School of Computer Science
Carnegie Mellon

Probabilistic Graphical Models

Bayesian nonparametrics: Dirchlet Process, Indian Buffet Process

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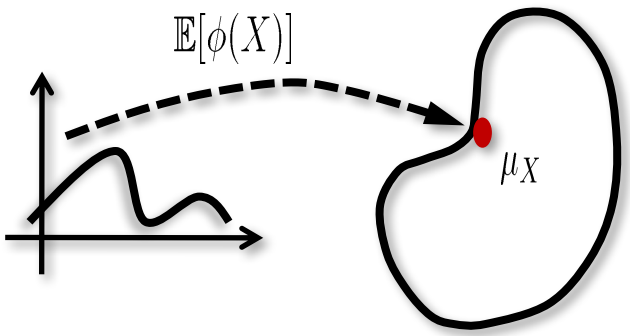
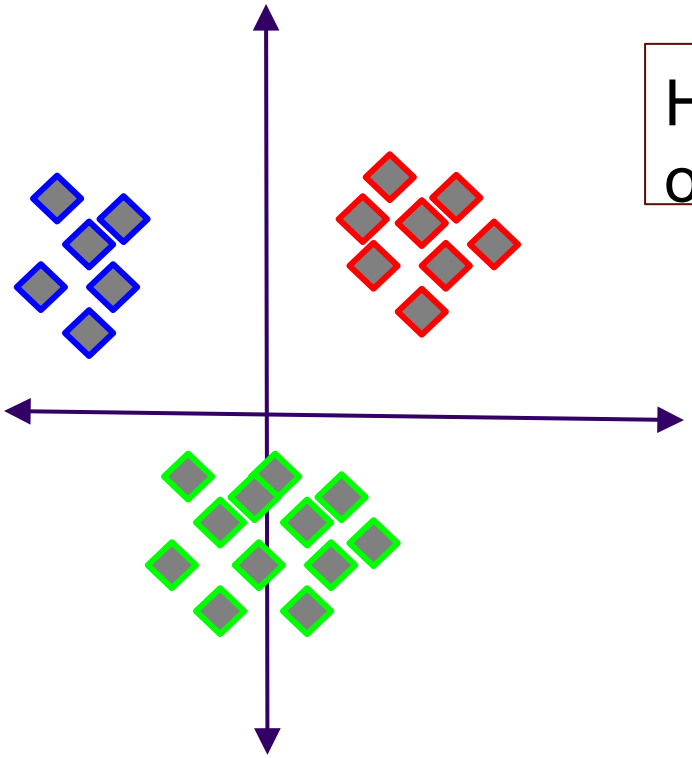
Lecture 23, April 12, 2017



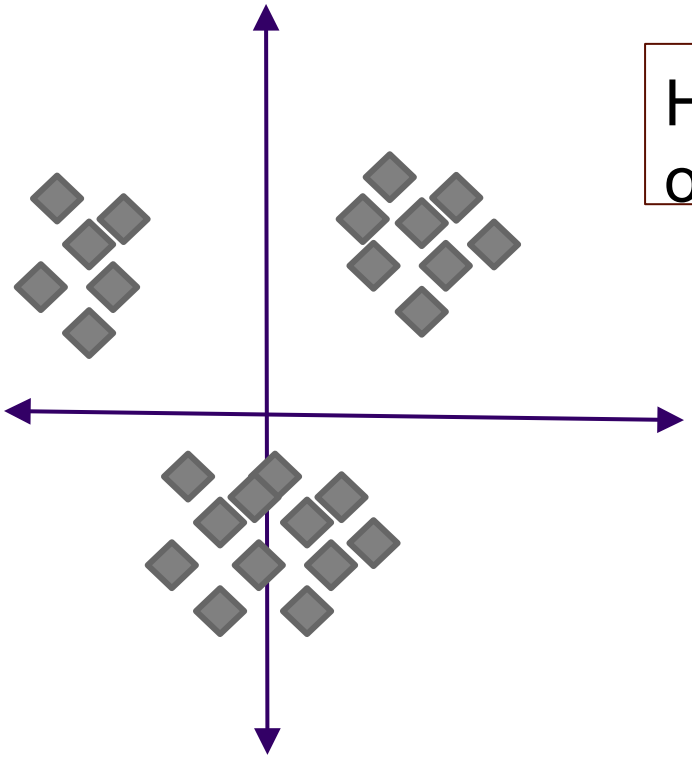
Motivation via Clustering



How to pick number of cluster?

Points on a 2D plane with color as attribute

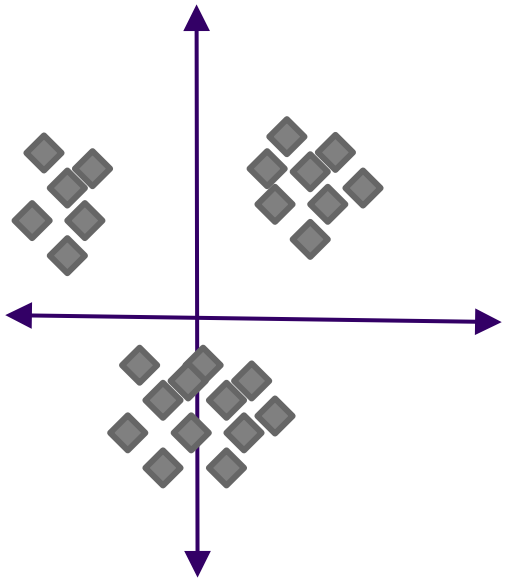
Motivation via Clustering



How to pick number of cluster?

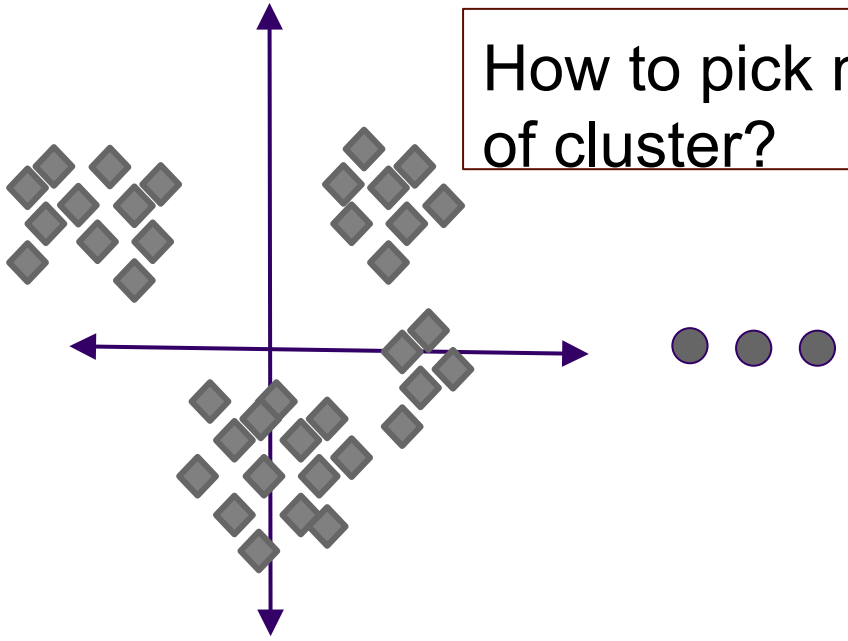
Points on a 2D plane
without color as attribute

Motivation via Clustering



T=1

Streaming Data



T=2

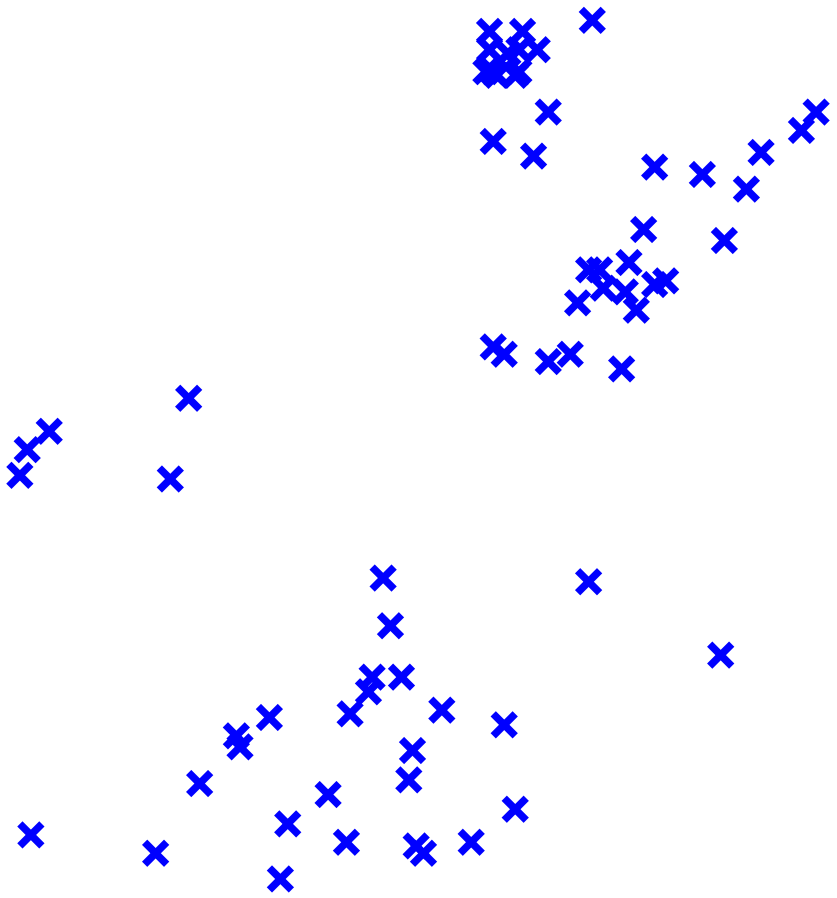
How to pick number of cluster?

Clustered data

- How to model this data?
- Mixture of Gaussians:

$$p(x_1, \ldots, x_N | \pi, \{\mu_k\}, \{\Sigma_k\}) = \prod_{n=1}^{\infty} \sum_{k=1}^{K} \pi_k \mathcal{N}(x_k | \mu_k, \Sigma_k)$$

- Parametric model: Fixed finite number of parameters.



Bayesian finite mixture model

- How to choose the mixing weights and mixture parameters?
- Bayesian choice: Put a prior on them and integrate out:

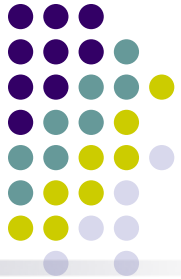
$$p(x_1, \ldots, x_N)$$

$$= \int \int \int \left(\prod_{n=1}^{\infty} \sum_{k=1}^{K} \pi_k \mathcal{N}(x_k | \mu_k, \Sigma_k) \right)$$

$$p(\pi) p(\mu_{1:K}) p(\Sigma_{1:K}) d\pi d\mu_{1:K} d\Sigma_{1:K}$$

- Where possible, use conjugate priors
  - Gaussian/inverse Wishart for mixture parameters
  - What to choose for mixture weights?

Parametric vs nonparametric



Parametric model:

- Assumes all data can be represented using a fixed, finite number of parameters.
  - Mixture of *K* Gaussians, polynomial regression.

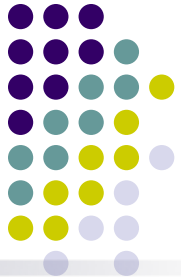
Nonparametric model:

- Number of parameters can grow with sample size.
- Number of parameters may be random.
  - Kernel density estimation.

Bayesian nonparametrics:

- Allow an *infinite* number of parameters *a priori.*
- A finite data set will only use a finite number of parameters.
- Other parameters are integrated out.

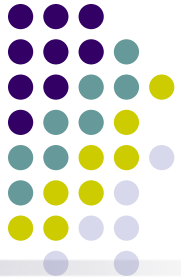
The Dirichlet distribution



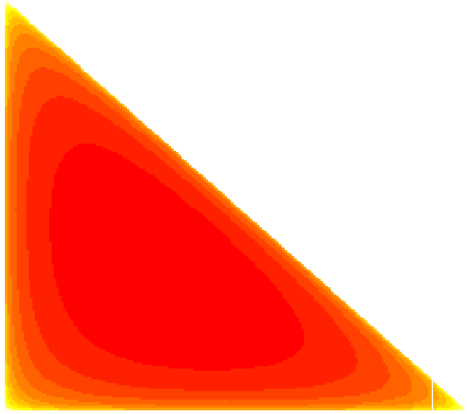
- The Dirichlet distribution is a distribution over the (*K-1*)-dimensional simplex.
- It is parametrized by a *K*-dimensional vector $(\alpha_1, \ldots, \alpha_K)$ such that $\alpha_k \geq 0, k = 1, \ldots, K$ and $\sum_k \alpha_k > 0$
- Its distribution is given by

$$\frac{\prod_{k=1}^{K} \Gamma(\alpha_k)}{\Gamma(\sum_{k=1}^{K} \alpha_k)} \prod_{k=1}^{K} \pi_k^{\alpha_k - 1}$$

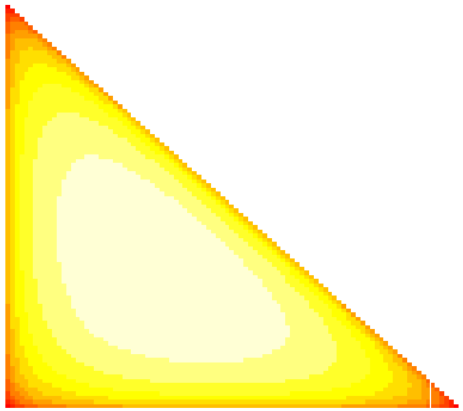
Samples from the Dirichlet distribution



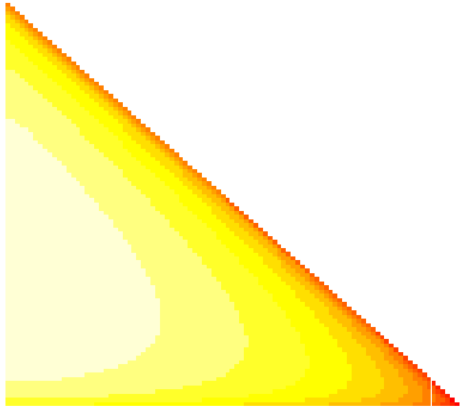
- If $\pi \sim \text{Dirichlet}(\alpha_1, \ldots, \alpha_K)$ then $\pi_k \geq 0$ for all *k*, and $\sum_{k=1}^{K} \pi_k = 1$.
- Expectation: $\mathbb{E}\left[(\pi_1, \ldots, \pi_K)\right] = \frac{(\alpha_1, \ldots, \alpha_K)}{\sum_k \alpha_k}$



