal{O}$-planes. In other words, a line is an $\mathcal{O}$-line if it is the intersection of two $\mathcal{O}$-planes. Note that every translation of an $\mathcal{O}$-line is an $\mathcal{O}$-line. Since every $\mathcal{O}$-plane is parallel to one of the planes of the orientation set $\mathcal{O}$, every $\mathcal{O}$-line is parallel to some line formed by the intersection of two elements of $\mathcal{O}$. For example, the intersections of the four planes of the orientation set in Figure 2(b) form six different lines through $o$ and every $\mathcal{O}$-line is parallel to one of these six lines.

Definition 3.2. **($\mathcal{O}$-halfspaces)** *A closed set is an* **$\mathcal{O}$-halfspace** *if the following conditions hold:*

*(1) Its intersection with every $\mathcal{O}$-line is empty, a ray, or a line.*

*(2) For every two parallel $\mathcal{O}$-lines whose intersection with the set is rays, these rays point in the same direction (rather than in opposite directions).*

Figures 3(b)–(d) provide examples of $\mathcal{O}$-halfspaces for the orthogonal-orientation set given in Figure 3(a). We use dashed lines to show infinite planar regions in the boundaries of these $\mathcal{O}$-halfspaces.

## 4 BASIC PROPERTIES OF $\mathcal{O}$-HALFSPACES

We give some simple properties of $\mathcal{O}$-halfspaces and compare them with properties of $\mathcal{O}$-halfplanes.

We readily conclude that Properties 1 and 2 given at the end of Section 2 hold for $\mathcal{O}$-halfspaces: every translation of an $\mathcal{O}$-halfspace is an $\mathcal{O}$-halfspace and every standard halfspace is an $\mathcal{O}$-halfspace. We next present an analog of Property 3.

Theorem 4.1. *If the orientation set $\mathcal{O}$ is not empty and the intersection of the elements of $\mathcal{O}$ is the*


Figure 4: Proof of Theorem 4.1.

*point $o$ (rather than a superset of $o$), then every $\mathcal{O}$-halfspace is connected and the intersection of an $\mathcal{O}$-halfspace with every $\mathcal{O}$-line and every $\mathcal{O}$-plane is also connected.*

**Sketch of a proof.** The intersection of an $\mathcal{O}$-halfspace with every $\mathcal{O}$-line is empty, a ray, or a line and, hence, such an intersection is always connected.

We show that the intersection of an $\mathcal{O}$-halfspace $P$ with every $\mathcal{O}$-plane $H$ is connected by demonstrating that every two points $p$ and $q$ of $P \cap H$ can be joined by a polygonal line in $P \cap H$. Since the intersection of the elements of $\mathcal{O}$ is a point, we may choose two $\mathcal{O}$-planes whose intersection with $H$ forms nonparallel $\mathcal{O}$-lines, say $l_1$ and $l_2$. We consider two parallel-to-$l_1$ $\mathcal{O}$-rays, with endpoints $p$ and $q$, that are contained in $P$ and point in the same direction (see Figure 4a). We choose a parallel-to-$l_2$ $\mathcal{O}$-line that intersects these two rays and denote the intersection points $x$ and $y$, respectively. The polygonal line $(p, x, y, q)$ is contained in $P \cap H$.

We use similar reasoning to show that every $\mathcal{O}$-halfspace $P$ is connected, by demonstrating that every two points $p$ and $q$ of $P$ can be joined by a polygonal line in $P$. Since the intersection of the elements of $\mathcal{O}$ is a point, we may choose an $\mathcal{O}$-line $l$ and an $\mathcal{O}$-plane $H$ that intersects $l$ and does not contain it (see Figure 4b). We consider two parallel-to-$l$ rays, with endpoints $p$ and $q$, that are contained in $P$ and point in the same direction. We choose an $\mathcal{O}$-plane $H'$, parallel to $H$, that intersects these two rays and denote the intersection points, respectively, $x$ and $y$. By the first part of the proof, we can join $x$ and $y$ by a polygonal line in $P \cap H'$; therefore, $p$ and $q$ can be joined by a polygonal line in $P$. □

If the intersection of the elements of $\mathcal{O}$ is a superset of $o$, then $\mathcal{O}$-halfspaces may be disconnected. In particular, if $\mathcal{O}$ is empty or contains only one plane, then there is no $\mathcal{O}$-lines and, hence, *every* set of points is an $\mathcal{O}$-halfspace. If $\mathcal{O}$ contains at least two planes and the intersection of these planes is a superset of $o$, then their intersection is a line (see Figure 2c) through $o$, and all $\mathcal{O}$-lines are parallel to this line. In this case, the union of several $\mathcal{O}$-lines is a disconnected $\mathcal{O}$-halfspace.

