Bayesian & Markov Networks: A unified view

Probabilistic Graphical Models (10-708)

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Reading: KF-Chap. 5.7, 5.8

- Auditing students: please fill out forms
- Recitation:
- questions:
Question: Is there a BN that is a perfect map for a given MN?

- The "diamond" MN

A

B

C

D

Question: Is there a BN that is a perfect map for a given MN?

A ⊥ C | {B,D}
B ⊥ D | {A,C}

A ⊥ C | {B,D}
B ⊥ D | A

A ⊥ C | {B,D}
B ⊥ D

- This MN does not have a perfect I-map as BN!
Question: Is there an MN that is a perfect I-map to a given BN?

• V-structure example

A

B

C

¬ (A ⊥ B | C)

¬ (A ⊥ B)

¬ (A ⊥ B | C)

¬ (A ⊥ B)

V-structure has no equivalent in MNs!
Instead of attempting perfect I-maps between BNs and MNs, we can try minimal I-maps.

Recall: H is a minimal I-map for G if

- \( I(H) \subseteq I(G) \)
- Removal of a single edge in H renders it not an I-map

Note: If H is a minimal I-map of G, H need not necessarily satisfy all the independence relationships in G.

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**Markov Blanket**

- **Markov Blanket** of \( X \) in a BN G:
  - \( MB_G(X) \) is the unique minimal set \( U \) of nodes in G such that \( (X \perp (\text{all other nodes}) | U) \) is guaranteed to hold for any distribution that factorizes over G.

- Defn (5.7.1): \( MB_G(X) \) is the set of nodes consisting of \( X \)'s parents, \( X \)'s children and other parents of \( X \)'s children.

- Idea: The neighbors of \( X \) in H --- the minimal I-map of G --- should be \( MB_G(X) \)!
Defn (5.7.3): The moral graph $M(G)$ of a BN $G$ is an undirected graph that contains an undirected edge between $X$ and $Y$ if:
- there is a directed edge between them in either direction
- $X$ and $Y$ are parents of the same node
Comment: this definition ensures $MB_G(X)$ is the set of neighbors of $X$ in $M(G)$

Corollary (5.7.4): The moral graph $M(G)$ of any BN $G$ is a minimal I-map for $G$
- Moralization turns each $(X, Pa(X))$ into a fully connected subset
  - CFDs associated with the network can be used as clique potentials
- The moral graph loses some independence information
Minimal I-maps from BNs to MNs: Perfect I-maps

- Proposition (5.7.5): If the BN $G$ is "moral", then its moralized graph $M(G)$ is a perfect I-map of $G$.

- Proof sketch:
  - $I(M(G)) \subseteq I(G)$ (from before)
  - The only independence relations that are potentially lost from $G$ to $M(G)$ are those arising from V-structures
  - Since $G$ has no V-structures (it is moral), no independencies are lost in $M(G)$

Soundness of $d$-separation

- Recall $d$-separation
  - Let $U = \{X, Y, Z\}$ be three disjoint sets of nodes in a BN $G$.
  - Let $G^+$ be the ancestral graph: the induced BN over $U \cup \text{ancestors}(U)$.
  - Then, $d$-$\text{sep}_{G}(X;Y|Z)$ iff $d$-$\text{sep}_{M(G^+)}(X;Y|Z)$

\[
\begin{align*}
D-\text{sep}_G(D;I \mid L) & \quad \text{sep}_{M(G^+)}(D;I \mid L) & \quad \text{Sep}_{M(G^+)}(D;I \mid S,A) \\
D-\text{sep}_G(D;I \mid S, A) & \quad & \\
\end{align*}
\]
Soundness of $d$-separation

- Why it works:

  \[ \text{G: } B \perp C \mid A \quad \text{M(G): } \neg(B \perp C \mid A) \quad \text{M(G^+): } B \perp C \mid A \]

  - Idea: Information \textit{blocked} through common children in G that are not in the conditioning variables, is simulated in M(G+) by ignoring all children.

Minimal I-maps from BNs to MNs:

\textbf{Summary}

- Moral Graph M(G) is a minimal I-map of G
- If G is moral, then M(G) is a perfect I-map of G
- $D\text{-sep}_G(X;Y\mid Z) \iff sep_{M(G^+)}(X;Y\mid Z)$
- \textit{Next:} minimal I-maps from MNs to BNs
Any BN I-map for an MN must add triangulating edges into the graph.