$\alpha = (0.01, 0.01, 0.01)$

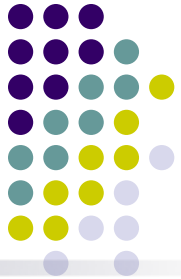


$\alpha = (100, 100, 100)$



$\alpha = (5, 50, 100)$

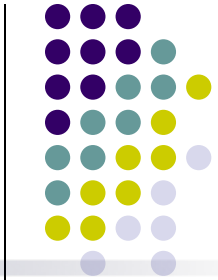
Conjugacy to the multinomial



- If $\theta \sim \text{Dirichlet}(\alpha_1, \ldots, \alpha_K)$ and $x_n \overset{iid}{\sim} \theta$

$$
\begin{aligned}
p(\pi|x_1, \ldots, x_n) &\propto p(x_1, \ldots, x_n|\pi)p(\pi) \\
&= \left(\frac{\prod_{k=1}^K \Gamma(\alpha_k)}{\Gamma(\sum_{k=1}^K \alpha_k)} \prod_{k=1}^K \pi_k^{\alpha_k - 1}\right)\left(\frac{n!}{m_1! \ldots m_K!}\pi_1^{m_1} \ldots \pi_K^{m_K}\right) \\
&\propto \frac{\prod_{k=1}^K \Gamma(\alpha_k + m_k)}{\Gamma(\sum_{k=1}^K \alpha_k + m_k)} \prod_{k=1}^K \pi_k^{\alpha_k + m_k - 1} \\
&= \text{Dirichlet}(\pi|\alpha_1 + m_1, \ldots, \alpha_K + m_K)
\end{aligned}
$$

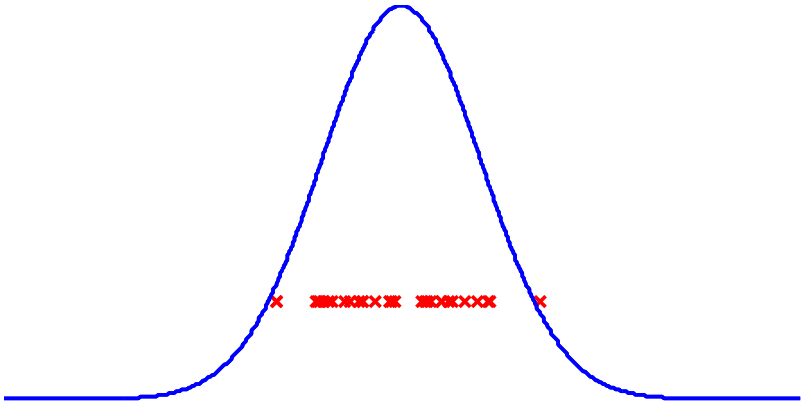
Distributions over distributions



- The Dirichlet distribution is a distribution over positive vectors that sum to one.
- We can associate each entry with a set of parameters
  - e.g. finite mixture model: each entry associated with a mean and covariance.
- In a Bayesian setting, we want these parameters to be *random.*
- We can combine the distribution over probability vectors with a distribution over parameters to get a **distribution over distributions over parameters.**

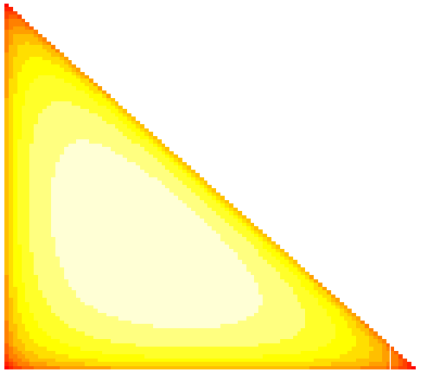
Example: finite mixture model

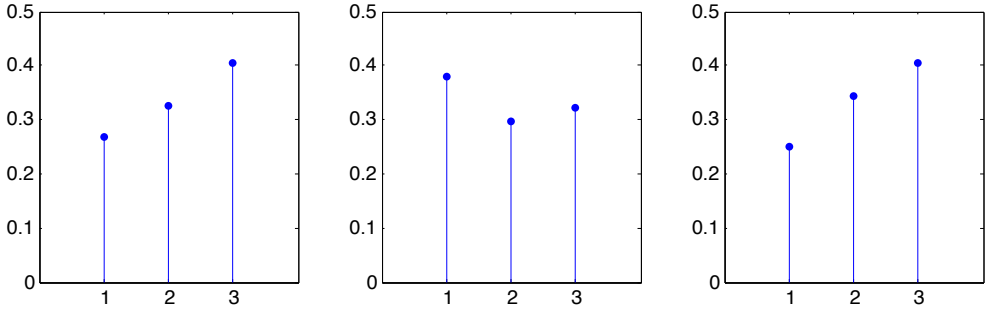
- Gaussian distribution: distribution over means.
  - Sample from a Gaussian is a real-valued number.



Example: finite mixture model

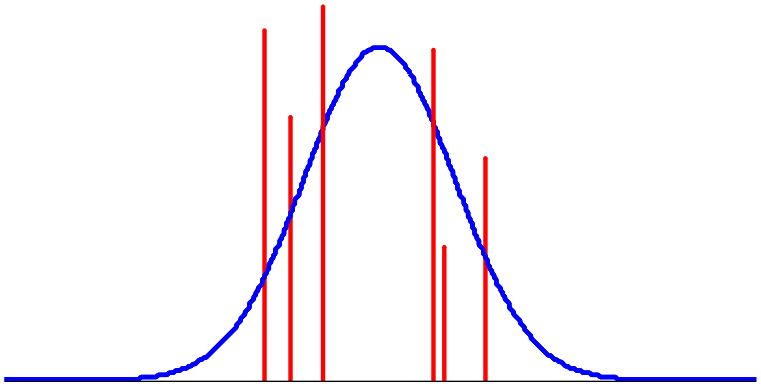
- Gaussian distribution: distribution over means.
  - Sample from a Gaussian is a real-valued number.
- Dirichlet distribution:
  - Sample from a Dirichlet distribution is a probability vector.



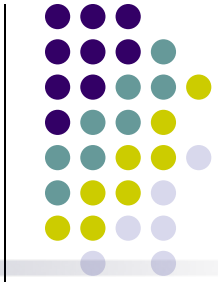


Example: finite mixture model

- Dirichlet prior
  - Each element of a Dirichlet-distributed vector is associated with a parameter value drawn from some distribution.
  - Sample from a Dirichlet prior is a probability distribution over parameters.



Properties of the Dirichlet distribution



- Relationship to gamma distribution: If $\eta_k \sim \text{Gamma}(\alpha_k, 1)$,

$$\frac{(\eta_1,\ldots,\eta_K)}{\sum_k \eta_k} \sim \text{Dirichlet}(\alpha_1,\ldots,\alpha_K)$$

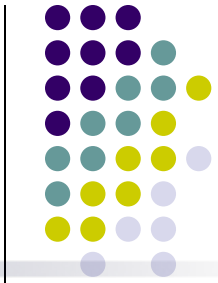
- If $\eta_1 \sim \text{Gamma}(\alpha_1, 1)$ and $\eta_2 \sim \text{Gamma}(\alpha_2, 1)$ then

$$\eta_1 + \eta_2 \sim \text{Gamma}(\alpha_1 + \alpha_2, 1)$$

- Therefore, if $(\pi_1 \ldots, \pi_K) \sim \text{Dirichlet}(\alpha_1,\ldots,\alpha_K)$ then

$$(\pi_1 + \pi_2, \pi_3, \ldots, \pi_K) \sim \text{Dirichlet}(\alpha_1 + \alpha_2, \alpha_3, \ldots, \alpha_K)$$

Properties of the Dirichlet distribution



- The beta distribution is a Dirichlet distribution on the 1-simplex.
- Let $(\pi_1 \ldots, \pi_K) \sim \text{Dirichlet}(\alpha_1, \ldots, \alpha_K)$ and

  $\theta \sim \text{Beta}(\alpha_1 b, \alpha_1(1-b)), 0 < b < 1.$
- Then

  $(\pi_1\theta, \pi_1(1-\theta), \pi_2, \ldots, \pi_K) \sim \text{Dirichlet}(\alpha_1 b_1, \alpha_1(1-b_1), \alpha_2, \ldots, \alpha_K)$
- More generally, if $\theta \sim \text{Dirichlet}(\alpha_1 b_1, \alpha_1 b_2, \ldots, \alpha_1 b_N), \sum_i b_i = 1.$ then

$$(\pi_1\theta_1, \ldots, \pi_1\theta_N, \pi_2, \ldots, \pi_K) \sim \text{Dirichlet}(\alpha_1 b_1, \ldots, \alpha_1 b_N, \alpha_2, \ldots, \alpha_K)$$

Properties of the Dirichlet distribution

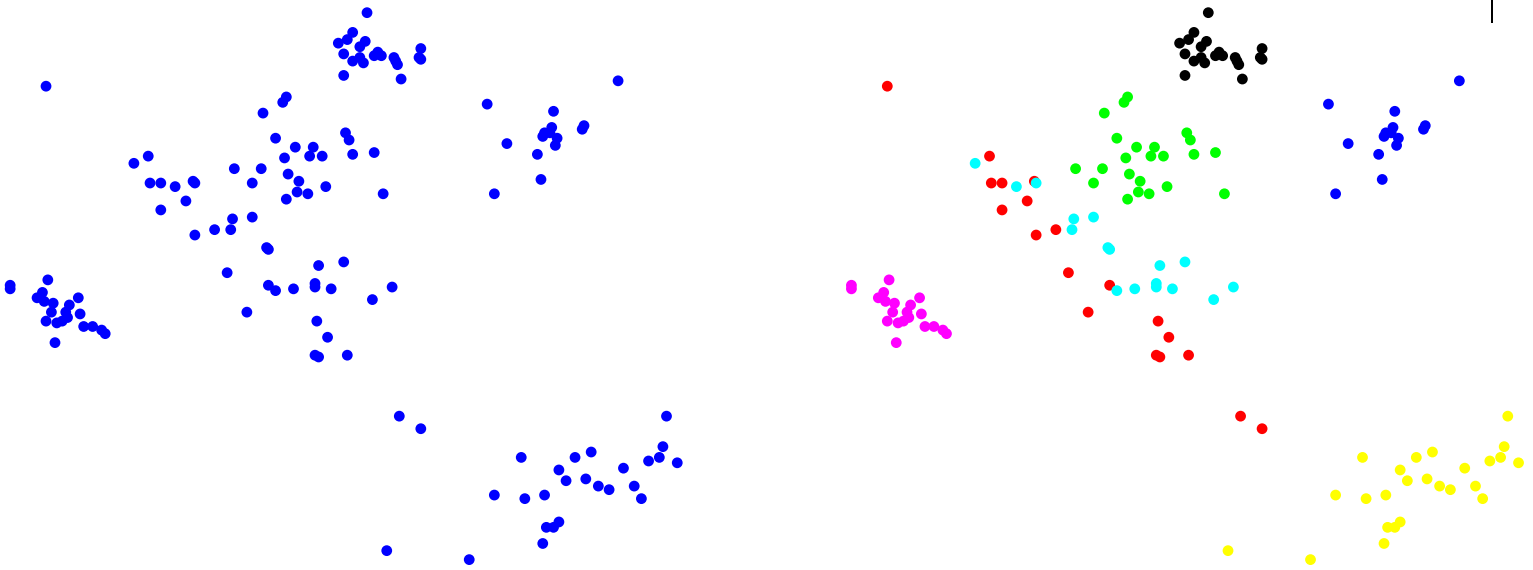
- Renormalization:

  If $(\pi_1 \ldots, \pi_K) \sim \text{Dirichlet}(\alpha_1, \ldots, \alpha_K)$

  then $\dfrac{(\pi_2, \ldots, \pi_K)}{\sum_{k=1}^{K} \pi_k} \sim ?$

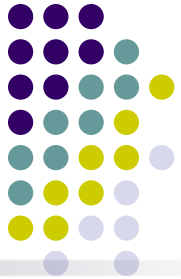
$$\frac{(\pi_2, \ldots, \pi_K)}{\sum_{k=1}^{K} \pi_k} \sim \text{Dirichlet}(\alpha_2, \ldots, \alpha_K)$$

Choosing the number of clusters



- Mixture of Gaussians – but how many components?
- What if we see more data – may find new components?

Bayesian nonparametric mixture models



- Make sure we always have more clusters than we need.
- Solution – infinite clusters *a priori*!

$$p(x_n|\pi, \{\mu_k\}, \{\Sigma_k\}) = \sum_{k=1}^{\infty} \pi_k \mathcal{N}(x_n|\mu_k, \Sigma_k)$$

- A finite data set will always use a finite – but *random* – number of clusters.
- How to choose the prior?
- We want something *like* a Dirichlet prior – but with an infinite number of components.

Constructing an appropriate prior

- Start off with $\pi^{(2)} = (\pi_1^{(2)}, \pi_2^{(2)}) \sim \text{Dirichlet}\left(\frac{\alpha}{2}, \frac{\alpha}{2}\right)$
- Split each component according to the splitting rule:

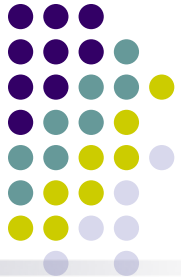
$$\theta_1^{(2)}, \theta_2^{(2)} \overset{iid}{\sim} \text{Beta}\left(\frac{\alpha}{2} \cdot \frac{1}{2}, \frac{\alpha}{2} \cdot \frac{1}{2}\right)$$

$$\pi^{(4)} = (\theta_1^{(2)}\pi_1^{(2)}, (1-\theta_1^{(2)})\pi_1^{(2)}, \theta_2^{(2)}\pi_2^{(2)}, (1-\theta_2^{(2)})\pi_2^{(2)})$$

$$\sim \text{Dirichlet}\left(\frac{\alpha}{4}, \frac{\alpha}{4}, \frac{\alpha}{4}, \frac{\alpha}{4}\right)$$

- Repeat to get $\pi^{(K)} \sim \text{Dirichlet}\left(\frac{\alpha}{K}, \ldots, \frac{\alpha}{K}\right)$
- As $K \to \infty$, we get a vector with infinitely many components

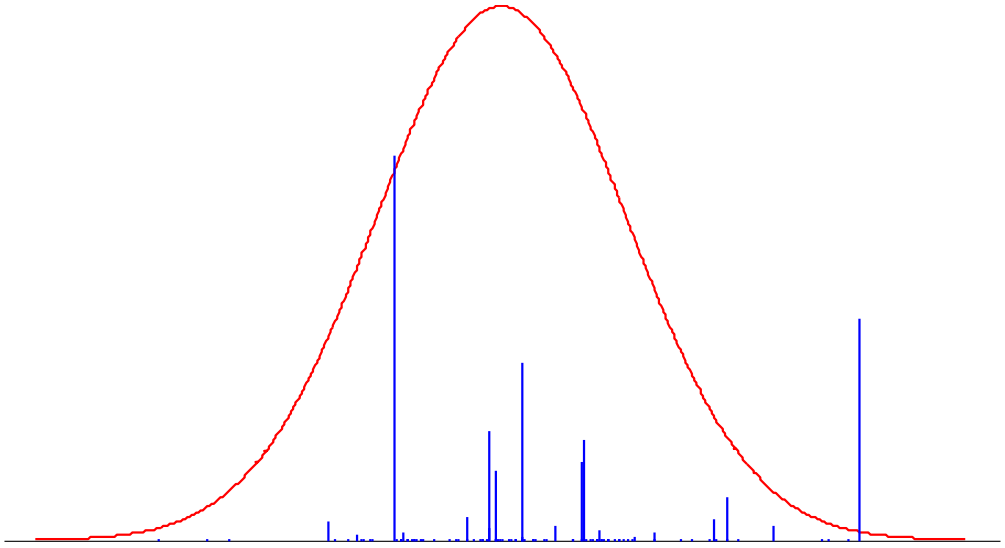
The Dirichlet process



- Let *H* be a distribution on some space Ω – e.g. a Gaussian distribution on the real line.
- Let $\pi \sim \lim_{K \to \infty} \text{Dirichlet}\left(\frac{\alpha}{K} \ldots, \frac{\alpha}{K}\right)$
- For $k = 1, \ldots, \infty$ let $\theta_k \sim H$.
- Then $G := \sum_{k=1}^{\infty} \pi_k \delta_{\theta_k}$ is an infinite distribution over *H*.
- We write $G \sim \text{DP}(\alpha, H)$

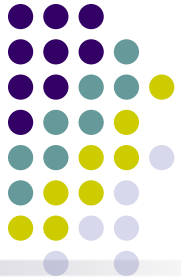
Samples from the Dirichlet process

- Samples from the Dirichlet process are *discrete*.
- We call the point masses in the resulting distribution, *atoms*.