The property of $\mathcal{O}$-halfspaces stated in Theorem 4.1 is called **$\mathcal{O}$-connectedness**. The notion of $\mathcal{O}$-connected sets is a three-dimensional analog of planar connected $\mathcal{O}$-convex sets.

Definition 4.2. **($\mathcal{O}$-connected sets)** *A closed set is* **$\mathcal{O}$-connected** *if it is connected and its intersection with every $\mathcal{O}$-line and every $\mathcal{O}$-plane is connected.*

$\mathcal{O}$-halfspaces are not the only $\mathcal{O}$-connected sets. For example, the set given in Figure 5(b) is $\mathcal{O}$-connected, even though it is not an $\mathcal{O}$-halfspace. In fact, every standard convex set is $\mathcal{O}$-connected. As another example, the set in Figure 5(c) is $\mathcal{O}$-connected for the orthogonal-orientation set in Figure 5(a). On the other hand, the set in Figure 5(d) is not $\mathcal{O}$-connected because it is disconnected, the set in Figure 5(e) is not $\mathcal{O}$-connected because its intersection with the dotted $\mathcal{O}$-line is disconnected, and the set in Figure 5(f) is not $\mathcal{O}$-connected because its intersection with the dotted $\mathcal{O}$-plane is disconnected. We have studied properties of $\mathcal{O}$-connected sets as a part of our exploration of $\mathcal{O}$-convexity.[3,5]

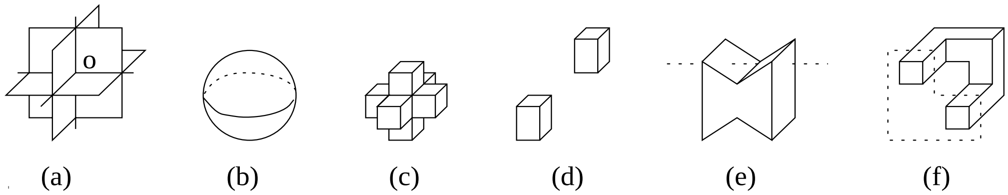


Figure 5: Sets (b,c) are $\mathcal{O}$-connected, whereas sets (d–f) are *not* $\mathcal{O}$-connected.

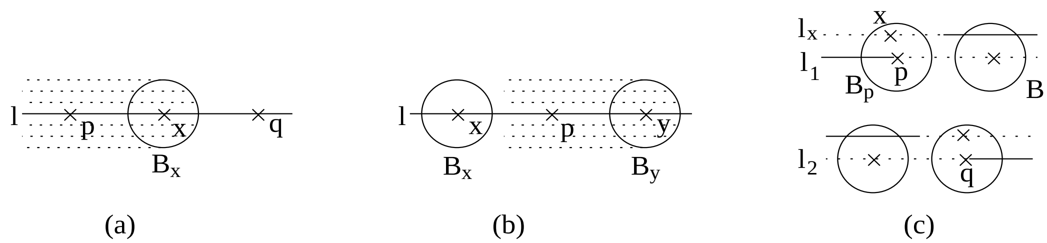


Figure 6: Proof of Theorem 5.1.

# 5 COMPLEMENTATION

We now demonstrate that, for every $\mathcal{O}$-halfspace, the closure of its complement is also an $\mathcal{O}$-halfspace. This result generalizes the observation that the complement of a standard halfspace is a halfspace.

THEOREM 5.1. *The closure of the complement of an $\mathcal{O}$-halfspace is an $\mathcal{O}$-halfspace.*

**Sketch of a proof.** We consider an $\mathcal{O}$-halfspace $P$ and denote the closure of its complement by $Q$. We first show, by contradiction, that the intersection of $Q$ with every $\mathcal{O}$-line $l$ is empty, a ray, or a line. If not, then $Q \cap l$ is either disconnected or a segment.