Minimal I-maps from MNs to BNs: chordal graphs

- Defn (5.7.11): Let $X_1-X_2-\ldots-X_k-X_1$ be a loop in a graph. A chord in a loop is an edge connecting $X_i$ and $X_j$ for non-consecutive $\{X_i, X_j\}$. An undirected graph $H$ is chordal if any loop $X_{i_1}-X_{i_2}-\ldots-X_{i_k}-X_{i_1}$ for $K \geq 4$ has a chord.

- Defn (5.7.12): A directed graph $G$ is chordal if its underlying undirected graph is chordal.
Thm (5.7.13): Let \( H \) be an MN, and \( G \) be any BN minimal I-map for \( H \). Then \( G \) can have no immoralities.
- Intuitive reason: Immoralities introduce additional independencies that are not in the original MN.
- (cf. proof for theorem 5.7.13 in K&F)

Corollary (5.7.14): Let \( K \) be any minimal I-map for \( H \). Then \( K \) is necessarily chordal!
- Because any non-triangulated loop of length at least 4 in a Bayesian network graph necessarily contains an immorality.

Process of adding edges also called *triangulation*.

Thm (5.7.15): Let \( H \) be a non-chordal MN. Then there is no BN \( G \) that is a perfect I-map for \( H \).

Proof:
- Minimal I-map \( G \) for \( H \) is chordal.
- It must therefore have additional directed edges not present in \( H \).
- Each additional edge eliminates some independence assumptions.
- Hence proved.
Clique trees (1)

- **Notation:**
  - Let $H$ be a connected undirected graph. Let $C_1, \ldots, C_k$ be the set of maximal cliques in $H$.
  - Let $T$ be a tree structured graph whose nodes are $C_1, \ldots, C_k$.
  - Let $C_i, C_j$ be two cliques in the tree connected by an edge. Define $S_{ij} = C_i \cap C_j$ be the sep-set between $C_i$ and $C_j$.
  - Let $W_{<i,j} = \text{Variables}(C_i) - \text{Variables}(S_{ij})$ --- the residue set.

Clique trees (2)

- A tree $T$ is a **clique tree** for $H$ if:
  - Each node corresponds to a clique in $H$ and each maximal clique in $H$ is a node in $T$.
  - Each sepset $S_{ij}$ separates $W_{<i,j}$ and $W_{<j,i}$.

- Every undirected chordal graph $H$ has a clique tree $T$.
  - Proof by induction (cf. Theorem 5.7.17 in K&F).
  - Example in next slide.
Example

- Example chordal graph and its clique tree

I-maps of MN as BN:

- **Thm (5.7.19):** Let $H$ be a chordal MN. Then there exists a BN such that $I(H) = I(G)$.
- **Proof sketch:**
  - Since $H$ is an MN, it has a clique tree
  - Number the nodes consistent with clique ordering
I-maps of MN as BN:

- **Thm (5.7.19):** Let $H$ be a chordal MN. Then there exists a BN such that $I(H) = I(G)$.

- **Proof sketch (contd):**
  - For each node $X_i$, let $C_i$ be the first clique it occurs in.
  - Define $Pa(X_i) = \var{C_i} - X_i \cap \{X_1, \ldots, X_{i-1}\}$

  - $G$ and $H$ have the same edges
  - All parents of each $X_i$ are in the same clique node
    - $\Rightarrow$ they are connected
    - $\Rightarrow$ no immoralities in $G$

Minimal I-maps from MNs to BNs:

**Summary**

- A minimal I-map BN of an MN is chordal
  - Obtained by triangulating the MN

- If the MN is chordal, there is a perfect BN I-map for the MN
  - Obtained from the corresponding clique-tree

- Next: Hybrids of BNs and MNs
  - Partially Directed Acyclic Graphs
Partially Directed Acyclic Graphs

- Also called chain graphs
- Nodes can be disjointly partitioned into several chain components
- An edge within the same chain component must be undirected
- An edge between two nodes in different chain components must be directed

Chain components:
\{A\}, \{B\}, \{C,D,E\}, \{F,G\}, \{H\}, \{I\}

Summary

- Investigated the relationship between BNs and MNs
  - They represent different families of independence assumptions
  - Chordal graphs can be represented in both

- Chain networks: superset of both BNs and MNs