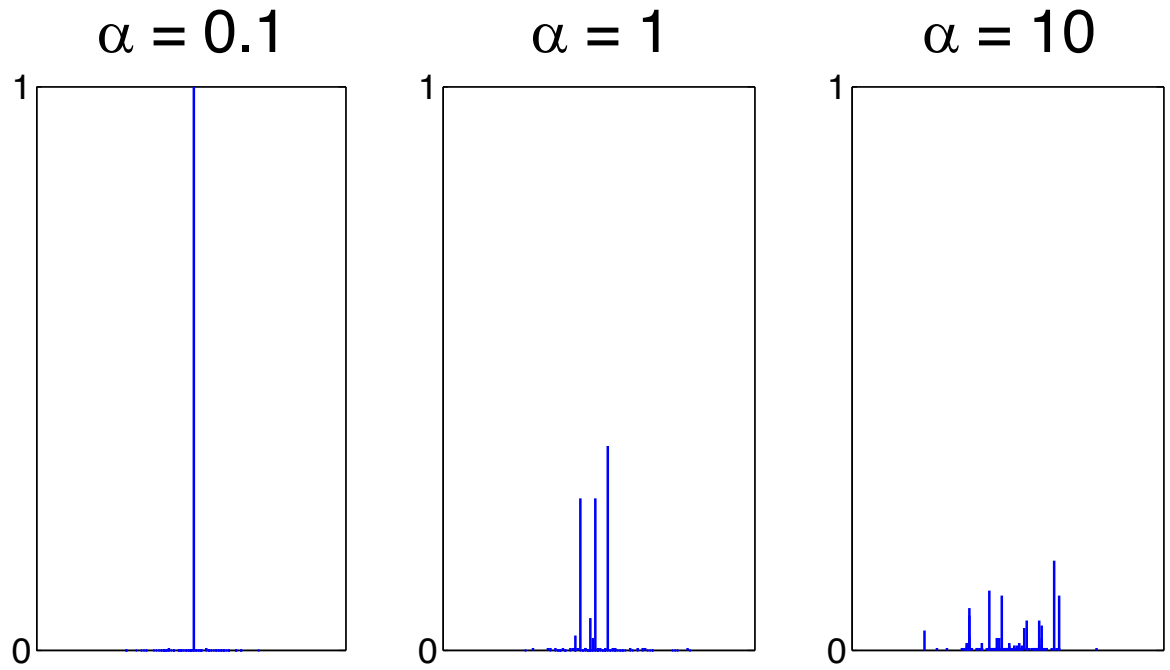


- The *base measure* H determines the *locations* of the atoms.

Samples from the Dirichlet process

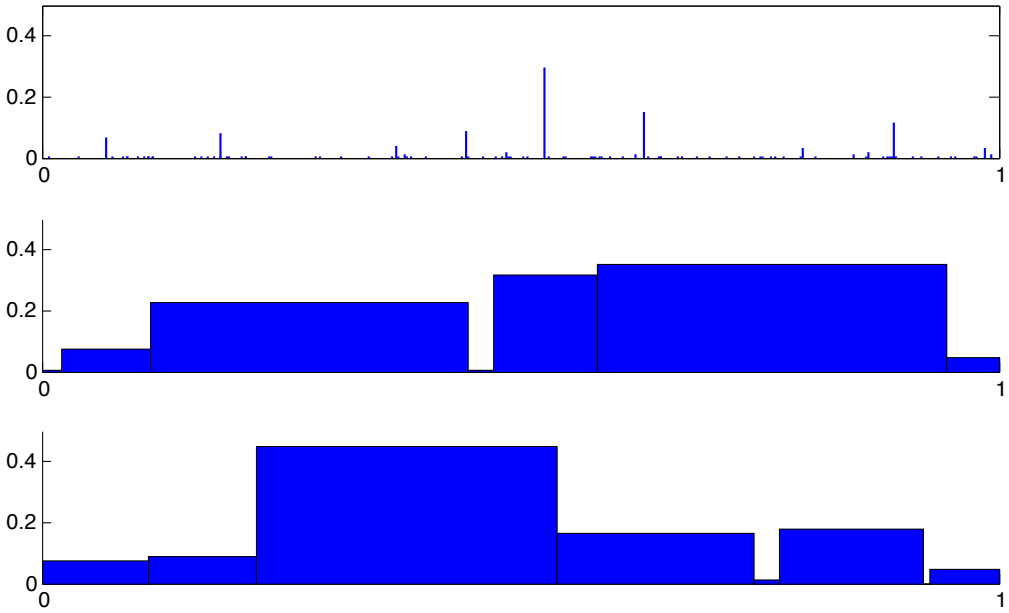


- The *concentration parameter* α determines the distribution over atom sizes.
- Small values of α give *sparse* distributions.



Properties of the Dirichlet process

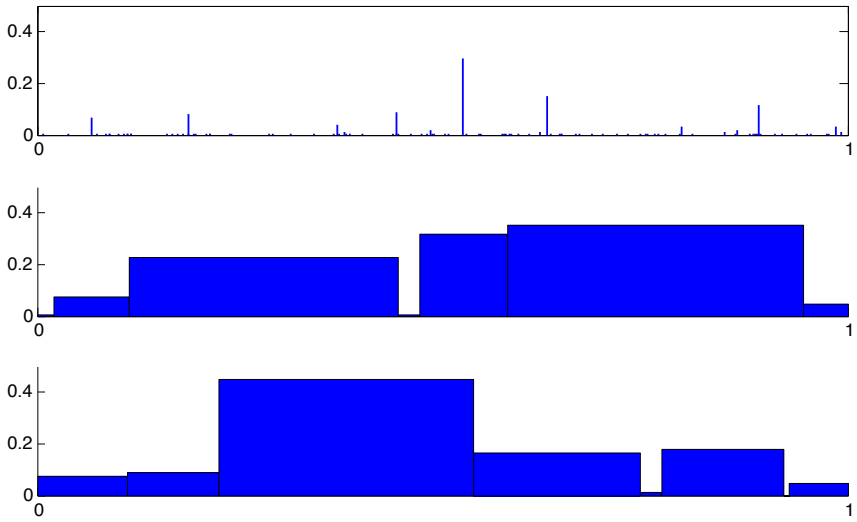
- For any partition $A_1,\ldots,A_K$ of Ω, the total mass assigned to each partition is distributed according to $Dir(\alpha H(A_1),\ldots,\alpha H(A_K))$



Definition: Finite marginals

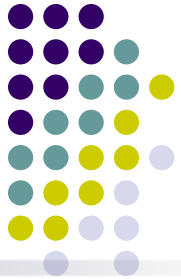
- A Dirichlet process is the unique distribution over probability distributions on some space Ω, such that for any finite partition $A_1,\ldots,A_K$ of Ω,

$$(P(A_1),\ldots,P(A_K)) \sim \text{Dirichlet}(\alpha H(A_1),\ldots,\alpha H(A_K)).$$



[Ferguson, 1973]

Conjugacy of the Dirichlet process



- Let $A_1,\ldots,A_K$ be a partition of Ω, and let H be a measure on Ω. Let $P(A_k)$ be the mass assigned by $G \sim \mathrm{DP}(\alpha, H)$ to partition A_k. Then $(P(A_1),\ldots,P(A_K)) \sim \mathrm{Dirichlet}(\alpha H(A_1),\ldots,\alpha H(A_K))$.

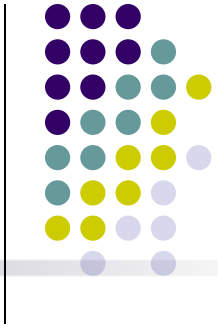
- If we see an observation in the J^{th} segment, then

$$(P(A_1),\ldots,P(A_j),\ldots,P(A_K)|X_1 \in A_j)$$
$$\sim \mathrm{Dirichlet}(\alpha H(A_1),\ldots,\alpha H(A_j)+1,\ldots,\alpha H(A_K)).$$

- This must be true for *all possible partitions of* Ω.
- This is only possible if the posterior of G, given an observation x, is given by

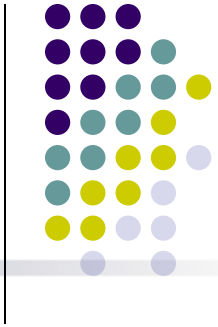
$$G|X_1 = x \sim \mathrm{DP}\left(\alpha+1, \frac{\alpha H + \delta_x}{\alpha+1}\right)$$

Predictive distribution



- The Dirichlet process clusters observations.
- A new data point can either join an existing cluster, or start a new cluster.
- Question: What is the predictive distribution for a new data point?
- Assume H is a continuous distribution on Ω. This means for every point θ in Ω, $H(\theta) = 0$.
- First data point:
  - Start a new cluster.
  - Sample a parameter θ_1 for that cluster.

Predictive distribution



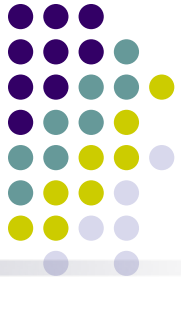
- We have now split our parameter space in two: the singleton θ_1, and everything else.
- Let π_1 be the atom at θ_1.
- The combined mass of all the other atoms is $\pi_* = 1-\pi_1$.
- *A priori*, $(\pi_1, \pi_*) \sim \text{Dirichlet}(0, \alpha)$
- A posteriori, $(\pi_1, \pi_*)|X_1 = \theta_1 \sim \text{Dirichlet}(1, \alpha)$

Predictive distribution

- If we integrate out π_1 we get

$$
\begin{aligned}
P(X_2 = \theta_k | X_1 = \theta_1) &= \int P(X_2 = \theta_k | (\pi_1, \pi_*)) P((\pi_1, \pi_* | X_1 = \theta_1) d\pi_1 \\
&= \int \pi_k \text{Dirichlet}((\pi_1, 1 - \pi_1) | 1, \alpha) d\pi_1 \\
&= \mathbb{E}_{\text{Dirichlet}(1,\alpha)} [\pi_k] \\
&= \begin{cases} \frac{1}{1+\alpha} & \text{if } k = 1 \\ \frac{\alpha}{1+\alpha} & \text{for new } k. \end{cases}
\end{aligned}
$$

Predictive distribution



- Lets say we choose to start a new cluster, and sample a new parameter $\theta_2 \sim H$. Let π_2 be the size of the atom at θ_2.
- A posteriori, $(\pi_1, \pi_2, \pi_*)|X_1 = \theta_1, X_2 = \theta_2 \sim \text{Dirichlet}(1, \alpha)$.
- If we integrate out $\pi = (\pi_1, \pi_2, \pi_*)$ we get

$$
\begin{aligned}
&P(X_3 = \theta_k | X_1 = \theta_1, X_2 = \theta_2) \\
&\qquad = \int P(X_3 = \theta_k | \pi) P(\pi | X_1 = \theta_1, X_2 = \theta_2) d\pi \\
&\qquad = \mathbb{E}_{\text{Dirichlet}(1,1,\alpha)}[\pi_k] \\
&\qquad = \begin{cases} \frac{1}{2+\alpha} & \text{if } k = 1 \\ \frac{1}{2+\alpha} & \text{if } k = 2 \\ \frac{\alpha}{2+\alpha} & \text{for new } k. \end{cases}
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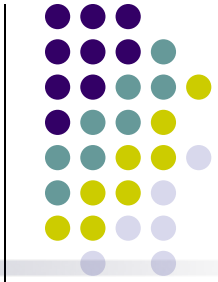
$$

Predictive distribution



- In general, if m_k is the number of times we have seen $X_i=k$, and K is the total number of observed values,

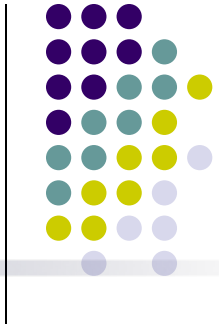
$$P(X_{n+1} = \theta_k | X_1, \ldots, X_n) = \int P(X_{n+1} = \theta_k | \pi) P(\pi | X_1, \ldots, X_n) d\pi$$

$$= \mathbb{E}_{\text{Dirichlet}(m_1, \ldots, m_K, \alpha)} [\pi_k]$$

$$= \begin{cases} \frac{m_k}{n+\alpha} & \text{if } k \leq K \\ \frac{\alpha}{n+\alpha} & \text{for new cluster.} \end{cases}$$

- We tend to see observations that we have seen before – *rich-get-richer property.*
- We can always add new features – *nonparametric.*

Polya urn scheme

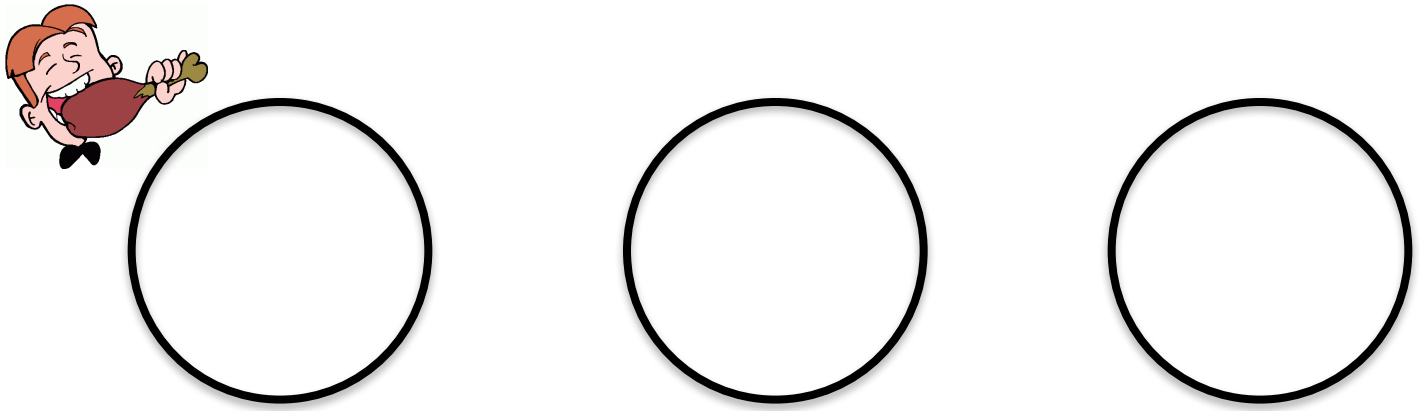


- The resulting distribution over data points can be thought of using the following urn scheme.
- An urn initially contains a black ball of mass α.
- For n=1,2,… sample a ball from the urn with probability proportional to its mass.
- If the ball is black, choose a previously unseen color, record that color, and return the black ball plus a unit-mass ball of the new color to the urn.
- If the ball is not black, record it's color and return it, plus another unit-mass ball of the same color, to the urn

[Blackwell and MacQueen,1973]

Chinese restaurant process

- The distribution over partitions can be described in terms of the following restaurant metaphor:
- The first customer enters a restaurant, and picks a table.