If $Q \cap l$ is disconnected, we may choose points $p, q \in l$ that are in $Q$, a point $x \in l$ between them that is in the interior of $P$, and a ball $B_x \subseteq P$ centered at $x$ (see Figure 6a). We assume, for convenience, that $p$ is to the left of $x$. Since $P$ is an $\mathcal{O}$-halfspace, either all left-pointed or all right-pointed rays with endpoints in $B_x$ are contained in $P$. If the left-pointed rays (shown by dotted lines in Figure 6a) are in $P$, then $p$ is in $P$'s interior. If the right-pointed rays are in $P$, then $q$ is in $P$'s interior. Thus, $p$ or $q$ is *not* in $Q$, which yields a contradiction.

If $Q \cap l$ is a segment, then we choose a point $p \in l$ in $Q$; points $x, y \in l$ in the interior of $P$, on different sides of $Q$; and balls $B_x, B_y \subseteq P$ centered at $x$ and $y$, respectively (see Figure 6b). Either all left-pointed or all right-pointed rays with endpoints in $B_x \cup B_y$ are contained in $P$. Therefore, $p$ is in the interior of $P$, which means that $p$ is *not* in $Q$, giving a contradiction.

We next show, by contradiction, that the rays formed by the intersection of $Q$ with parallel $\mathcal{O}$-lines always point in the same direction. Suppose that the intersection of $Q$ with two parallel $\mathcal{O}$-lines, $l_1$ and $l_2$, forms rays that point in opposite directions. We denote the endpoints of these rays $p$ and $q$, respectively. In Figure 6(c), we show the rays by solid lines. Note that $p$ and $q$ are in the boundary of $P$ and the dotted parts of the lines are in the interior of $P$.

We choose a point in the dotted part of $l_1$ and a ball $B \subseteq P$ centered at this point. Let $B_p$ be the same-sized ball centered at $p$, and $x \in B_p$ be a point *not* in $P$. We consider the $\mathcal{O}$-line $l_x$ through $x$ parallel to $l_1$. The intersection of $l_x$ with $P$ is empty, a ray, or a line. Since $x$ is not in $P$ and, on

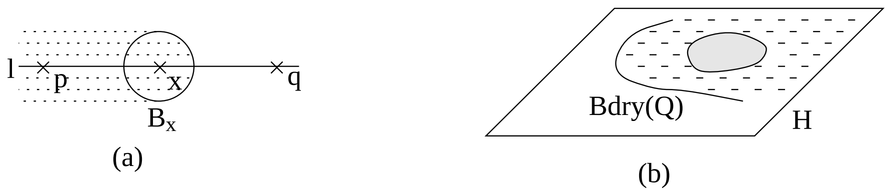


Figure 7: Proof of Theorem 6.1.

the other hand, $l_x$ intersects the ball $B \subseteq P$, we conclude that the intersection of $l_x$ with $P$ is a ray pointing to the right. Using a similar construction with $l_2$, we get an $\mathcal{O}$-line whose intersection with $P$ is a ray pointing to the left (see Figure 6c), contradicting the assumption that $P$ is an $\mathcal{O}$-halfspace. □

# 6 BOUNDARY CONVEXITY

The boundary of a standard halfspace is a plane, which is a convex set. We generalize this boundary-convexity observation to $\mathcal{O}$-halfspaces.

THEOREM 6.1. *If the orientation set $\mathcal{O}$ is not empty and the intersection of the elements of $\mathcal{O}$ is the point $o$ (rather than a superset of $o$), then the boundary of every $\mathcal{O}$-halfspace is $\mathcal{O}$-connected.*

**Sketch of a proof.** We consider an arbitrary $\mathcal{O}$-halfspace $P$ and show, by contradiction, that its boundary is connected and the boundary's intersection with every $\mathcal{O}$-line and every $\mathcal{O}$-plane is also connected. First, suppose that $P$'s boundary is disconnected. Since $P$ is connected (Theorem 4.1), its closed complement in this case is disconnected. On the other hand, the closed complement of $P$ is an $\mathcal{O}$-halfspace (Theorem 5.1), contradicting the connectedness of $\mathcal{O}$-halfspaces.

