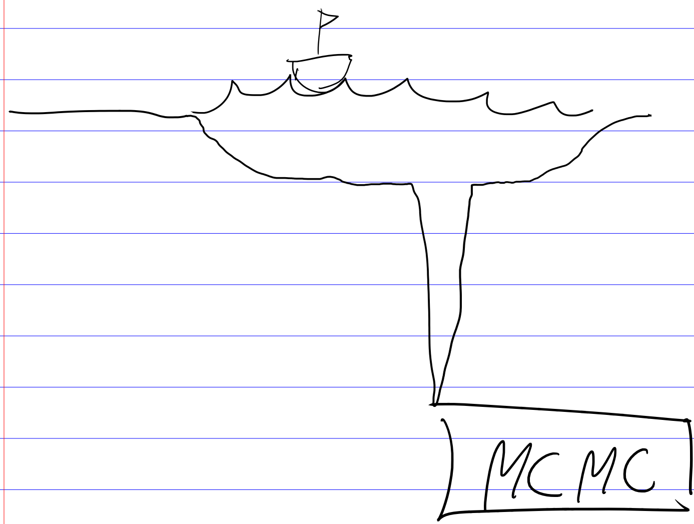

Setting

Goal: Draw samples $\{\vec{x}^{(t)}\}_{t=1}^{T}$ from $p(\vec{x})$

Assume: $p(\vec{x}) = \dfrac{\tilde{p}(\vec{x})}{Z}$ — can compute

Metropolis Alg.

- Choose a proposal $q(\vec{x}\,|\,\vec{x}')$
  s.t. $q(x, x') = q(x', x)$ (symmetric)
- Choose initial state $x^{(1)}$
- For $t = 1 \ldots T$:
  - Propose a new state $x \sim q(x\,|\,x^{(t)})$
  - Accept x w/prob.
    $$a = A(x \leftarrow x^{(t)}) = \min\left(1, \frac{\tilde{p}(x)}{\tilde{p}(x^{(t)})}\right)$$
  - If $\underbrace{x \text{ is accepted}}_{a \,>\, u \,\sim\, \mathrm{Uniform}(0,1)}$: $x^{(t+1)} = x$
  - Otherwise: $x^{(t+1)} = x^{(t)}$

Metropolis-Hastings Alg.

- The proposal q need not be symmetric
- Identical to Metropolis alg., but accept w/prob.

$$A(x \leftarrow x^{(t)}) = \min\left(1, \frac{\tilde{p}(x)}{\tilde{p}(x^{(t)})} \frac{q(x^{(t)}\,|\,x)}{q(x\,|\,x^{(t)})}\right)$$

- M is a special of M-H w/ q symmetric.

Gibbs Sampling

- Instead, draw samples of one-dimension from "full conditionals" $p(x_i \mid \{x_j\}_{j\neq i})$ and always accept.

$$\boxed{|\vec{x}| = N}$$

- Let $x^{(1)}$ be the initial state.
  $t = 1$
- while true:

  for $i = 1 \ldots N$:
    sample $x_i^{(t+1)} \sim p(x_i \mid \{x_j^{(t)}\}_{j\neq i})$
    set $\vec{x}_{-i}^{(t+1)} = \vec{x}_{-i}^{(t)}$
  set $t = t+1$

Gibbs as M-H

- Gibbs Sampling is just a special case of M-H

$$A(x \leftarrow x^{(t)}) = \min\left(1, \frac{\tilde{p}(x)}{p(x^{(t)})} \frac{q(x^{(t)} \mid x)}{q(x \mid x^{(t)})}\right)$$

$$= \min\left(1, \frac{p(x)}{p(x^{(t)})} \frac{p(x_i^{(t)} \mid \vec{x}_{-i})}{p(x_i \mid \vec{x}_{-i}^{(t)})}\right)$$

$$= \min\left(1, \frac{p(x_i \mid \vec{x}_{-i})\, p(\vec{x}_{-i})}{p(x_i^{(t)} \mid \vec{x}_{-i}^{(t)})\, p(\vec{x}_{-i}^{(t)})} \cdot \frac{p(x_i^{(t)} \mid \vec{x}_{-i})}{p(x_i \mid \vec{x}_{-i}^{(t)})}\right)$$

$$\boxed{x_{-i}^{(t)} = x_{-i}}$$

$$= \min(1, 1)$$

Markov Chains

Markov chain is a random process
↳ gives series of r.v.s $\vec{x}^{(1)}, \ldots, \vec{x}^{(T)}$

- first-order MC: $p(\vec{x}^{(t+1)} \mid \vec{x}^{(t)} \ldots x^{(1)}) = p(\vec{x}^{(t+1)} \mid \vec{x}^{(t)})$
- second-order MC: $p($ '' $) = p(\vec{x}^{(t+1)} \mid \vec{x}^{(t)}, \vec{x}^{(t-1)})$

Transition Probabilities

Trans Probs: $p(x^{(t+1)} \mid x^{(t)}) = R_t(x_{t+1} \leftarrow x_t)$

Homogeneous MC: $R_t \triangleq R \Rightarrow$ the trans probs are the same for all t

Invariant Dist (Required Property #1)

Consider the marginal of $x^{(t+1)}$

$$p(x^{(t+1)}) = \sum_{x^{(t)}} p(x^{(t+1)} \mid x^{(t)})\, p(x^{(t)})$$ true by induction

We say $p^*(x)$ is invariant (aka. stationary) w.r.t. the transition matrix (aka. our Markov chain) R if:

$$p^*(x) = \sum_{x'} R(x \leftarrow x')\, p^*(x')$$

OR

$$p^*(x) = \int_x R(x \leftarrow x')\, p^*(x')$$

Equilibrium Dist (Required Property #2)

Given an arbitrary initial dist $p^{(1)}(x)$

Suppose $p^*(x)$ is invariant wrt R

$p^*(x)$ is the equilibrium distribution if

$$\lim_{t \to \infty} p^{(t)}(x) = p^*(x)$$

if it exists, there can only be one

(aka. the MC is ergodic)

Sufficient Conditions

R satisfies detailed balance if:

$$R(x' \leftarrow x)\, p^*(x) = R(x \leftarrow x')\, p^*(x')$$

and $\Rightarrow$ $p^*(x)$ is invariant wrt R.

If $p^*(x)$ is invariant wrt R, and

$$v = \min_{\substack{x, x' \\ \text{s.t. } p^*(x') > 0}} \frac{R(x' \leftarrow x)}{p^*(x)} > 0$$

then p^* is the equilibrium distribution for R.

Transition Matrix for M-H

$$R(x \leftarrow x^{(t)}) = \underline{q(x \mid x^{(t)})} \cdot \min\left(1, \underline{\frac{p(x)}{p(x^{(t)})} \frac{q(x^{(t)} \mid x)}{q(x \mid x^{(t)})}}\right)$$

<u>Ex:</u> Prove that M-H satisfies detailed balance.