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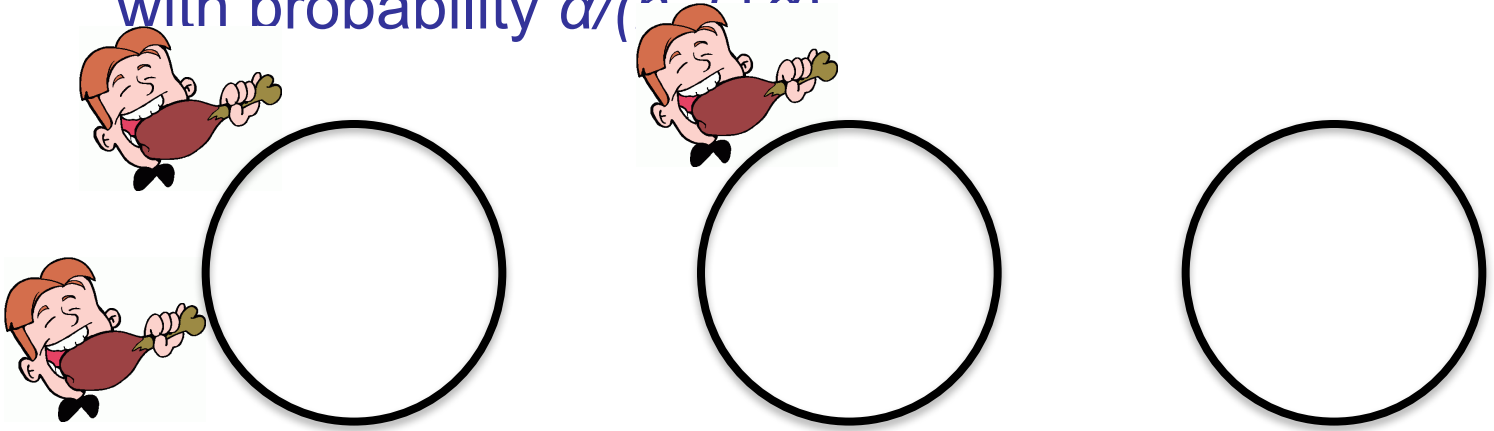

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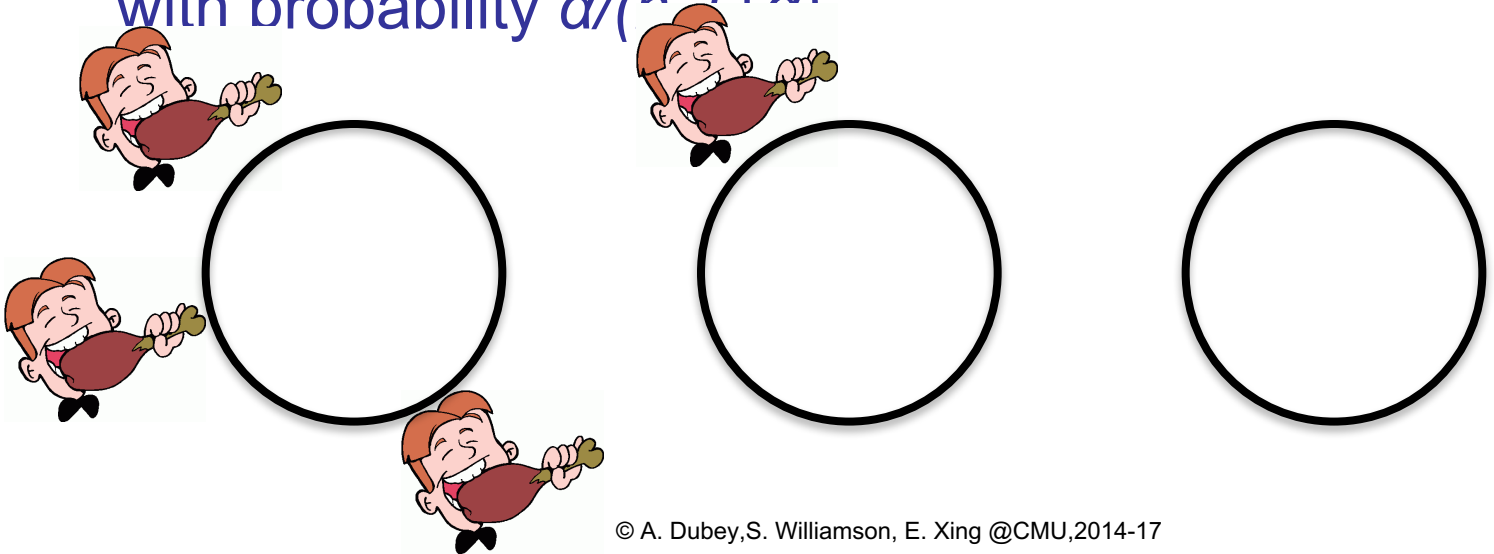

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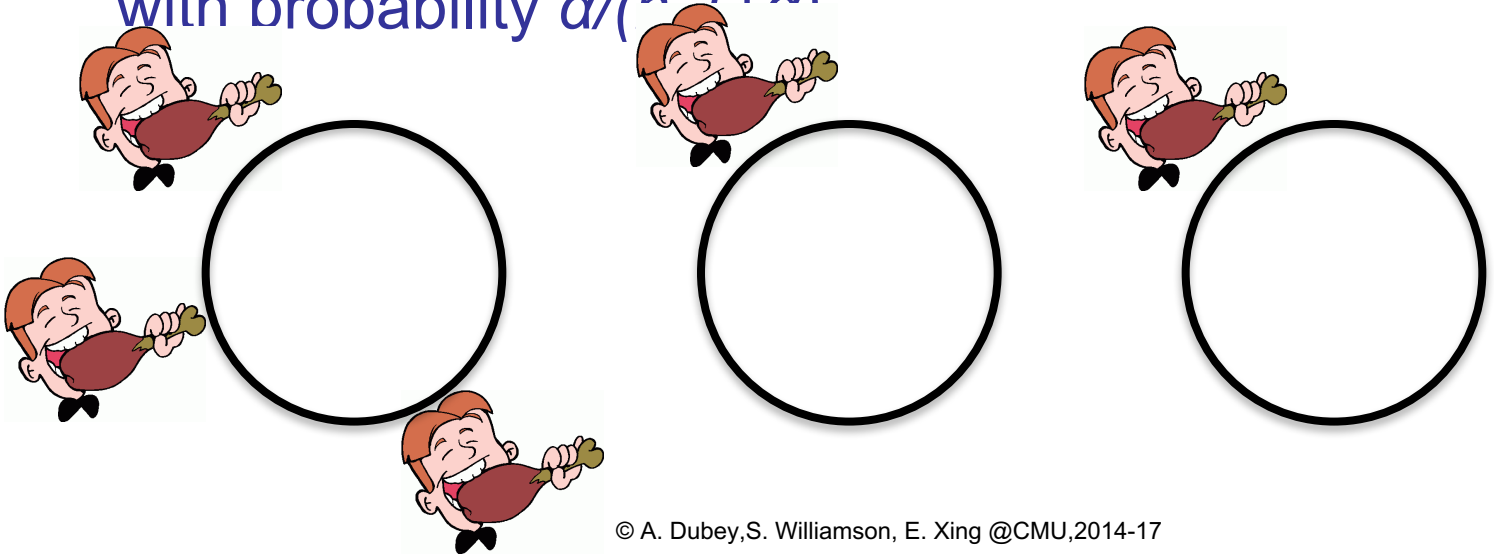
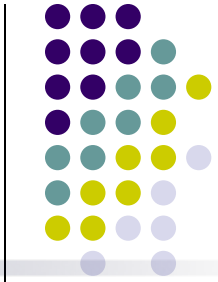
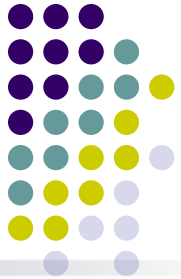


Exchangeability



- An interesting fact: the distribution over the clustering of the first *N* customers *does not depend on the order in which they arrived*.
- Homework: Prove to yourself that this is true.
- However, the customers are not independent – they tend to sit at popular tables.
- We say that distributions like this are *exchangeable*.
- De Finetti's theorem: If a sequence of observations is exchangeable, there must exist a distribution given which they are iid.
- The customers in the CRP are iid given the underlying Dirichlet process – by integrating out the DP, they become dependent.

Stick breaking construction



- We can represent samples from the Dirichlet process exactly.
- Imagine a stick of length 1, representing total probability.
- For k=1,2,…
  - Sample a beta(1,α) random variable b_k.
  - Break off a fraction b_k of the stick. This is the k^{th} atom size
  - Sample a random location for this atom.
  - Recurse on the remaining stick.

$$G := \sum_{k=1}^{\infty} \pi_k \delta_{\theta_k}$$

$$\pi_k := b_k \prod_{j=1}^{k-1} (1 - b_k)$$

$$b_k \sim \text{Beta}(1, \alpha)$$

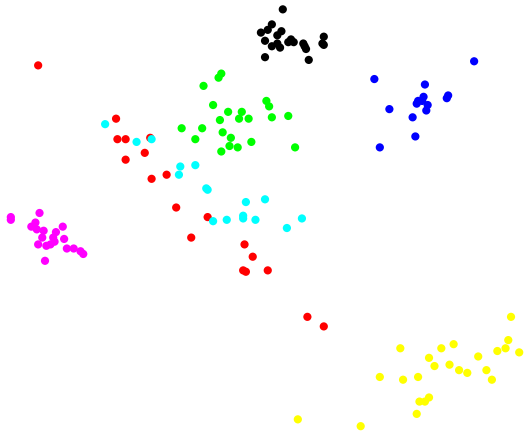
[Sethuraman, 1994]

Inference in the DP mixture model

$$G := \sum_{k=1}^{\infty} \pi_k \delta_{\theta_k} \sim \mathrm{DP}(\alpha, H)$$

$$\phi_n \sim G$$

$$x_n \sim f(\phi_n)$$



Inference: Collapsed sampler

- We can integrate out *G* to get the CRP.
- Reminder: Observations in the CRP are exchangeable.
- Corollary: When sampling any data point, we can always rearrange the ordering so that it is the last data point.
- Let z_n be the cluster allocation of the *n*th data point.
- Let *K* be the total number of instantiated clusters.
- Then

$$p(z_n = k | x_n, z_{-n}, \phi_{1:K}) \propto \begin{cases} m_k f(x_n|\phi_k) & k \leq K \\ \alpha \int_\Omega f(x_n|\phi) H(d\phi) & k = K+1 \end{cases}$$

- If we use a conjugate prior for the likelihood, we can often integrate out the cluster parameters

Problems with the collapsed sampler

- We are only updating one data point at a time.
- Imagine two “true” clusters are merged into a single cluster – a single data point is unlikely to “break away”.
- Getting to the true distribution involves going through low probability states ➔ mixing can be slow.
- If the likelihood is not conjugate, integrating out parameter values for new features can be difficult.
- Neal [2000] offers a variety of algorithms.
- Alternative: Instantiate the latent measure.

Inference: Blocked Gibbs sampler

- Rather than integrate out *G*, we can instantiate it.
- Problem: *G* is infinite-dimensional.
- Solution: Approximate it with a truncated stick-breaking process:

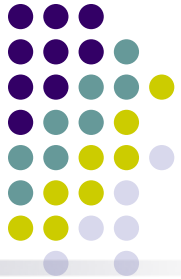
$$G^K := \sum_{k=1}^{K} \pi_k \delta_{\theta_k}$$

$$\pi_k = b_k \prod_{j=1}^{k-1} (1 - b_j)$$

$$b_k \sim \text{Beta}(1, \alpha), k = 1, \ldots, K-1$$

$$b_K = 1$$

Inference: Blocked Gibbs sampler



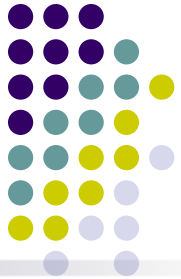
- Sampling the cluster indicators:

$$p(z_n = k|\text{rest}) \propto \pi_k f(x_n|\theta_k)$$

- Sampling the stick breaking variables:
  - We can think of the stick breaking process as a sequence of binary decisions.
  - Choose $z_n = 1$ with probability b_1.
  - If $z_n \neq 1$, choose $z_n = 2$ with probability b_2.
  - *etc..*

$$b_k|\text{rest} \sim \text{Beta}\left(1 + m_k, \alpha + \sum_{j=k+1}^{K} m_j\right)$$

Inference: Slice sampler



- Problem with batch sampler: Fixed truncation introduces error.
- Idea:
  - Introduce *random truncation*.
  - If we marginalize over the random truncation, we recover the full model.
- Introduce a uniform random variable u_n for each data point.
- Sample indicator z_n according to

$$p(z_n = k|\text{rest}) = I(\pi_k > u_n) f(x_n|\theta_k)$$

- Only a **finite** number of possible values.