Next suppose that the intersection of $P$'s boundary with some $\mathcal{O}$-line $l$ is disconnected. We may then select points $p, q \in l$ that are in the boundary and a point $x \in l$ between them that is *not* in the boundary (see Figure 7a). We assume, for convenience, that $p$ is to the left of $x$. Since the intersection of $P$ and $l$ is connected, $x$ is in the interior of $P$ and we can choose a circle $B_x \subseteq P$ centered at $x$. Either all left-pointed or all right-pointed rays with endpoints in $B_x$ are contained in $P$. If the left-pointed rays (shown by dotted lines in Figure 7a) are in $P$, then $p$ is in $P$'s interior. If the right-pointed rays are in $P$, then $q$ is in $P$'s interior. Thus, $p$ or $q$ is not in the boundary, yielding a contradiction.

Finally, we show that the intersection of $P$'s boundary with every $\mathcal{O}$-plane $H$ is connected. Let $Q$ be the intersection of $P$ with $H$, and Bdry($Q$) be the boundary of $Q$ in the plane $H$ (rather than in the whole space). Note that Bdry($Q$) is wholly contained in the boundary of $P$. We next observe that $Q$ is an $\mathcal{O}$-halfplane with respect to the $\mathcal{O}$-lines contained in $H$. Reasoning similar to the first part of the proof shows that Bdry($Q$) is connected.

Suppose that the intersection of $P$'s boundary with $H$ is disconnected. Then, this intersection has a component that is disconnected from the boundary of $Q$ (see Figure 7(b), where this component is shown by the shaded region). The component is contained in $Q$; therefore, it is surrounded in the plane $H$ by interior points of $P$ (the interior points are shown by the dashed region in Figure 7b).

We now consider the intersection of $H$ with the closure of $P$'s complement. This intersection contains Bdry($Q$) and the shaded component, which is disconnected from Bdry($Q$). The intersection

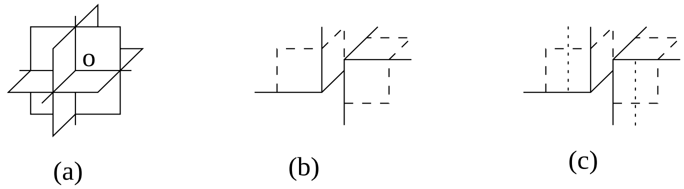


Figure 8: Set (b) is *not* an $\mathcal{O}$-halfspace, even though its boundary is $\mathcal{O}$-connected.

does *not* contain any interior points of $P$. Therefore, the intersection is disconnected. On the other hand, the closure of $P$'s complement is an $\mathcal{O}$-halfspace (Theorem 5.1), which implies that the intersection of the $\mathcal{O}$-plane $H$ with the closure of $P$'s complement is connected (Theorem 4.1), yielding a contradiction. □

The converse of this result does not hold: a set with an $\mathcal{O}$-connected boundary may not be an $\mathcal{O}$-halfspace. We show such a set in Figure 8(b). This set consists of two rectangular polyhedral angles ("quadrants"), which touch each other along one of their faces. The set's boundary is $\mathcal{O}$-connected for the orthogonal-orientation set given in Figure 8(a). The set, however, is *not* an $\mathcal{O}$-halfspace, since the dotted rays (Figure 8c), formed by its intersection with vertical $\mathcal{O}$-lines, point in opposite directions.

# 7 CONCLUSIONS

We have described generalized halfspaces in the theory of restricted-orientation convexity and demonstrated that the properties of these generalized halfspaces are similar to the properties of standard halfspaces. These results extend our previous exploration of $\mathcal{O}$-convex and $\mathcal{O}$-connected sets.[3,5]

In the full paper, we extend the notion of $\mathcal{O}$-halfspaces to higher-dimensional space.[2] For $d$ dimensions, we define $\mathcal{O}$ as a set of $(d-1)$-dimensional planes through a fixed point $o$. The $(d-1)$-dimensional planes that are parallel to elements of $\mathcal{O}$ are called $\mathcal{O}$-planes and the lines formed by the intersections of $\mathcal{O}$-planes are called $\mathcal{O}$-lines. We define higher-dimensional $\mathcal{O}$-halfspaces in the same way as in three dimensions (see Definition 3.2). The notion of $\mathcal{O}$-connected sets in $d$ dimensions is somewhat more complex.[3] A set is **$\mathcal{O}$-connected** if its intersection with every flat (affine variety) formed by the intersection of $\mathcal{O}$-planes is empty or connected. We have demonstrated that all our results hold in higher dimensions.[2]