Inference: Slice sampler

- The conditional distribution for u_n is just:

$$u_n|\text{rest} \sim \text{Uniform}[0, \pi_{z_n}]$$

- Conditioned on the u_n and the z_n, the π_k can be sampled according to the block Gibbs sampler.

- Only need to represent a finite number K of components such that

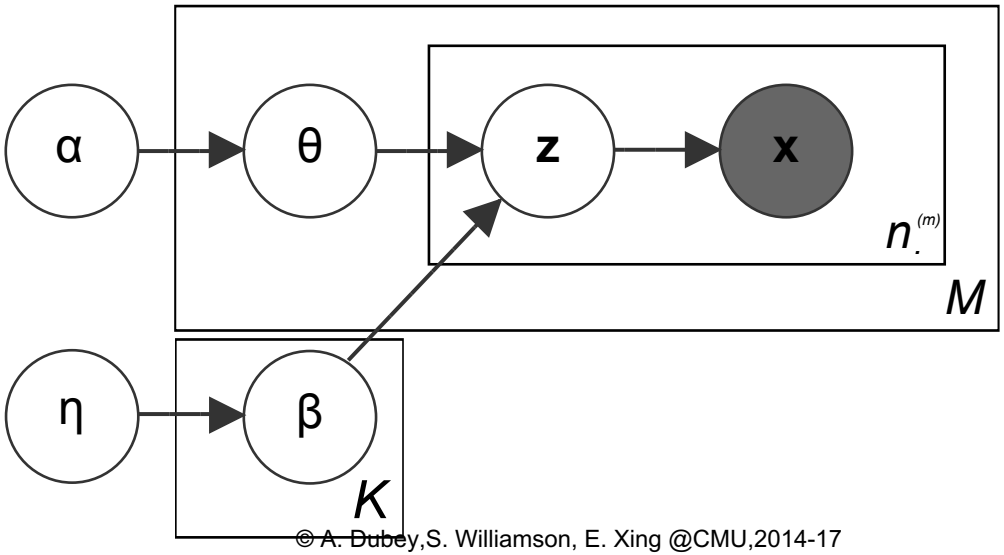
$$1 - \sum_{k=1}^{K} \pi_k < min(u_n)$$

Topic models

- Topic models describe documents using a distribution over features.
- Each feature is a distribution over words
- Each document is represented as a collection of words (usually unordered – “bag of words” assumption).
- The words within a document are distributed according to a document-specific mixture model
  - Each word in a document is associated with a feature.
- The features are shared between documents.
- The features learned tend to give high probability to semantically related words – “topics”

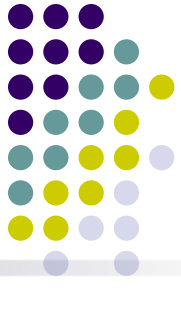
Latent Dirichlet allocation

- For each topic $k=1,\ldots,K$
  - Sample a distribution over words, $\beta \sim \text{Dir}(\eta_1,\ldots,\eta_V)$
- For each document $m=1,\ldots,M$
  - Sample a distribution over topics, $\theta_m \sim \text{Dir}(\alpha_1,\ldots,\alpha_K)$
  - For each word $n=1,\ldots,N_m$
    - Sample a topic $z_{mn} \sim \text{Discrete}(\theta_m)$
    - Sample a word $w_{mk} \sim \text{Discrete}(\beta_z)$



Blei et al, 2002

"Topics" found by LDA



"Arts"	"Budgets"	"Children"	"Education"
NEW	MILLION	CHILDREN	SCHOOL
FILM	TAX	WOMEN	STUDENTS
SHOW	PROGRAM	PEOPLE	SCHOOLS
MUSIC	BUDGET	CHILD	EDUCATION
MOVIE	BILLION	YEARS	TEACHERS
PLAY	FEDERAL	FAMILIES	HIGH
MUSICAL	YEAR	WORK	PUBLIC
BEST	SPENDING	PARENTS	TEACHER
ACTOR	NEW	SAYS	BENNETT
FIRST	STATE	FAMILY	MANIGAT
YORK	PLAN	WELFARE	NAMPHY
OPERA	MONEY	MEN	STATE
THEATER	PROGRAMS	PERCENT	PRESIDENT
ACTRESS	GOVERNMENT	CARE	ELEMENTARY
LOVE	CONGRESS	LIFE	HAITI

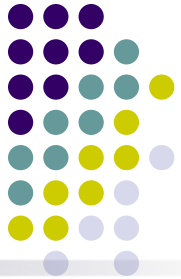
The William Randolph Hearst Foundation will give $1.25 million to Lincoln Center, Metropolitan Opera Co., New York Philharmonic and Juilliard School. "Our board felt that we had a real opportunity to make a mark on the future of the performing arts with these grants an act every bit as important as our traditional areas of support in health, medical research, education and the social services," Hearst Foundation President Randolph A. Hearst said Monday in announcing the grants. Lincoln Center's share will be $200,000 for its new building, which will house young artists and provide new public facilities. The Metropolitan Opera Co. and New York Philharmonic will receive $400,000 each. The Juilliard School, where music and the performing arts are taught, will get $250,000. The Hearst Foundation, a leading supporter of the Lincoln Center Consolidated Corporate Fund, will make its usual annual $100,000 donation, too.

Image from Blei et al, 2002

Constructing a topic model with infinitely many topics

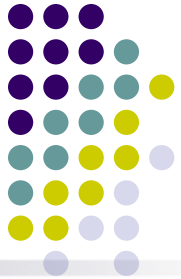
- LDA: Each distribution is associated with a distribution over *K* topics.
- Problem: How to choose the number of topics?
- Solution:
  - Infinitely many topics!
  - Replace the Dirichlet distribution over topics with a Dirichlet process!
- Problem: We want to make sure the topics are *shared* between documents

Sharing topics



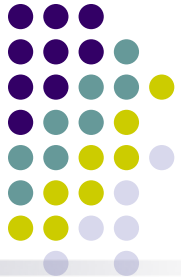
- In LDA, we have *M* independent samples from a Dirichlet distribution.
- The weights are different, but the topics are fixed to be the same.
- If we replace the Dirichlet distributions with Dirichlet processes, each atom of each Dirichlet process will pick a topic *independently* of the other topics.

Sharing topics



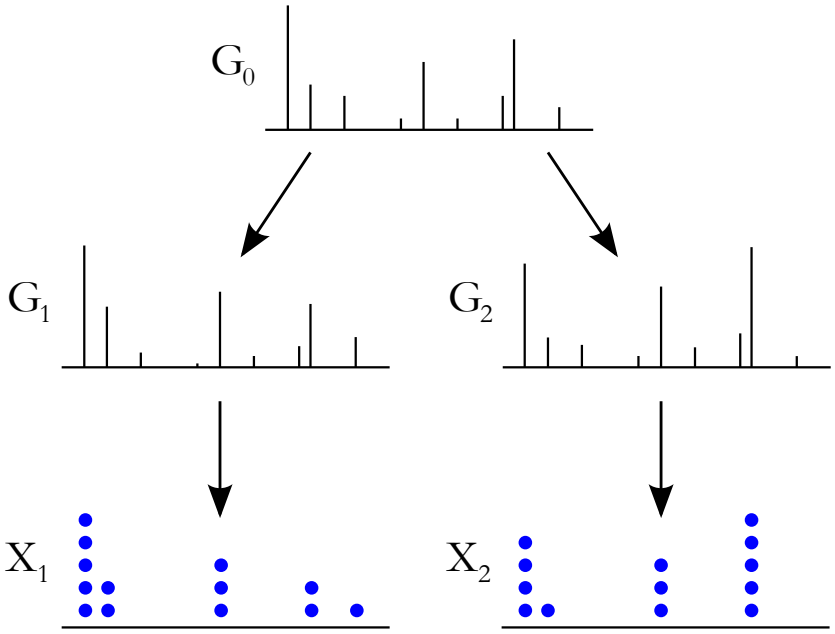
- Because the base measure is *continuous*, we have zero probability of picking the same topic twice.
- If we want to pick the same topic twice, we need to use a *discrete* base measure.
- For example, if we chose the base measure to be $H = \sum_{k=1}^{K} \alpha_k \delta_{\beta_k}$ then we would have LDA again.
- We want there to be an infinite number of topics, so we want an *infinite, discrete* base measure.
- We want the location of the topics to be random, so we want an *infinite, discrete, random* base measure.

Hierarchical Dirichlet Process (Teh et al, 2006)

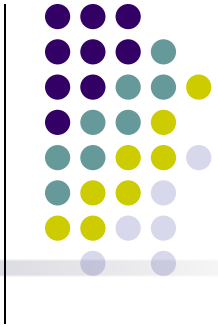


- Solution: Sample the base measure from a Dirichlet process!

$$G_0 \sim \mathrm{DP}(\gamma, H)$$
$$G_m \sim \mathrm{DP}(\alpha, G_0)$$



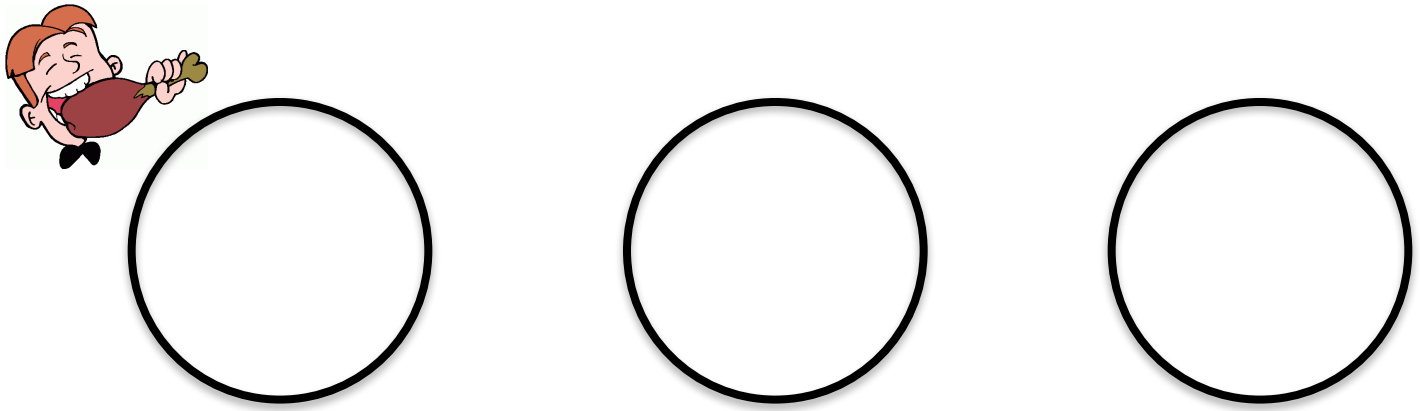
Chinese restaurant franchise



- Imagine a *franchise* of restaurants, serving an infinitely large, global menu.
- Each table in each restaurant orders a single dish.
- Let n_{rt} be the number of customers in restaurant r sitting at table t.
- Let m_{rd} be the number of tables in restaurant r serving dish d.
- Let $m_{.d}$ be the number of tables, across *all* restaurants, serving dish d.

Chinese restaurant franchise

- Customers enter the restaurants, and sit at tables according to the Chinese restaurant process
  - The first customer enters a restaurant, and picks a table.



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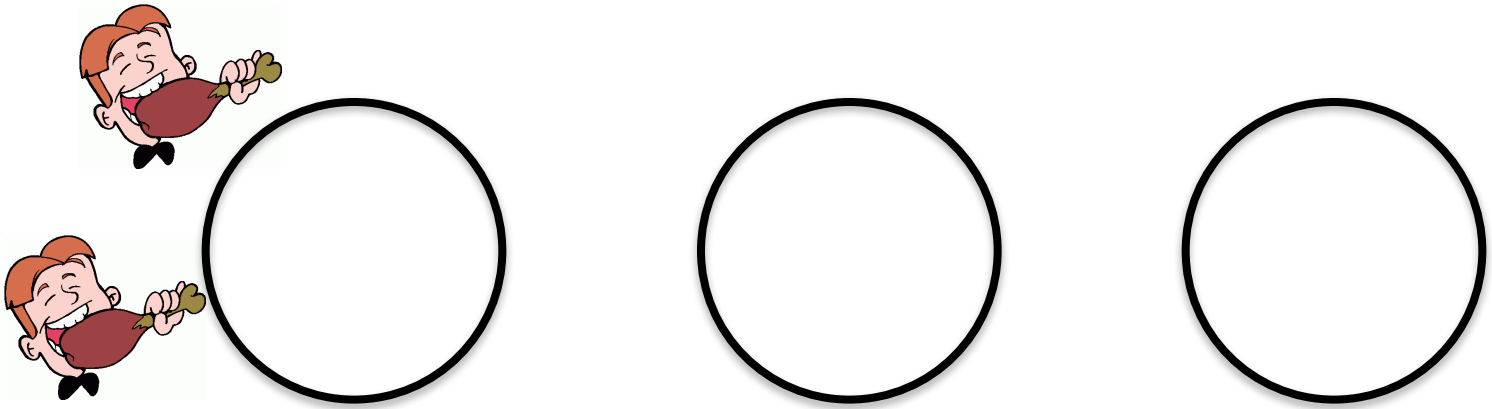

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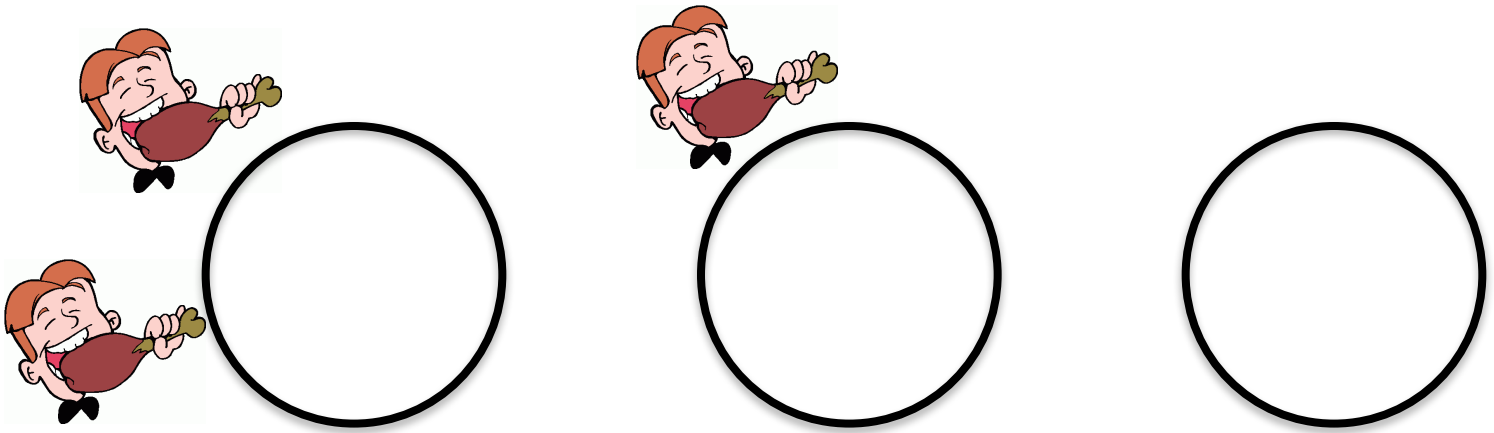

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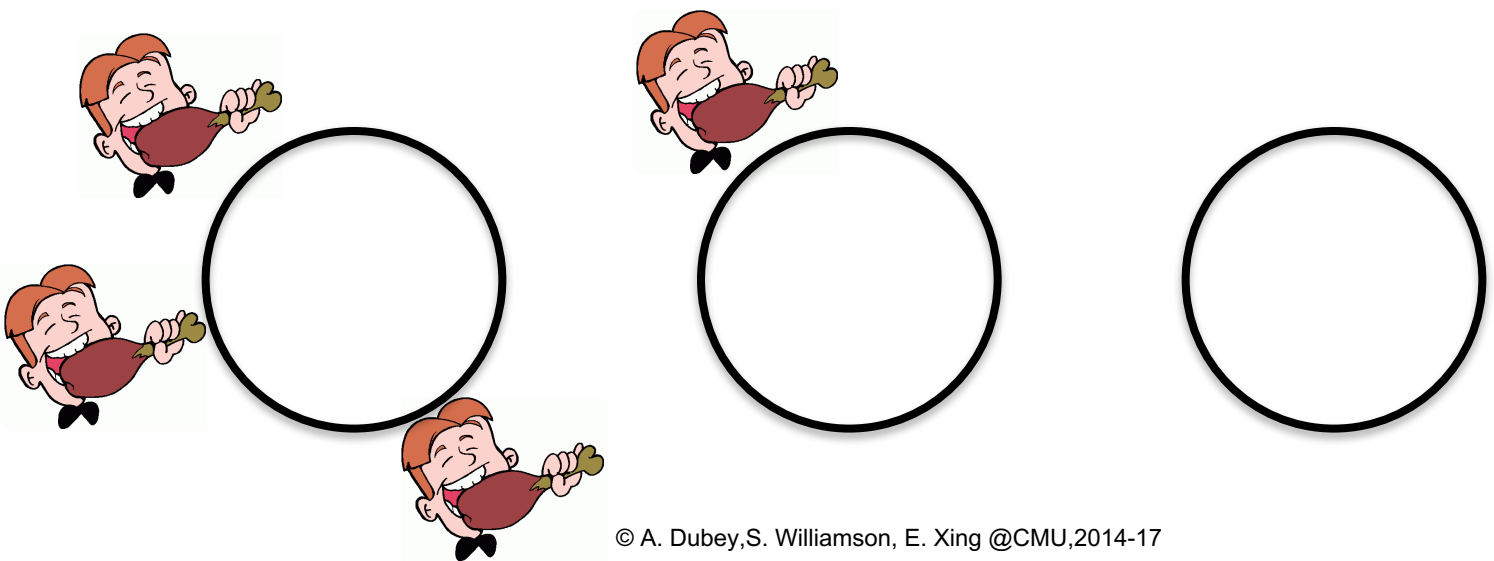

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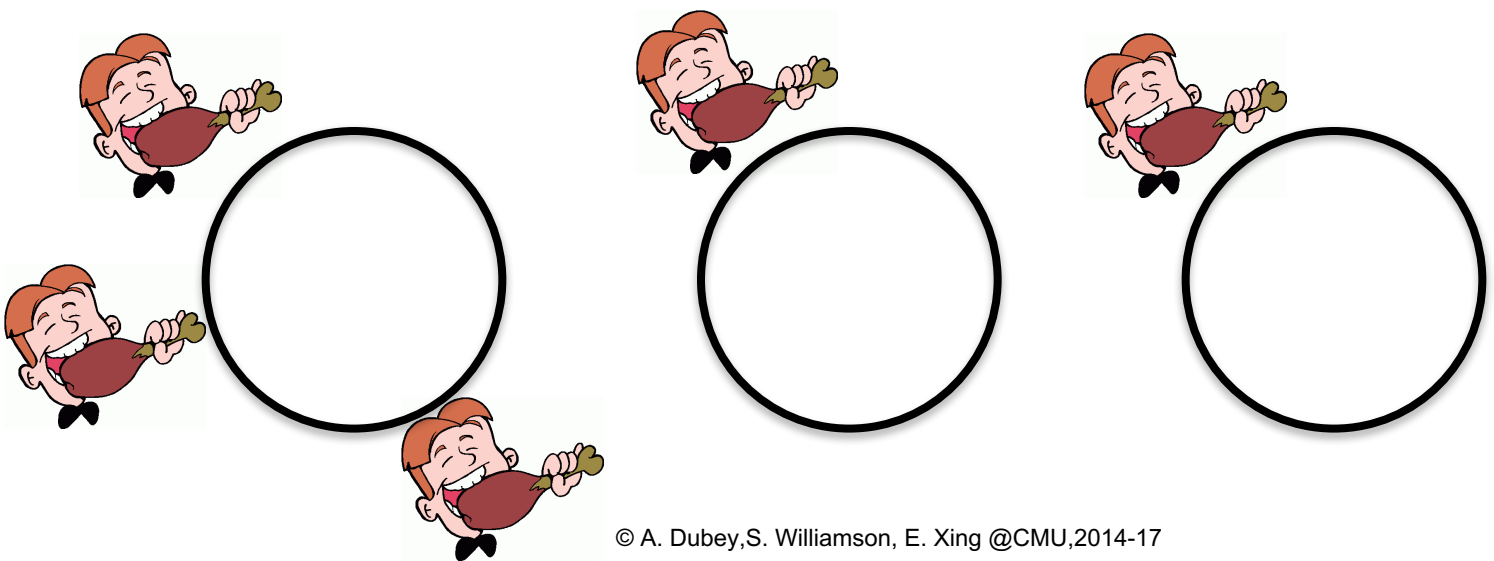
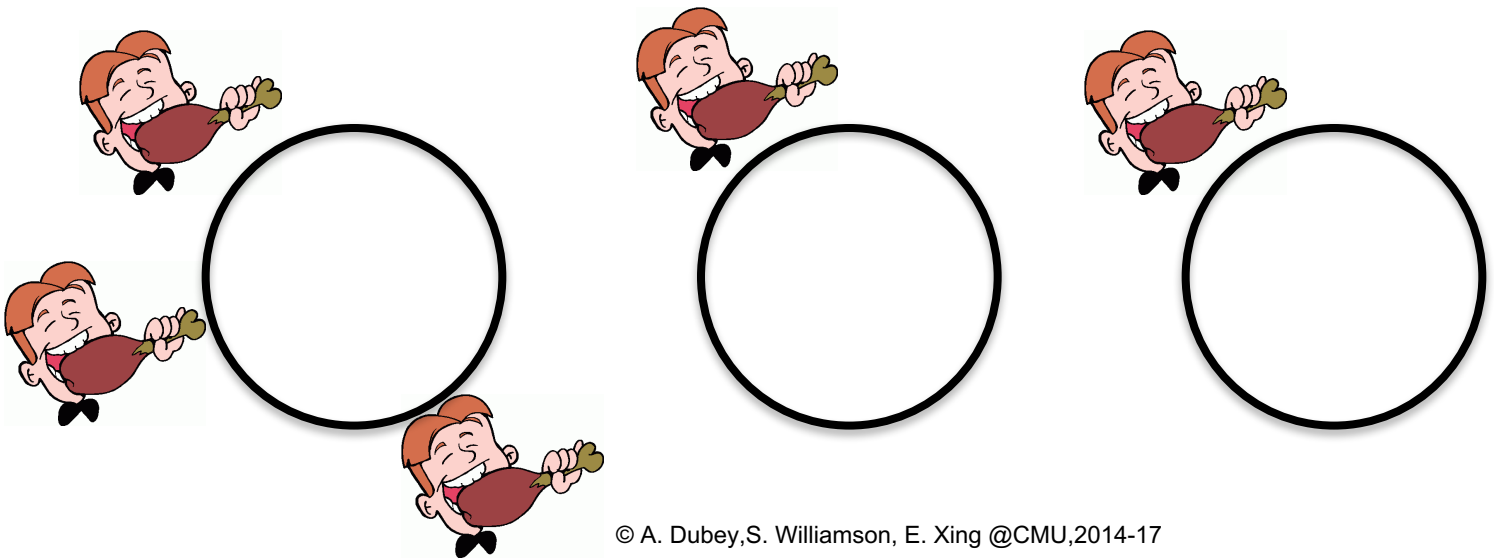


Chinese restaurant franchise

- Each *table* in each restaurant picks a *dish*, with probability proportional to the number of times it has been served across *all* restaurants.

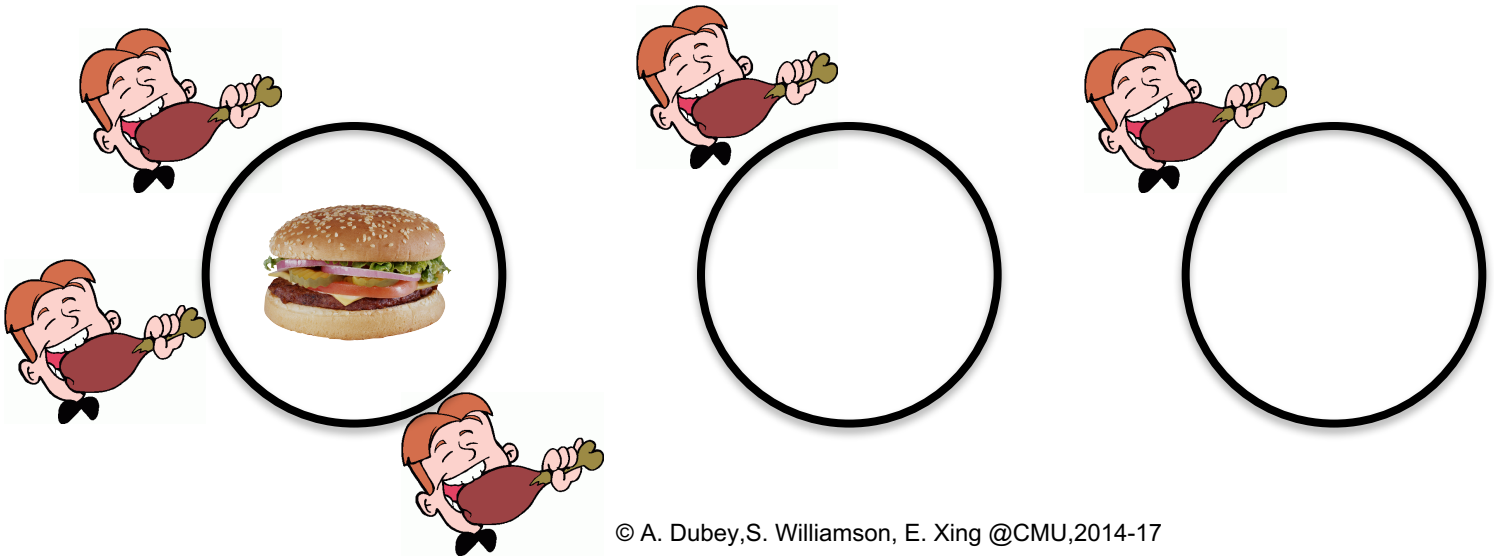
$$p(\text{table } t \text{ chooses dish } d|\text{previous tables}) = \begin{cases} \frac{m_d}{T+\gamma} & \text{for an existing table} \\ \frac{\gamma}{T+\gamma} & \text{for a new table} \end{cases}$$



Chinese restaurant franchise

- Each *table* in each restaurant picks a *dish*, with probability proportional to the number of times it has been served across *all* restaurants.

$$p(\text{table } t \text{ chooses dish } d|\text{previous tables}) = \begin{cases} \frac{m_d}{T+\gamma} & \text{for an existing table} \\ \frac{\gamma}{T+\gamma} & \text{for a new table} \end{cases}$$



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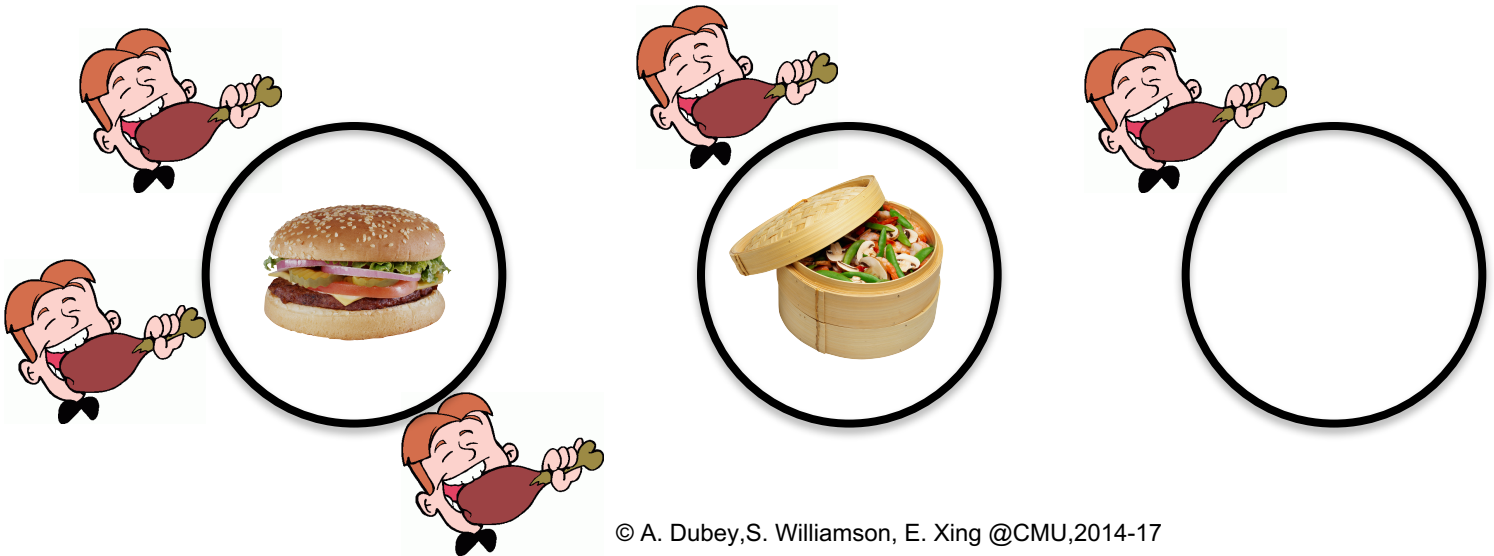