The work presented here leaves some unanswered questions. For example, we have not generalized Property 4 of $\mathcal{O}$-halfplanes (see the end of Section 2) to three dimensions. We conjecture that *every $\mathcal{O}$-connected set is the intersection of $\mathcal{O}$-halfspaces that contain it.* We further conjecture that, *for every two $\mathcal{O}$-connected sets $P$ and $Q$ that do not intersect, there exists an $\mathcal{O}$-halfspace that contains $P$ and does not intersect $Q$.* As another example of an open problem, we conjecture that, *if $\mathcal{O}_1$ is an infinite orientation set and $\mathcal{O}_2$ is the closure of $\mathcal{O}_1$, then these two orientation sets give rise to the same restricted-orientation halfspaces.* That is, a set is an $\mathcal{O}_1$-halfspace if and only if it is an $\mathcal{O}_2$-halfspace.

# 8 ACKNOWLEDGMENTS

This work was supported under grants from the Natural Sciences and Engineering Research Council of Canada, the Information Technology Research Centre of Ontario, and the Research Grant Committee of Hong Kong.

# 9 REFERENCES

[1] W. Fenchel. Convexity through the ages. In P. M. Gruber and J. M. Wills, editors, *Convexity and Its Applications*. Birkhäuser Verlag, Boston, MA, 1988.

[2] Eugene Fink and Derick Wood. Generalized halfspaces in restricted-orientation convexity. Unpublished manuscript, 1995.

[3] Eugene Fink and Derick Wood. Fundamentals of restricted-orientation convexity. *Information Sciences*, 1996. To appear.

[4] Eugene Fink and Derick Wood. Generalizing halfspaces. In *Proceedings of the Eighth Canadian Conference on Computational Geometry*, 1996.

[5] Eugene Fink and Derick Wood. Three-dimensional restricted-orientation convexity. In *Proceedings of the Eighth Canadian Conference on Computational Geometry*, 1996.

[6] Branko Grünbaum, Victor Klee, M. A. Perles, and G. C. Shephard. *Convex Polytopes*. John Wiley & Sons, New York, NY, 1967.

[7] Ralf Hartmut Güting. Stabbing $C$-oriented polygons. *Information Processing Letters*, 16:35–40, 1983.

[8] Victor Klee. What is a convex set? *American Mathematical Monthly*, 78:616–631, 1971.

[9] D. Y. Montuno and Alain Fournier. Finding the $x$-$y$ convex hull of a set of $x$-$y$ polygons. Technical Report 148, University of Toronto, Toronto, Ontario, 1982.

[10] Tina M. Nicholl, D. T. Lee, Y. Z. Liao, and C. K. Wong. Constructing the X-Y convex hull of a set of X-Y polygons. *BIT*, 23:456–471, 1983.

[11] Thomas Ottmann, Eljas Soisalon-Soininen, and Derick Wood. On the definition and computation of rectilinear convex hulls. *Information Sciences*, 33:157–171, 1984.

[12] Franco P. Preparata and Michael I. Shamos. *Computational Geometry*. Springer-Verlag, New York, NY, 1985.

[13] Gregory J. E. Rawlins. *Explorations in Restricted-Orientation Geometry*. PhD thesis, University of Waterloo, Ontario, 1987. Technical Report CS-89-48.

[14] Gregory J. E. Rawlins and Derick Wood. Ortho-convexity and its generalizations. In Godfried T. Toussaint, editor, *Computational Morphology*, pages 137–152. Elsevier Science Publishers B. V., North-Holland, 1988.

[15] Gregory J. E. Rawlins and Derick Wood. Restricted-orientation convex sets. *Information Sciences*, 54:263–281, 1991.

[16] Sven Schuierer. *On Generalized Visibility.* PhD thesis, Universität Freiburg, Germany, 1991.

[17] Peter Widmayer, Y. F. Wu, and C. K. Wong. Distance problems in computational geometry for fixed orientations. In *Proceedings of the First Annual ACM Symposium on Computational Geometry*, pages 186–195, 1985.