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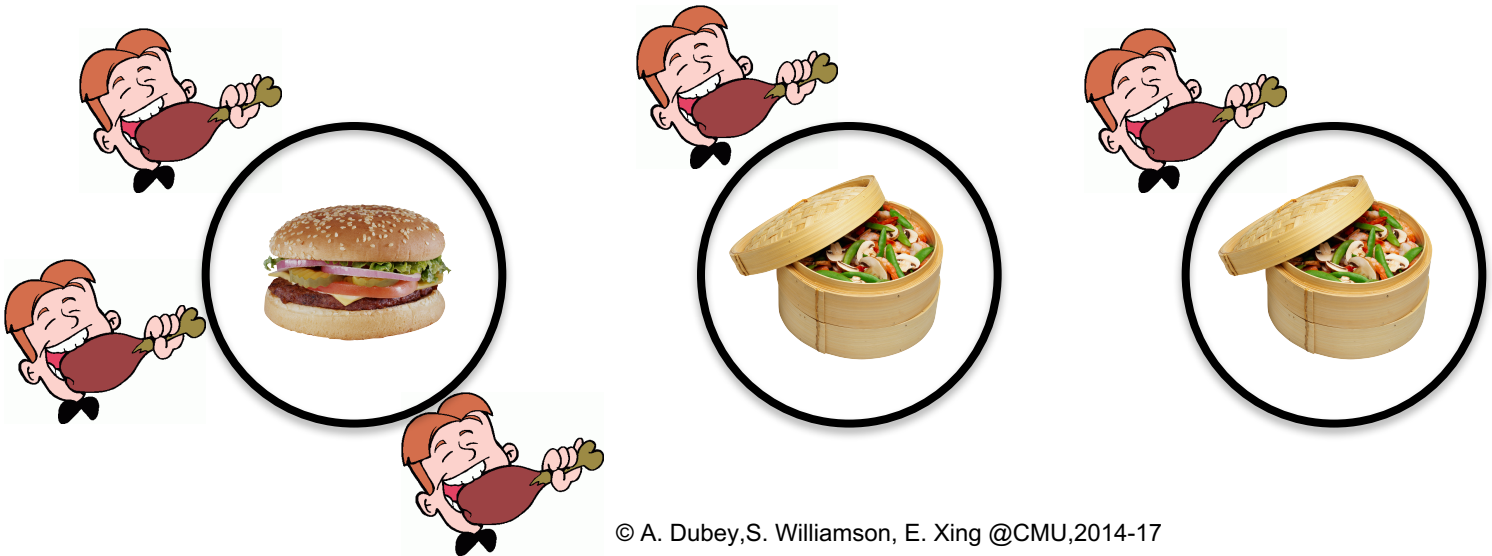


An infinite topic model

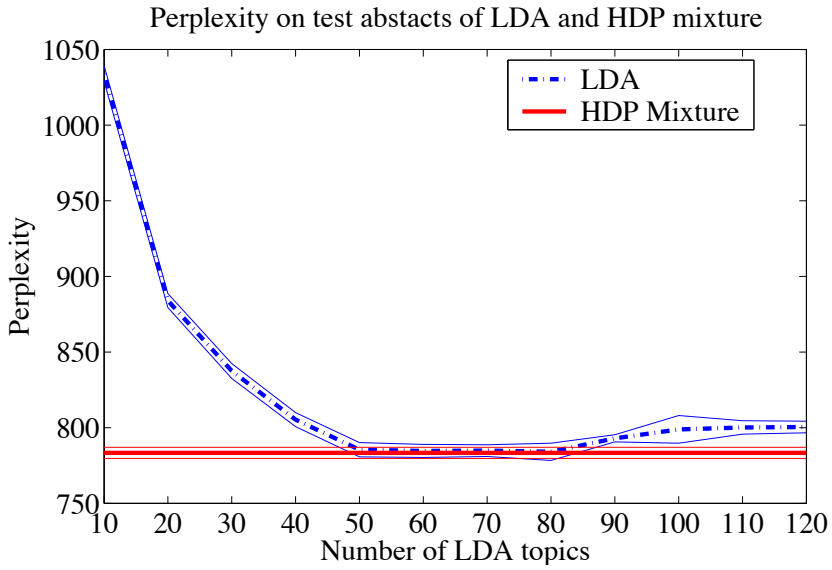
- Restaurants = documents; dishes = topics.

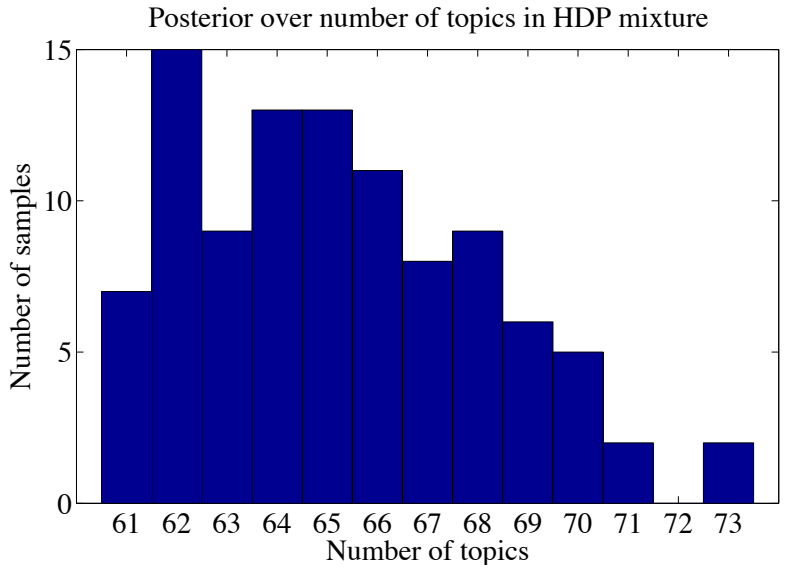
- Let *H* be a V-dimensional Dirichlet distribution, so a sample from *H* is a distribution over a vocabulary of *V* words.
- Sample a global distribution over topics,

$$G_0 := \sum_{k=1}^{\infty} \pi_k \delta_{\beta_k} \sim \mathrm{DP}(\alpha, H)$$

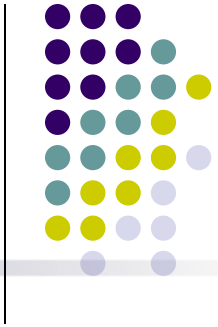
- For each document *m=1,…,M*
  - Sample a distribution over topics, $G_m \sim \mathrm{DP}(\gamma, G_0)$.
  - For each word *n=1,…,N_m*
    - Sample a topic $\phi_{mn} \sim \mathrm{Discrete}(G_0)$.
    - Sample a word $w_{mk} \sim \mathrm{Discrete}(\phi_{mn})$.

The “right” number of topics



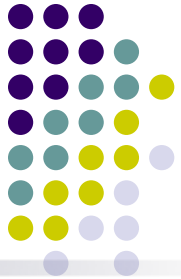


Limitations of a simple mixture model



- The Dirichlet distribution and the Dirichlet process are great if we want to cluster data into non-overlapping clusters.
- However, DP/Dirichlet mixture models cannot share features between clusters.
- In many applications, data points exhibit properties of multiple latent features
  - Images contain multiple objects.
  - Actors in social networks belong to multiple social groups.
  - Movies contain aspects of multiple genres.

Latent variable models

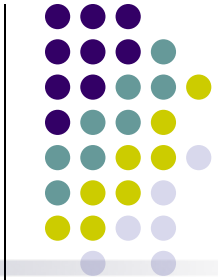


- Latent variable models allow each data point to exhibit *multiple* features, to *varying degrees*.
- Example: Factor analysis

  $\mathbf{X} = \mathbf{W}\mathbf{A}^T + \varepsilon$

  - Rows of **A** = latent features
  - Rows of **W** = datapoint-specific weights for these features
  - ε = Gaussian noise.
- Example: LDA
  - Each document represented by a *mixture* of features.

Infinite latent feature models



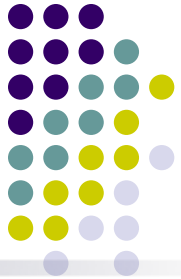
- Problem: How to choose the number of features?
- Example: Factor analysis

  $\mathbf{X} = \mathbf{W}\mathbf{A}^T + \varepsilon$

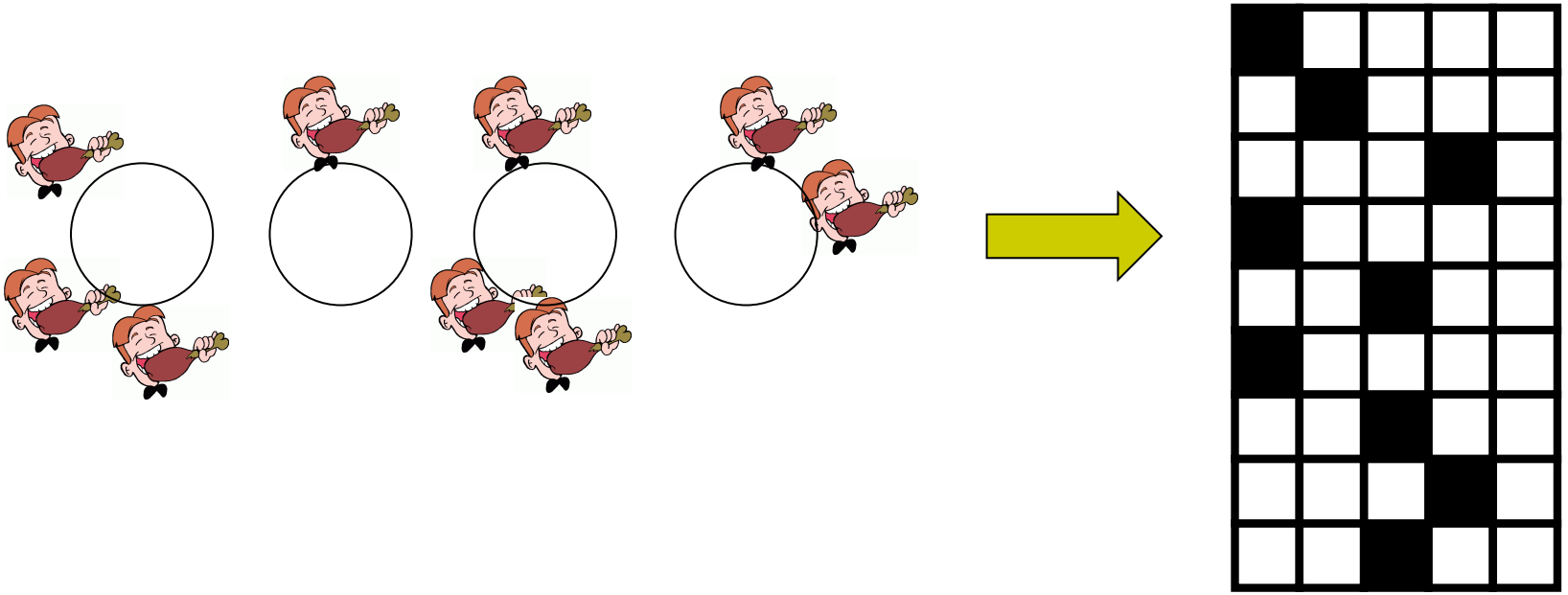
- Each column of **W** (and row of **A**) corresponds to a feature.
- Question: Can we make the number of features *unbounded* a posteriori, as we did with the DP?
- Solution: allow *infinitely many* features a priori – ie let **W** (or **A**) have infinitely many columns (rows).
- Problem: We can't represent infinitely many features!
- Solution: make our infinitely large matrix *sparse*.

Griffiths and Ghaharamani, 2006

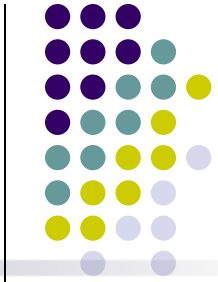
The CRP: A distribution over binary matrices



- Recall that the CRP gives us a distribution over *partitions* of our data.
- We can represent this as a distribution over *binary matrices*, where each row corresponds to a data point, and each column to a cluster.



A sparse, finite latent variable model



- We want a *sparse* model – so let

$$\mathbf{X} = \mathbf{W}\mathbf{A}^T + \epsilon$$

$$\mathbf{W} = \mathbf{Z} \odot \mathbf{V}$$

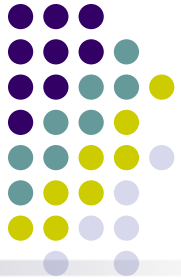
for some sparse matrix **Z**.

- Place a *beta-Bernoulli prior* on **Z**:

$$\pi_k \sim \text{Beta}\left(\frac{\alpha}{K}, 1\right), k = 1, \ldots, K$$

$$z_{nk} \sim \text{Bernoulli}(\pi_k), n = 1, \ldots, N.$$

A sparse, finite latent variable model



- If we integrate out the π_k, the marginal probability of a matrix **Z** is:

$$p(\mathbf{Z}) = \prod_{k=1}^{K} \int \left(\prod_{n=1}^{N} p(z_{nk}|\pi_k) \right) p(\pi_k) d\pi_k$$

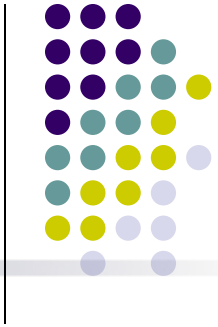
$$= \prod_{k=1}^{K} \frac{B(m_k + \alpha/K, N - m_k + 1)}{B(\alpha/K, 1)}$$

$$= \prod_{k=1}^{K} \frac{\alpha}{K} \frac{\Gamma(m_k + \alpha/K)\Gamma(N - m_k + 1)}{\Gamma(N + 1 + \alpha/K)}$$

where $m_k = \sum_{n=1}^{N} z_{nk}$

- This is *exchangeable* (doesn't depend on the order of the rows or columns

A sparse, finite latent variable model



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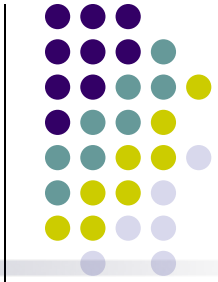
- How is this sparse?

An equivalence class of matrices



- We can naively take the infinite limit by taking *K* to infinity
- Because all the columns are equal in expectation, as *K* grows we are going to have more and more empty columns.
- We do not want to have to represent infinitely many empty columns!
- Define an *equivalence class* [**Z**] of matrices where the non-zero columns are all to the left of the empty columns.
- Let *lof(.)* be a function that maps binary matrices to *left-ordered* binary matrices – matrices ordered by the binary number made by their rows.

Left-ordered matrices

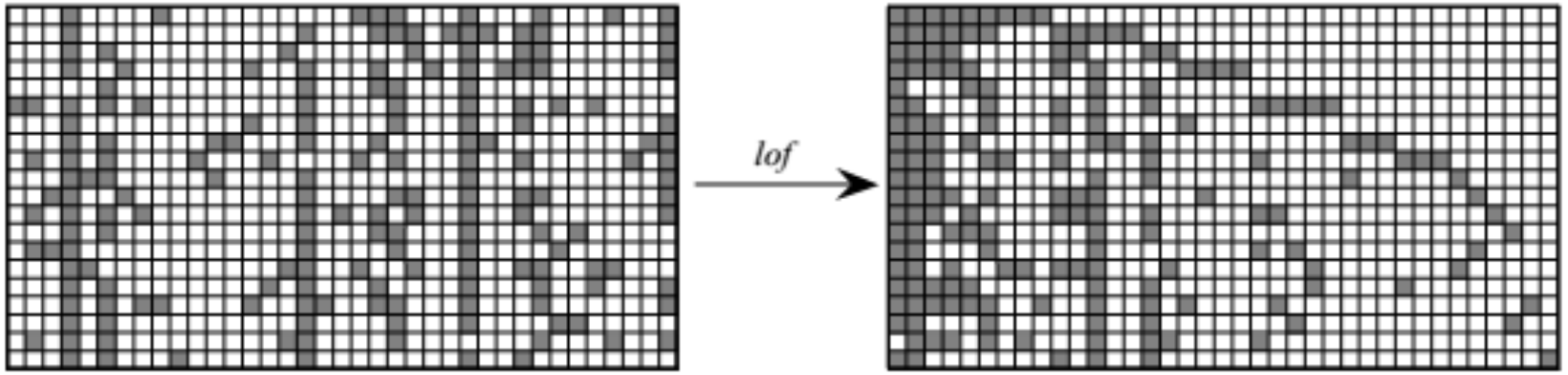
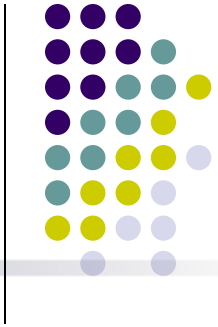


Figure 5: Binary matrices and the left-ordered form. The binary matrix on the left is transformed into the left-ordered binary matrix on the right by the function $lof(\cdot)$. This left-ordered matrix was generated from the exchangeable Indian buffet process with $\alpha = 10$. Empty columns are omitted from both matrices.

Image from Griffiths and Ghahramani, 2011

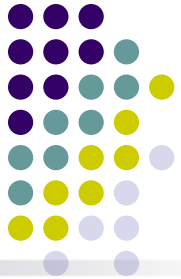
How big is the equivalence set?



- All matrices in the equivalence set [**Z**] are equiprobable (by exchangeability of the columns), so if we know the size of the equivalence set, we know its probability.
- Call the vector $(z_{1k}, z_{2,k}, \ldots, z_{(n-1)k})$ the *history* of feature *k* at data point *n* (a number represented in binary form).
- Let K_h be the number of features possessing history *h*, and let K_+ be the total number of features with non-zero history.
- The total number of lof-equivalent matrices in [**Z**] is

$$\binom{K}{K_0 \cdots K_{2^N-1}} = \frac{K!}{\prod_{n=0}^{2^N-1} K_n!}$$

Probability of an equivalence class of finite binary matrices.



- If we know the size of the equivalence class [**Z**], we can evaluate its probability:

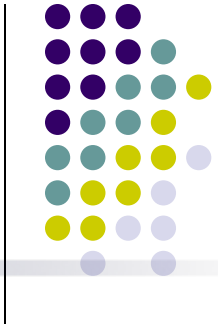
$$p([\mathbf{Z}]) = \sum_{\mathbf{Z}\in[\mathbf{Z}]} p(\mathbf{Z})$$

$$= \frac{K!}{\prod_{n=0}^{2^N-1} K_n!} \prod_{k=1}^{K} \frac{\alpha}{K} \frac{\Gamma(m_k+\alpha/K)\Gamma(N-m_k+1)}{\Gamma(N+1+\alpha/K)}$$

$$= \frac{\alpha^{K_+}}{\prod_{n=1}^{2^N-1} K_n!} \frac{K!}{K_0! K^{K_+}} \left(\frac{N!}{\prod_{j=1}^{N} j+\alpha/K} \right)^K$$

$$\cdot \prod_{k=1}^{K_+} \frac{(N-m_k)! \prod_{j=1}^{m_k-1}(j+\alpha/K)}{N!}$$

Taking the infinite limit



- We are now ready to take the limit of this finite model as *K* tends to infinity:

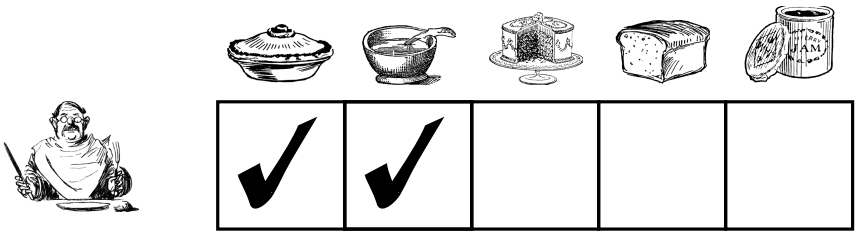
$$\frac{\alpha^{K_+}}{\prod_{n=1}^{2^N-1} K_n!} \frac{K!}{K_0! K^{K_+}} \left(\frac{N!}{\prod_{j=1}^{N} j + \frac{\alpha}{K}} \right)^K \prod_{k=1}^{K_+} \frac{(N-m_k)! \prod_{j=1}^{m_k-1} (j + \frac{\alpha}{K})}{N!}$$

$$\downarrow K \to \infty$$

$$\frac{\alpha^{K_+}}{\prod_{n=1}^{2^N-1} K_n!} \qquad 1 \qquad \exp\{-\alpha H_N\} \qquad \prod_{k=1}^{K_+} \frac{(N-m_k)!(m_k-1)!}{N!}$$

Predictive distribution: The Indian buffet process

- We can describe this model in terms of the following restaurant analogy.
  - A customer enters a restaurant with an infinitely large buffet
  - He helps himself to Poisson(α) dishes.



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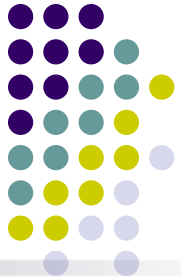

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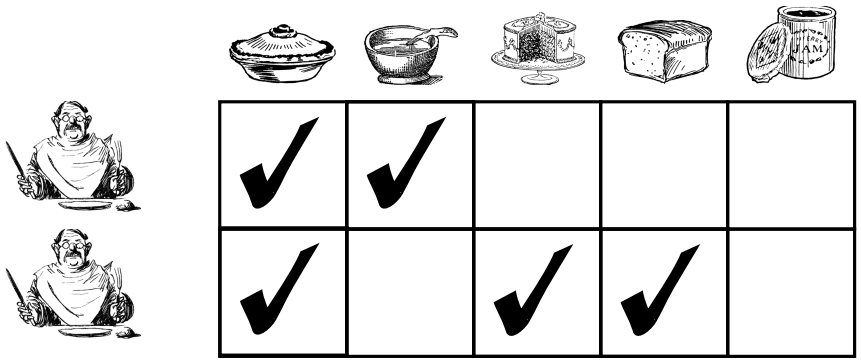
  - The n^{th} customer enters the restaurant
  - He helps himself to each dish with probability m_k/n
  - He then tries Poisson(α/n) new dishes



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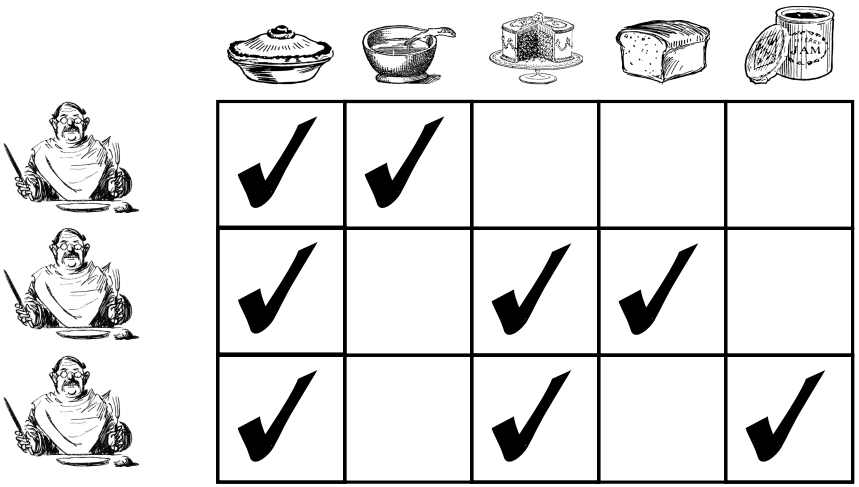
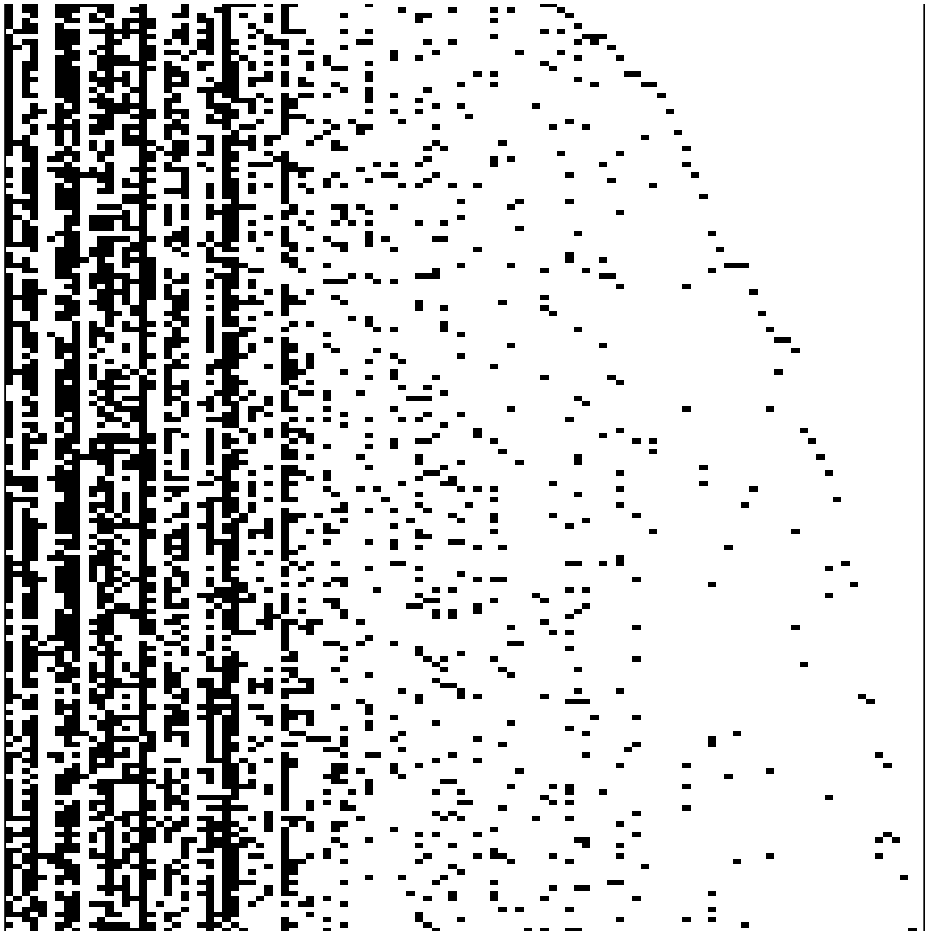
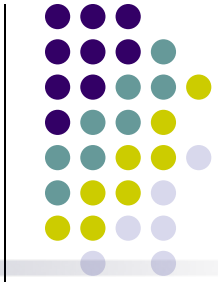


Example



Proof that the IBP is lof-equivalent to the infinite beta-Bernoulli model



- What is the probability of a matrix **Z**?
- Let $K_1^{(n)}$ be the number of new features in the n^{th} row.

$$p(\mathbf{Z}) = \prod_{n=1}^{N} p(\mathbf{z}_n | \mathbf{z}_{1:(n-1)})$$

$$= \prod_{n=1}^{N} \text{Poisson}\left(K_1^{(n)} \middle| \frac{\alpha}{n}\right) \prod_{k=1}^{K_+} \left(\frac{\sum_{i=1}^{n-1} z_{ik}}{n}\right)^{z_{nk}} \left(\frac{n - \sum_{i=1}^{n-1} z_{ik}}{n}\right)^{1-z_{nk}}$$

$$= \prod_{n=1}^{N} \left(\frac{\alpha}{n}\right)^{K_1^{(n)}} \frac{1}{K_1^{(n)}!} e^{-\alpha/n} \prod_{k=1}^{K_+} \left(\frac{\sum_{i=1}^{n-1} z_{ik}}{n}\right)^{z_{nk}} \left(\frac{n - \sum_{i=1}^{n-1} z_{ik}}{n}\right)^{1-z_{nk}}$$

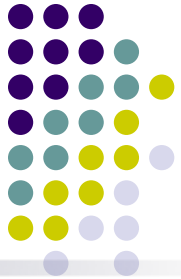
$$= \frac{\alpha^{K_+}}{\prod_{n=1}^{N} K_1^{(n)}!} \exp\{-\alpha H_N\} \prod_{k=1}^{K_+} \frac{N - m_k)!(m_k - 1)!}{N!}$$

- If we include the cardinality of [**Z**], this is the same as before

Properties of the IBP

- "Rich get richer" property – "popular" dishes become more popular.
- The number of nonzero entries for each row is distributed according to Poisson(α) – due to exchangeability.
- Recall that if $x_1 \sim \text{Poisson}(\alpha_1)$ and $x_2 \sim \text{Poisson}(\alpha_2)$, then $(x_1+x_2) \sim \text{Poisson}(\alpha_1+\alpha_2)$
  - The number of nonzero entries for the whole matrix is distributed according to Poisson($N\alpha$).
  - The number of non-empty columns is distributed according to Poisson(αH_N)

Building latent feature models using the IBP



- We can use the IBP to build latent feature models with an unbounded number of features.
- Let each column of the IBP correspond to one of an *infinite* number of features.
- Each row of the IBP selects a *finite subset* of these features.
- The **rich-get-richer** property of the IBP ensures features are shared between data points.
- We must pick a *likelihood model* that determines **what the features look like** and **how they are combined**.